16.76

(16.3): $\Delta Q = mc\Delta T \longrightarrow \frac{\Delta Q}{\Delta t} = mc\frac{\Delta T}{\Delta t} \xrightarrow[\Delta t \to 0]{\lim} \frac{dQ}{dt} = mc\frac{dT}{dt}$

(16.5): $H = -kA\frac{\Delta T}{\Delta x} = \frac{dQ}{dt}$ (Heat loss rate)

$$\Rightarrow mc\frac{dT}{dt} = -kA\frac{\Delta T}{\Delta x} \Rightarrow \frac{dT}{dt} = \underbrace{\frac{-kA}{mc\Delta x}}_{\text{constants}}\Delta T \Rightarrow \boxed{\frac{dT}{dt} \propto \Delta T}$$

Newton's Law of cooling

(Rate of change of temperature is proportional to the temperature difference between the house and its surroundings)

$$\frac{dT}{dt} = -\frac{1}{mc}\cdot\frac{kA}{\Delta x}\cdot\Delta T = -\frac{1}{C}\cdot\frac{1}{R}\cdot\Delta T = -\frac{35}{6.5\cdot10^{6}\cdot6.67\cdot10^{-3}} = -8.07\cdot10^{-4}\,\frac{{}^{o}K}{s}$$

→ Heat capacity $C \equiv mc = 6.5\cdot10^{6}\,\frac{J}{{}^{o}K}$

→ Thermal resistance: $R \equiv \frac{\Delta x}{kA} = 6.67\cdot10^{-3}\,\frac{{}^{o}K}{W}$

→ Temperature difference: $\Delta T = 20^{o}C - (-15^{o}C) = 35^{o}K$

How long would it take to go from $20^{o}C$ to $0^{o}C$ freezing point: $\left(\frac{dT}{dt} = 8.07\cdot10^{-4}\,\frac{{}^{o}K}{s}\right)$

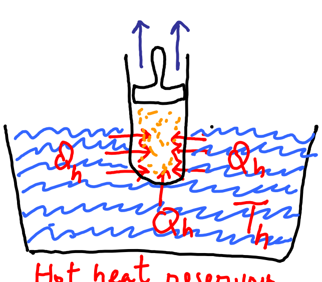
$$t = \frac{\Delta T}{\frac{dT}{dt}} = \frac{20\,{}^{o}K}{8.07\cdot10^{-4}\,\frac{{}^{o}K}{s}} = 24790\,s\cdot\frac{1h}{3600s} = 6.88\ \text{hrs}.$$

Ch 19 2nd Law of Thermodynamics

↳ Heat engine

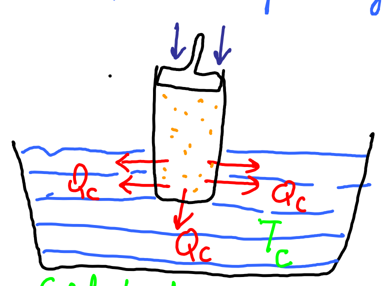
↳ Heat reservoir: source of heat that stays at constant temperature

Heat engine: placing an ideal gas in a cylinder + piston in thermal contact with a hot heat reservoir @ T_h, then in thermal contact with a cold heat reservoir @ T_c, then repeating this cycle.



Hot heat reservoir

As heat transfers in, gas KE is increased, its temperature & pressure are increased → gas expands, piston is pushed up, work done by gas W is positive.

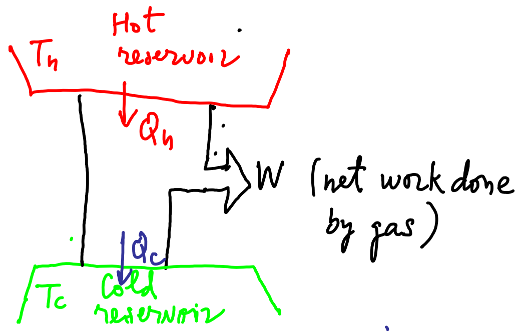


Cold heat reservoir

To prepare gas for further work on piston, it is brought in thermal contact with a cold reservoir. As heat transfers out gas KE is decreased, so are its temperature & pressure → gas compresses, piston comes back down, This continues until the gas reaches TD equilibrium with hot reservoir @ T_h

work done by gas is negative. This continues until the gas reaches TD equilibrium with cold reservoir @ T_c

