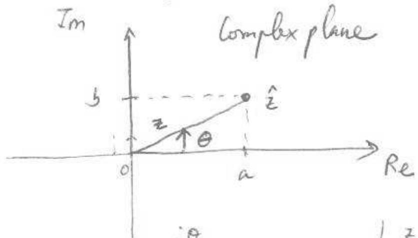
Circuit Analysis II (Linear)

Complex numbers: ① $\hat{z} = a + jb$ $\quad \begin{cases} Re[\hat{z}] = a \\ Im[\hat{z}] = b \end{cases}$

rectangular coordinates

$j = \sqrt{-1}$



② $\hat{z} = z e^{j\theta}$

polar coordinates

$\begin{cases} z = \text{magnitude of } \hat{z} \\ \theta = \text{angle, or phase, w.r.t. to Re axis} \end{cases}$

(Here $\hat{z}$ is located in the first quadrant)

① → ② : $z = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

② → ① : $a = z\cos\theta$, $b = z\sin\theta$ → ① $\hat{z} = z\cos\theta + j z\sin\theta$

& ② → $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

de Moivre Formula.

Consequence: $1 e^{j\frac{\pi}{2}} = j$ or phase of j is 90° ($\frac{\pi}{2}$ radians)

Vector space: $(1,0)$ $(0,1)$

Real axis Imaginary axis: j

Polar coordinates for $\hat{z}_1 = -3 + j4$ $\begin{cases} z_1 = 5 \\ \theta_1 = -53.13° \end{cases}$ $\hat{z}_1 = 5\, e^{-j\,53.13°}$ (sec. quadrant)

$\hat{z}_2 = 3 - j4 \longrightarrow \hat{z}_2 = 5\, e^{-j\,53.15°}$ (fourth quadrant)

Sinusoids : $x(t) = \underbrace{A}_{\text{amplitude}} \cos(\underbrace{\omega}_{\text{angular frequency} = 2\pi f} t + \underbrace{\theta}_{\text{phase}})$

$$= \underbrace{\mathrm{Re}\left[A\, e^{j(\omega t + \theta)} \right]}$$

Sinusoid can be associated with a vector in the complex plane

$$\cos(\omega t + \theta_1) = \cos \omega t$$

$$\cos(\omega t + \theta_2) = \cos\left[\omega \left(t + \frac{\theta_2}{\omega} \right) \right]$$

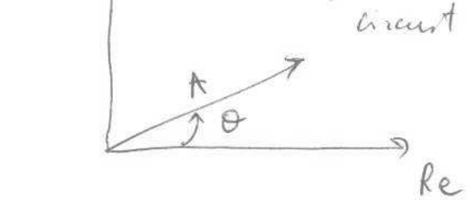
time shift $= \dfrac{\pi}{4 \cdot 5} =$

$$\frac{\theta_2}{\omega} = \frac{\pi}{\omega}$$

$$x(t) = \mathrm{Re}\left[e^{j\omega t} \left(A\, e^{j\theta} \right) \right]$$

$\boxed{\text{Phasor}}$

Sinusoid can also be associated with this $\boxed{\text{vector}}$ in complex plane (if we leave out time-dependence factor; same ω thru-out circuit)



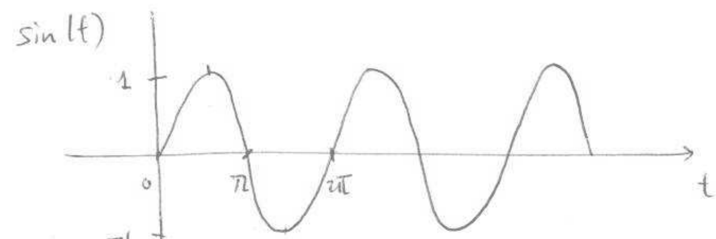
- Phase change b/w two sinusoids using phasors:

$$v_1(t) = 6 \sin(100t + 60^\circ)$$
$$v_2(t) = -6 \cos(100t + 30^\circ)$$

Trig. identities: $\cos\alpha = \sin(\alpha + \frac{\pi}{2})$

$$\cos \omega t = \sin(\omega t + 90^\circ)$$

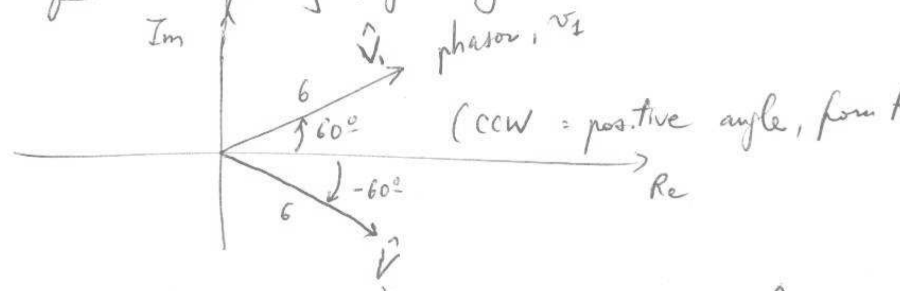
$$v_2(t) = -6 \sin(100t + 120^\circ)$$



if shifted π to the left = change of sign.

$$v_2(t) = 6 \sin(100t + \underset{\uparrow}{300^\circ})$$

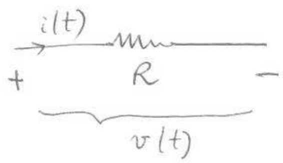
Let's see their relative phase by drawing the phasor diagram (ignoring frequency and time)



Then $\underset{\text{signal}}{v_1}$ or $\underset{\text{phasor}}{\hat{V}_1}$ is 120° ahead of $\underset{\text{signal}}{v_2}$ or $\underset{\text{phasor}}{\hat{V}_2}$.

Phasors & AC Circuit Analysis (Linear Circuits)

R



: $\boxed{R = \frac{v(t)}{i(t)}}$

AC circuit : $i(t) = I \cos(\omega t + \theta_I)$

$\updownarrow$ same (linear circuit)

b/c of Ohm's law : $v(t) = V \cos(\omega t + \theta_V)$

$i(t) = Re\left[\underbrace{I e^{j\theta_I}}_{\text{phasor } \hat{I}} e^{j\omega t}\right]$ (can represent $i(t)$ by $\hat{I} = I e^{j\theta_I}$)

$v(t) = Re\left[\underbrace{V e^{j\theta_V}}_{\text{phasor } \hat{V}} e^{j\omega t}\right]$ (can represent $v(t)$ by $\hat{V} = V e^{j\theta_V}$)

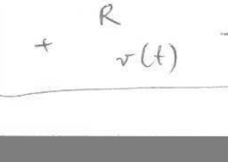
$$R = \frac{v(t)}{i(t)} = \frac{Re\left[\hat{V} \cancel{e^{j\omega t}}\right]}{Re\left[\hat{I} \cancel{e^{j\omega t}}\right]} = \frac{Re[\hat{V}]}{Re[\hat{I}]} = Re\left[\frac{\hat{V}}{\hat{I}}\right] = Re\left[\frac{V}{I} e^{j(\theta_V - \theta_I)}\right]$$

$\Rightarrow \theta_V - \theta_I = 0$ or voltage & current are in phase at R

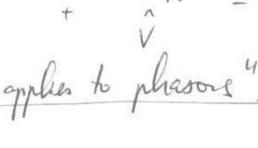
or $\boxed{\theta_V = \theta_I}$

With this result I can write : $R = \frac{\hat{V}}{\hat{I}} = \hat{Z}_R$

(impedance at a resistance is R)



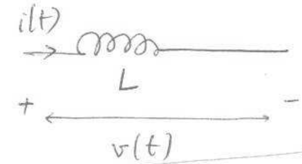
$R = \frac{v(t)}{i(t)}$; also :



$R = \frac{\hat{V}}{\hat{I}}$

"Ohm's law also applies to phasors" !!