Heat engine diagram:



Efficiency of Heat Engine:

1) 1st Law of T.D: $\Delta U_{\text{Heat engine}} = Q - W$ (Heat absorbed minus work done by gas)

$$= Q_h - Q_c - W$$

2) Heat reservoirs are at constant temperature → heat transfers in/out of Heat engine under isothermal processes → ideal gas $\begin{cases} \text{Monoatomic}: \Delta U = \frac{3}{2} k \Delta T \cdot N \\ \text{Diatomic}: \Delta U = \frac{5}{2} k \Delta T \cdot N \end{cases}$

$\Delta T = 0 \Rightarrow \Delta U_{\text{Heat engine}} = 0$

$\Rightarrow Q_h - Q_c - W = 0$ or $\boxed{Q_h - Q_c = W}$

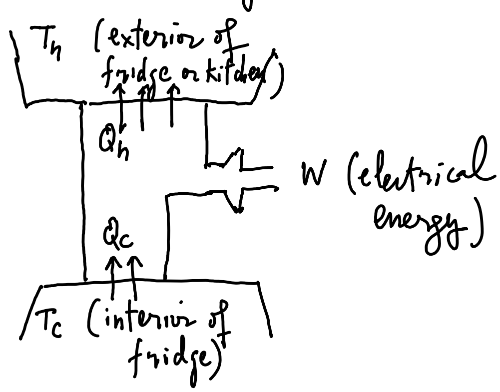
3) Efficiency of a heat engine using ideal gas:

$$e \equiv \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} \Rightarrow e = 1 - \frac{|Q_c|}{|Q_h|}$$

$e < 1$

$|Q_c| < |Q_h| \rightarrow \frac{|Q_c|}{|Q_h|} < 1 \rightarrow \boxed{e < 1} \rightarrow$ 2nd Law of TD: it is impossible to build a heat engine working in cycles that extracts heat from a hot reservoir (and returning some of it to a cold reservoir) that can deliver 100 % of work or $e < 1$

Reversed heat engine (refrigerators)



2nd Law of TD: it is impossible to transfer heat from a cold reservoir to a hot reservoir without requiring any work W (29)

Coefficient of performance $C.O.P. \equiv \frac{Q_c}{W}$

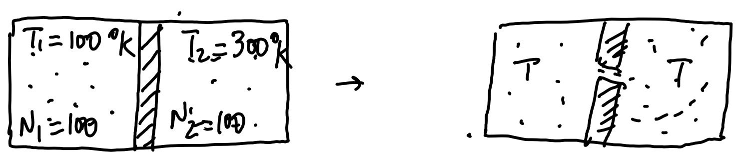
3rd Law TD:

↳ Entropy: $\Delta S \equiv \int_1^2 \frac{dQ}{T}$

↳ (i) Degree of disorder

(ii) The entropy of a closed system (no external assistance) can never decrease or $\Delta S \geq 0$

Example: two gases at different temperatures once they are mixed up they would never separate back to their original temperatures (without external assistance)

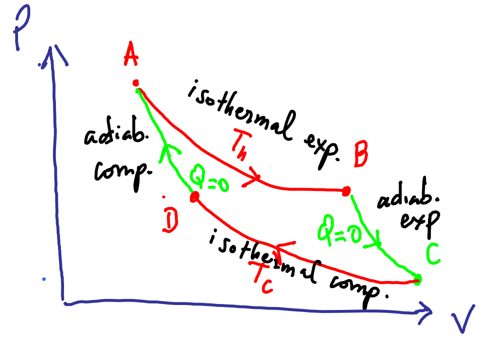


Hotter molecules tend to go left, some of cold molecules get pushed to the right, they will arrive at same final temperature T. This mix ~ higher level of disorder ($\Delta S > 0$)

Heat engines
- Carnot engines: max efficiency $e_{max} = 1 - \frac{T_c}{T_h}$ (will derive)
- Otto Cycle engines: $e < e_{max}$

Carnot Engines: (i) a type of heat engine, follos 4 reversible processes (two isothermal, two adiabatic)

(ii) $e_{Carnot} = e_{max} = 1 - \frac{T_c}{T_h}$.



Q_h: heat absorbed from hot reservoir during isothermal expansion $A \to B$

$$Q_h = W = nRT_h \cdot \ln\left(\frac{V_B}{V_A}\right)$$

$$\Delta U = 0 = Q - W \text{ (isothermal)}$$

Q_c: heat ejected to cold reservoir during isothermal compression $C \to D$

$$Q_c = nRT_c \ln\left(\frac{V_D}{V_C}\right)$$

(since $V_D < V_C \to Q_c < 0$ heat loss to cold reservoir)

$B \to C$: adiabatic expansion : $T_B V_B^{\gamma-1} = T_C V_C^{\gamma-1} \Rightarrow \left(\frac{V_B}{V_C}\right)^{\gamma-1} = \frac{T_c}{T_B} = \frac{T_c}{T_h}$

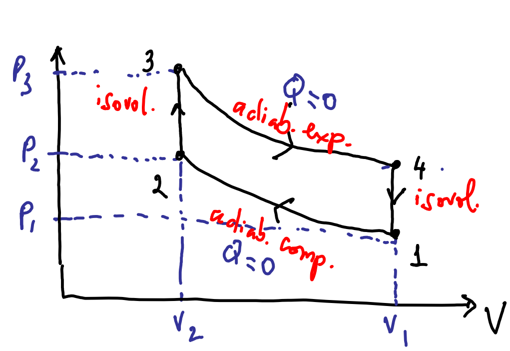
$D \to A$: adiabatic compression: $T_D V_D^{\gamma-1} = T_A V_A^{\gamma-1} \Rightarrow \left(\frac{V_D}{V_A}\right)^{\gamma-1} = \frac{T_A}{T_D} = \frac{T_h}{T_c}$

$\frac{V_B}{V_C} = \frac{V_A}{V_D}$ or $\boxed{\frac{V_B}{V_A} = \frac{V_C}{V_D}}$

$$\boxed{e_{Carnot} = e_{max} = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{|\cancel{nR}T_c \ln\left(\frac{V_D}{V_C}\right)|}{|\cancel{nR}T_h \ln\left(\frac{V_B}{V_A}\right)|} = 1 - \frac{T_c}{T_h}\frac{|\ln\left(\frac{V_D}{V_C}\right)|}{|\ln\left(\frac{V_B}{V_A}\right)|} = 1 - \frac{T_c}{T_h}}$$

<u>Otto Cycle Engine:</u> (i) Another type of heat engine. It follows four reversible processes : 2 adiabatic, 2 isovolumic

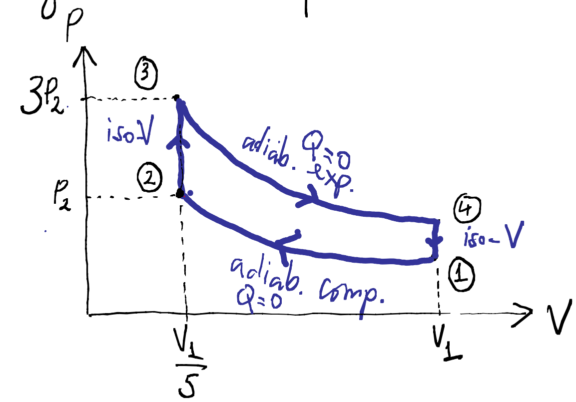
(ii) $e_{otto} < e_{max}$



Applications of reversible processes in Ch18 to the calculations of heat engine efficiencies

19.53 Given a gasoline engine operating in the Otto Cycle (2 adiabatic + 2 isovolumic processes) and given its PV diagram, find its efficiency $e = \frac{W}{Q_h}$

Step 1: Diagram with information: PV diagram is given



$\gamma = \frac{C_p}{C_v}$ specific heat ratio

$e = \frac{W}{Q_h}$

Step 2: Relevant equations: eqs. for iso-V & adiabatic processes.

iso-V ④→① ($P_1 < P_4$)

$$e = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{|Q_c|}{|Q_h|} < 1$$

iso-V ②→③ ($P_3 = 3P_2$)