At L :



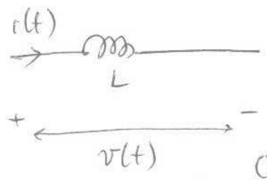
$$\boxed{v(t) = L\,\frac{di}{dt}}$$

AC i & v

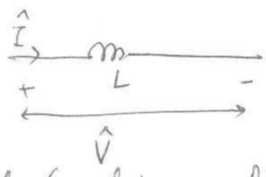
$$\mathrm{Re}\left[\hat{V}\cancel{e^{j\omega t}}\right] = L\,\frac{d}{dt}\,\mathrm{Re}\left[\hat{I}\,e^{j\omega t}\right] = L\,\mathrm{Re}\left[\frac{d}{dt}\left(\hat{I}e^{j\omega t}\right)\right]$$

$$= L\,\mathrm{Re}\left[\hat{I}\,j\omega\,\cancel{e^{j\omega t}}\right]$$

$\rightarrow \hat{V} = Lj\omega\,\hat{I}$: Now, with this result, I can define a impedance at the inductor as $\boxed{\hat{Z}_L = \frac{\hat{V}}{\hat{I}} = j\omega L}$



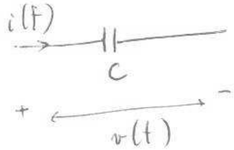
$v(t) = L\,\frac{di}{dt}$ or



$\hat{Z}_L = j\omega L$

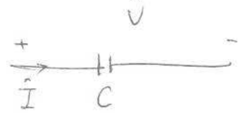
Can use Ohm's law arithmetic (complex) even at an inductor in AC circuits !!

at C: Derive $\hat{Z}_C$ (extra credit: 0.5 one HW set)



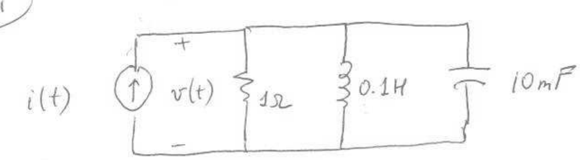
$$i(t) = C\frac{dv}{dt}$$

$$\rightarrow \; Z_c = \frac{1}{j\omega C} = \frac{\hat{V}}{\hat{I}}$$

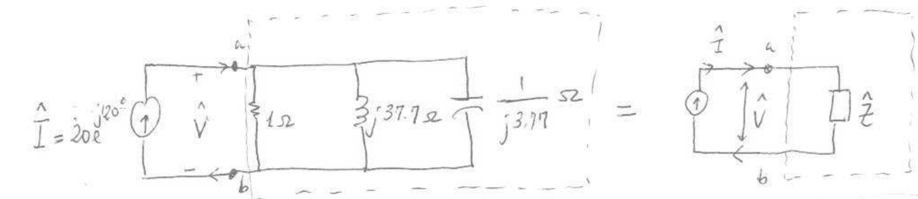


$$Z_c = (j\omega C)^{-1}$$

(7.21) $i(t) = 20\cos(377t + 120^\circ)$ A $\rightarrow$ find $v(t)$ via frequency domain analysis.



$\downarrow$ F.D.



$$\hat{V} = 20e^{j120^\circ} \cdot \hat{Z} = 20e^{j120^\circ}\left(1 \,\|\, \frac{10}{j37.7 - j\frac{1}{3.77}}\right) = 20e^{j120^\circ}\left(1 \,\|\, \frac{10}{j37.43}\right)$$