Heat reservoirs @ constant T:
$\Delta U = 0 = Q_h - Q_c - W$

(i) Heat exchange during iso-volumic processes: $c_v = \frac{1}{n}\frac{Q}{\Delta T}$ or $\boxed{Q = nc_v \Delta T}$

$$\left\{\begin{aligned} Q_h &= Q_{23} \underset{\text{constant vol.}}{=} nc_v(T_3 - T_2) \\ Q_c &= Q_{41} = nc_v(T_1 - T_4) \end{aligned}\right\} \Rightarrow e = 1 - \frac{|T_1 - T_4|}{|T_3 - T_2|}$$

Now we need to relate these temperatures to γ

Adiabatic processes
$TV^{\gamma-1} = \text{constant}$

(ii) $T \cdot V^{\gamma-1}$ is constant during adiabatic processes: ①→② & ③→④

$$\left\{\begin{aligned} T_1 V_1^{\gamma-1} &= T_2 V_2^{\gamma-1} \\ T_3 V_3^{\gamma-1} &= T_4 V_4^{\gamma-1} \end{aligned}\right.$$

{ comes from $P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$ & $P_1 = \frac{nRT_1}{V_1}$; $P_2 = \frac{nRT_2}{V_2}$

There are some information on the volumes: $\left\{\begin{aligned} V_4 &= V_1 \\ V_3 &\underset{\text{iso-V}}{=} V_2 \overset{\text{given}}{=} \frac{V_1}{5} \end{aligned}\right.$

$$\left\{\begin{aligned} T_1 V_1^{\gamma-1} &= T_2 \left(\frac{V_1}{5}\right)^{\gamma-1} \\ T_4 \cancel{V_1^{\gamma-1}} &= T_3 \left(\frac{\cancel{V_1}}{5}\right)^{\cancel{\gamma-1}} \end{aligned}\right\} \Rightarrow \boxed{\frac{T_1}{T_4} = \frac{T_2}{T_3}}$$

$\frac{T_4}{T_3} = \frac{1}{5^{\gamma-1}}$ or $T_3 = T_4\, 5^{\gamma-1}$

Rewrite equation for e with these results:

$$e_{otto} = 1 - \frac{|T_1 - T_4|}{|T_3 - T_2|} = 1 - \frac{T_4 \cancel{\left|\frac{T_1}{T_4} - 1\right|}}{T_3 \cancel{\left|1 - \frac{T_2}{T_3}\right|}} = 1 - \frac{T_4}{T_3}$$

a) $$\boxed{e_{otto} = 1 - \frac{1}{5^{\gamma - 1}} = 1 - 5^{1-\gamma}}$$ $\left(\frac{1}{a} = a^{-1}\right)$

b) Write T_{max} (highest temperature in the Otto Cycle) in terms of T_{min} (min temp. in the Cycle)

T_1 lowest $\rightarrow T_{min} = T_1$

adiab. comp. $T_2 > T_1$

iso-V $(P_3 = 3P_2)$ $T_3 > T_2$ $\rightarrow T_{max} = T_3$

adiab. exp. $(V_4 = 5V_3)$ $T_4 < T_3$

& $P_4 > P_1 \rightarrow T_4 > T_1$

Write T_3 in term of T_1

(i) Adiabatic $3 \rightarrow 4$: $T_3 = T_4 5^{\gamma-1}$ (since $V_3 = \frac{V_1}{5}$ & $V_4 = V_1$)

(ii) Adiabatic $3 \rightarrow 4$ & $1 \rightarrow 2$: $\frac{T_1}{T_4} = \frac{T_2}{T_3}$

(iii) Ideal gas equation: $PV = nRT \rightarrow \boxed{\frac{T_2}{T_3} = \frac{\frac{P_2 V_2}{\cancel{nR}}}{\frac{P_3 V_3}{\cancel{nR}}} = \frac{P_2}{P_3} = \frac{1}{3}}$

$P_3 = 3P_2$ given

iso-V (2)→(3) $V_2 = V_3$

$\overset{(ii)}{\Rightarrow} \frac{T_1}{T_4} = \frac{T_2}{T_3} \overset{(iii)}{=} \frac{1}{3} \Rightarrow \boxed{T_4 = 3T_1}$

$\overset{(i)}{\Rightarrow} T_3 = T_4 5^{\gamma-1} = 3T_1 \cdot 5^{\gamma-1} = 3 \cdot 5^{\gamma-1} T_1$ or $\boxed{T_{max} = 3 \cdot 5^{\gamma-1} \cdot T_{min}}$

c) Compare e_{otto} to e_{Carnot} where the Carnot Engine work between same T_{max} & T_{min} (temps for hot & cold reservoirs, respectively)

$$e_{Carnot} = 1 - \frac{T_c}{T_h} = 1 - \frac{T_{min}}{T_{max}} = 1 - \frac{1}{3 \cdot 5^{\gamma-1}} = 1 - \frac{5^{1-\gamma}}{3}$$

$$e_{otto} \overset{a)}{=} 1 - 5^{1-\gamma} < e_{Carnot} = e_{max}$$

Conforms 2nd law of TD

Entropy calculation in different processes:

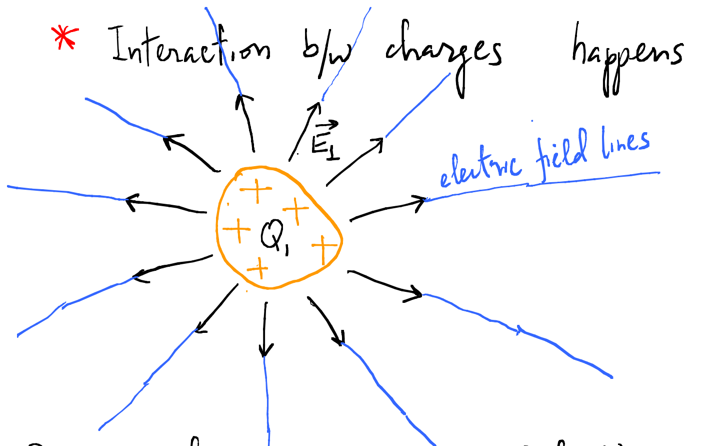
$$\Delta S \equiv \int_1^2 \frac{dQ}{T} \begin{cases} \text{Isothermal}: & \Delta S = \frac{1}{T}\int_1^2 dQ = \frac{Q_2 - Q_1}{T} = \frac{\Delta Q}{T} \quad \text{(change in entropy is heat change per unit temp)} \\ \text{Isovolumic}: & dQ = nc_v dT \Rightarrow \Delta S = nc_v \int_1^2 \frac{dT}{T} = nc_v \ln\left(\frac{T_2}{T_1}\right) \\ \text{Isobaric}: & dQ = nc_p dT \Rightarrow \Delta S = nc_p \ln\left(\frac{T_2}{T_1}\right) \end{cases}$$

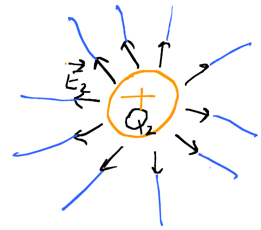
Ch 20: Electric Charge, Force, Field

An electric charge (Q_1) distribution creates an electric field ($\vec{E}_1$), when a probe (another charge Q_2) comes to its proximity it will feel an electric force by the original charge distribution

$$\boxed{\vec{F}_{12} = Q_2 \vec{E}_1}\ *$$

* Interaction b/w charges happens through their electric fields





Q_1 = a scalar (unit is C for Coulomb)
$\vec{E}_1$ = a vector

Electric field lines point away from a + charge distribution and they are more spread out away from the charge showing $\vec{E}_1$ is weaker away from the charge (stronger closer to charge)

Strength of field = density of field lines!

Q_2 = probe charge
↳ feels $\vec{F}_{12} = Q_2 \vec{E}_1$ (N)
↳ force by ① on ②
↳ unit: $\frac{N}{C}$

$\vec{E}_2$: electric field by Q_2
↳ Q_1 feels $\vec{F}_{21} = Q_1 \vec{E}_2$
↳ force by ② on ①

⇒ Both guest & host charges feel electric interaction!