$j = \sqrt{-1}$

$j^2 = -1$

$-j = e^{-j90^\circ}$

$$= 20e^{j120^\circ}\left(1 \,\|\, -j0.26\right) = 20e^{j120}\frac{-j0.26}{1 - j0.26}$$

$$= 20e^{j120^\circ}\frac{\boxed{0.26\,e^{-j90^\circ}}}{\underbrace{\sqrt{1^2 + 0.26^2}}_{1.033}\,e^{-j14.96^\circ}} = \frac{20 \cdot 0.26}{1.033}e^{j(120^\circ - 90^\circ + 14.96^\circ)}$$

$$= 5.03\,e^{j44.96} \text{ V} \approx 5e^{j45^\circ} \text{ V}$$

$$\rightarrow \boxed{v(t) = 5\cos(377t + 45^\circ) \text{ V}}$$

(8)

$$\rightarrow \left\{ \begin{array}{l} V = RI_1 + \omega L I_2 \\ 0 = RI_2 - \omega L I_1 \end{array} \right\rangle \text{ solve for } I_1 \ \& \ I_2$$

$$I_1 = \frac{R}{(\omega L)^2 + R^2} V \quad ; \quad I_2 = \frac{\omega L}{(\omega L)^2 + R^2} V$$

since we defined I_1 & I_2 as $\left\{ \begin{array}{l} I_1 = I \cos\theta \\ I_2 = -I \sin\theta \end{array} \right.$

$$\rightarrow \frac{I_2}{I_1} = -\tan\theta \rightarrow \theta = -\tan^{-1}\frac{I_2}{I_1} = -\tan^{-1}\frac{\omega L}{R}$$

$$I^2 = I_1^2 + I_2^2 \rightarrow I = \sqrt{\frac{R^2}{[(\omega L)^2 + R^2]^2} V^2 + \frac{(\omega L)^2}{[(\omega L)^2 + R^2]^2} V^2}$$

$$= \frac{V}{(\omega L)^2 + R^2} \sqrt{R^2 + (\omega L)^2} = \frac{V}{\sqrt{(\omega L)^2 + R^2}}$$

$$\Rightarrow i(t) = I \cos(\omega t + \theta)$$

$$\boxed{i(t) = \frac{V}{\sqrt{(\omega L)^2 + R^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)}$$

$\boxed{\text{Phasors}}$:

$\hat{Z} = R$

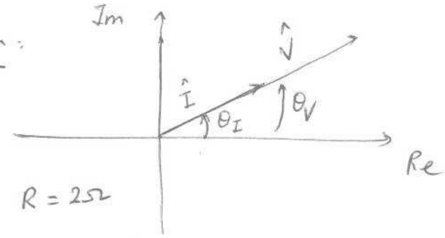
Ohm's law: $\hat{Z} = \frac{\hat{V}}{\hat{I}} = \frac{V}{I}e^{j(\theta_V-\theta_I)}$

(Complex numbers: $\hat{z}_1 = z_1 e^{j\theta_1}$; $\hat{z}_2 = z_2 e^{j\theta_2} \rightarrow \frac{\hat{z}_1}{\hat{z}_2} = \frac{z_1 e^{j\theta_1}}{z_2 e^{j\theta_2}} = \frac{z_1}{z_2}e^{j(\theta_1-\theta_2)}$

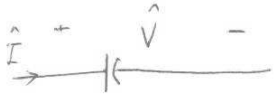
$$\underset{\text{(zero phase)}}{R} = \frac{V}{I}e^{j(\underbrace{\theta_V-\theta_I}_{0})}$$

→ "Voltage & current at a resistor are in <u>phase</u>."

Phasor diagram:



$R = 2\Omega$

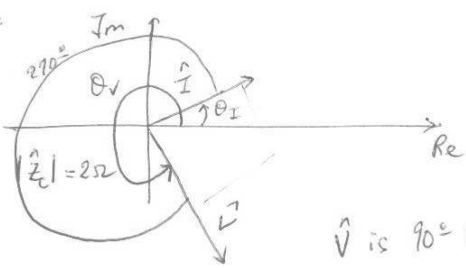


$\hat{Z} = \frac{1}{j\omega C}$

Ohm's law: $\hat{Z}_C = \frac{1}{j\omega C} = \frac{V}{I}e^{j(\theta_V-\theta_I)}$

$\rightarrow \theta_V - \theta_I = -90°$

Phasor diagram:



$\hat{V}$ is 90° behind $\hat{I}$ across a capacitor.

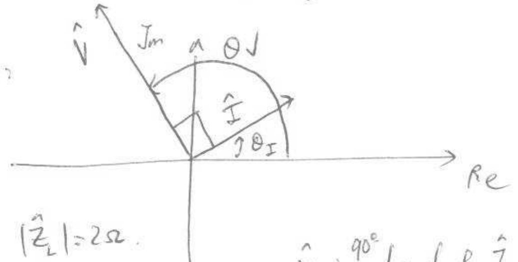
$\xrightarrow{\hat{I}}$ + $\hat{V}$ −

$\hat{Z}_L = j\omega L$

Ohm's law: $\hat{Z}_L = j\omega L = \frac{V}{I} e^{j(\theta_v - \theta_i)}$

$\theta_v - \theta_I = 90°$.

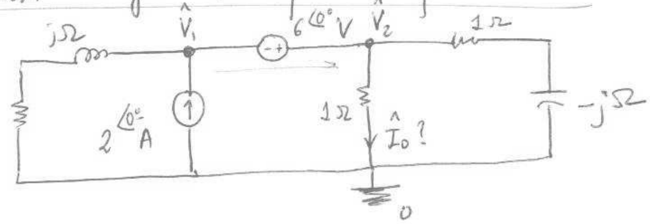
Phasor diagram:



$|\hat{Z}_L| = 2\Omega$.

$\hat{V}$ is 90° ahead of $\hat{I}$ across an inductor.

AC Circuit Analysis Techniques using Phasors:



1) Nodal Analysis: KCL assign a voltage at each node

At node 1: KCL:

$$+2 - \frac{\hat{V}_1}{1+j} - \underbrace{\left(\frac{\hat{V}_2}{1} + \frac{\hat{V}_2}{1-j}\right)}_{\text{This current is leaving node 1}} = 0$$

$$\hat{V}_2 = \hat{V}_1 + 6$$

Since we need to compute $\hat{I}_o$ → we need $\hat{V}_2$ → let's eliminate $\hat{V}_1$ → KCL:

$$2 - \frac{\hat{V}_2 - 6}{1+j} - \hat{V}_2 - \frac{\hat{V}_2}{1-j} = 0$$

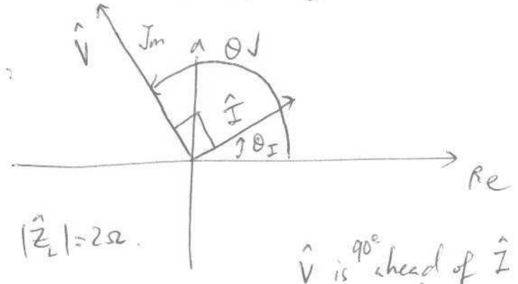
$\hat{I} \rightarrow$ $+\ \hat{V}\ -$

$\hat{Z}_L = j\omega L$

Ohm's law: $\hat{Z}_L = j\omega L = \frac{V}{I} e^{j(\theta_V - \theta_I)}$

$\theta_V - \theta_I = 90°$.

Phasor diagram:



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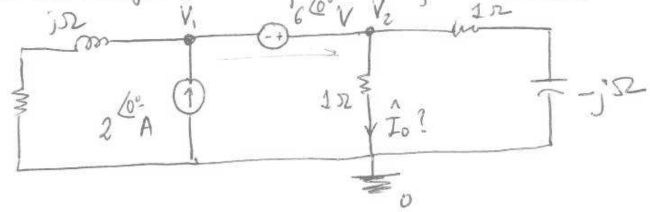
AC Circuit Analysis Techniques using Phasors:



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