Charges {
2 types: + and −; unit = charge of electron $e = 1.6 \cdot 10^{-19}$ C
e^+ e^-

- Charge of an electron is $-e$
- Charge of a proton is $+e$
- charge of a deuteron is $+2e$
- Charge of a H atom: 0

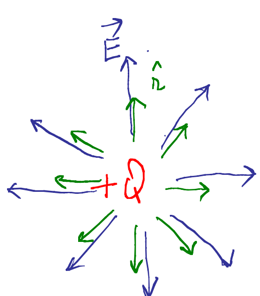
$-e$
$+e$

charge of H^+ is $+e$ (hydrogen ion)

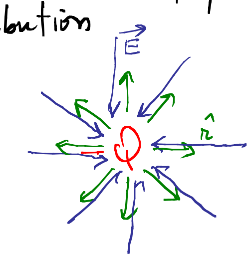
"Sources" of electric field

Charge Q → $\vec{E} = k \frac{Q}{r^2} \hat{r}$

- k : electric constant $= 9 \times 10^9 \frac{Nm^2}{C^2}$
- Q : net charge of the source
- r : separation from charge Q to field point or probe point (where the field is measured)
- $\hat{r}$: radial unit vector, points away from the charge distribution



$\vec{E} = +k \frac{Q}{r^2} \hat{r}$
points away from charge or repulsive



$\vec{E} = -k \frac{Q}{r^2} \hat{r}$
points toward charge or attractive

Electric

Sources: 2 types of charges

Fields: repulsive or attractive

$$\vec{E} = k\frac{Q}{r^2}\hat{r}$$

$k = 9\cdot10^9 \frac{Nm^2}{C^2}$

Gravitational

Source: one type of mass

Fields: always attractive

$$\vec{g} = -G\frac{M}{r^2}\hat{r}$$

$G = 6.67\cdot10^{-11} \frac{Nm^2}{kg^2}$

$\frac{k}{G} \sim 10^{20}$!

electric interactions are huge compared to gravitational interactions!

We feel grav. attraction, not much electric interaction

(one type, weak)

(two opposite types of charges balanced → no fields, although very strong)

Calculation of the electric field $\vec{E}$: 3 methods:

1) Vector superposition:
(ch. 20)
(any charge distribution)

$\begin{cases} \text{(i) } \vec{E} \text{ by two + charges} \\ \text{(ii) } \vec{E} \text{ by a ring of charge} \\ \text{(iii) } \vec{E} \text{ by an } \infty\text{-long line of charge} \end{cases}$

2) Use Gauss' Law: use symmetry to determine the Gaussian surface
(ch. 21)
(symmetric charge distribution: spherical, cylindrical, ...)

$$\hookrightarrow \begin{cases} \Phi_{G\text{-surface}} = \dfrac{q_{enclosed}}{\varepsilon_0} \quad ; \quad \varepsilon_0 = \dfrac{1}{4\pi k} = 8.85 \cdot 10^{-12} \dfrac{C^2}{Nm^2} \\ \phi = \oint \vec{E} \cdot d\vec{A} \quad \text{electric flux} \end{cases}$$

3) Use electric potential V, which is a <u>scalar</u> $\rightarrow$ arithmetic addition (advantage)
(Ch. 22)
(any charge distribution)

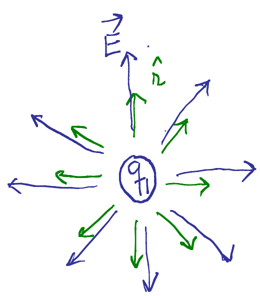
$$\hookrightarrow \vec{E} = -\vec{\nabla} V \quad ; \quad \vec{\nabla}: \text{"gradient operator"}$$

in Cartesian coords: $\vec{\nabla} = \hat{\imath}\dfrac{\partial}{\partial x} + \hat{\jmath}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$

Calculation of $\vec{E}$ using vector superposition:

1) $\vec{E}$ due to a single charge q_1 : $\vec{E} = k\dfrac{q_1}{r^2}\hat{r}$

$q_1 > 0$



$\vec{E} = +k\dfrac{q_1}{r^2}\hat{r}$

points away from charge or repulsive

(i) Field points in the radial direction away from a positive charge

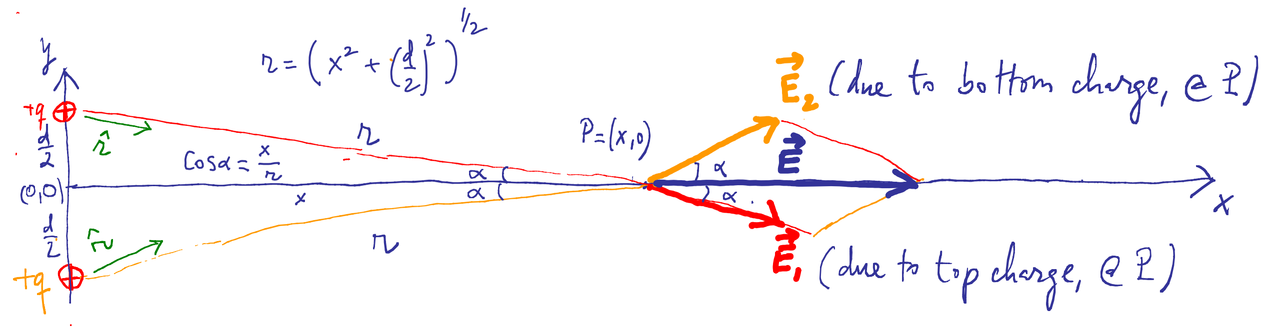
(ii) Field strength is proportional to the charge q_1

(iii) Field strength is inversely proportional to separation r^2

(both grav. & electric fields are <u>inverse-squared law</u>)

2) $\vec{E}$ due to <u>two positive charges</u> : <u>vector superposition</u>:

$\hookrightarrow$ along their line of symmetry (midline b/w the two charges)



a) $\vec{E}_1$ & $\vec{E}_2$ point along different directions

b) Both have the same field strength : $\frac{kq}{r^2}$

c) Total field due to both charges:

$$\vec{E}=\vec{E}_1+\vec{E}_2$$

(i) Graphical vector addition: in the parallelogram formed by two adding vectors, the diagonal is the sum vector

(ii) Mathematically using unit vectors + component

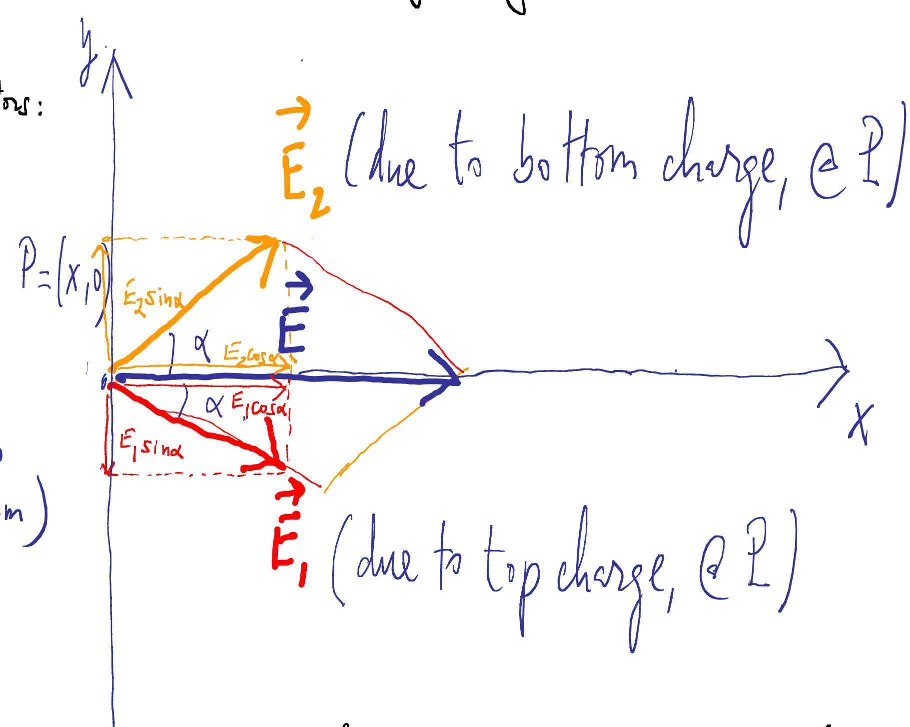
Using cartesian components & unit vectors:

$$\vec{E}_1=E_1\cos\alpha\,\hat{i}-E_1\sin\alpha\,\hat{j}$$

$+$ $E_1=E_2=\frac{kq}{r^2}$ same strength

$$\vec{E}_2=E_1\cos\alpha\,\hat{i}+E_1\sin\alpha\,\hat{j}$$

$$\vec{E}=2E_1\cos\alpha\,\hat{i}$$ (diagonal of parallelogram in the diagram)



Electric field due to two positive charges along their midline:

$$\vec{E}=2E_1\cos\alpha\,\hat{i}=\frac{2kq}{r^2}\cdot\frac{x}{r}\,\hat{i}=\frac{2kqx}{r^3}\,\hat{i}=\frac{2kqx}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{\frac{3}{2}}}\,\hat{i}\quad\left(\frac{N}{C}\right)$$

$E_1=E_2=k\frac{q}{r^2}$ $\quad\cos\alpha=\frac{x}{r}$ $\quad r=\left(x^2+\left(\frac{d}{2}\right)^2\right)^{\frac{1}{2}}$

(i) Total field points along midline; field strength is $\frac{2kqx}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}$

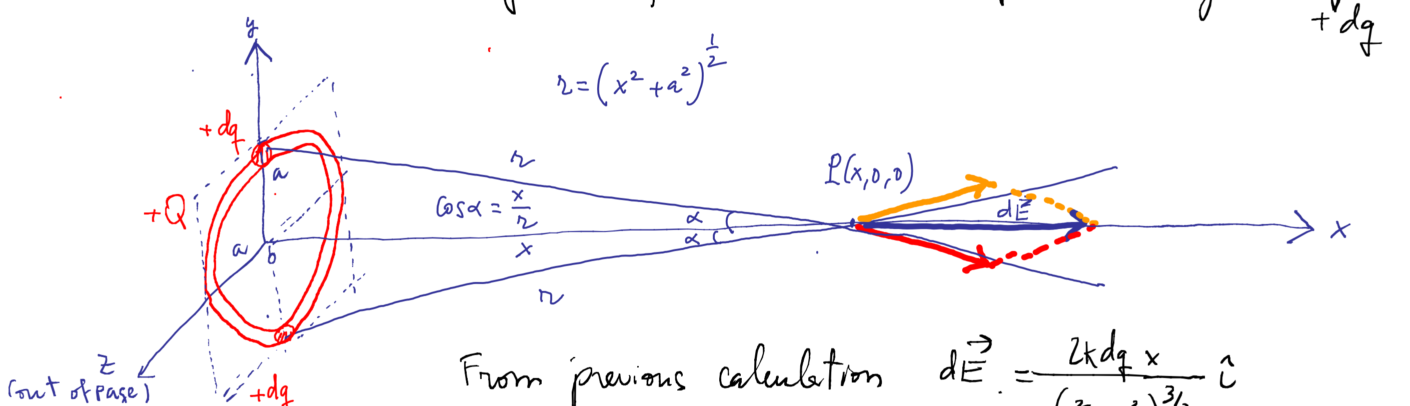
(ii) When P is very far away from the two charges $(x \to \infty)$

$$E \xrightarrow{x\to\infty} \frac{2kqx}{(x^2)^{3/2}} = \frac{2kqx}{x^3} = \frac{2kq}{x^2} = \frac{k(2q)}{x^2}$$

Very far away, the total field is that of a single charge of value $2q$ (this limit makes sense!)

3) Electric field due to a continuous ring of charge @ a point P along its axis: <u>vector superposition</u>:

↳ (i) Ring in YZ plane centered in the x-axis, radius a
(ii) Ring center is origin of coordinates
(iii) Total charge on ring is Q
(iv) Make use of total field due to two positive charges: $+dq \updownarrow 2a \; +dq$



From previous calculation $d\vec{E} = \frac{2k\,dq\,x}{(x^2+a^2)^{3/2}}\hat{\imath}$

Field due to entire ring: $\vec{E} = \int_{\text{half ring}} d\vec{E} = \frac{2kx}{(x^2+a^2)^{3/2}}\hat{\imath} \boxed{\int_{\text{half ring}} dq}$ $Q/2$

$$\boxed{\vec{E} = \frac{kQx}{(x^2+a^2)^{3/2}}\hat{\imath}}$$

↳ Very far away from ring $x >> a \rightarrow x^2+a^2 \simeq x^2$

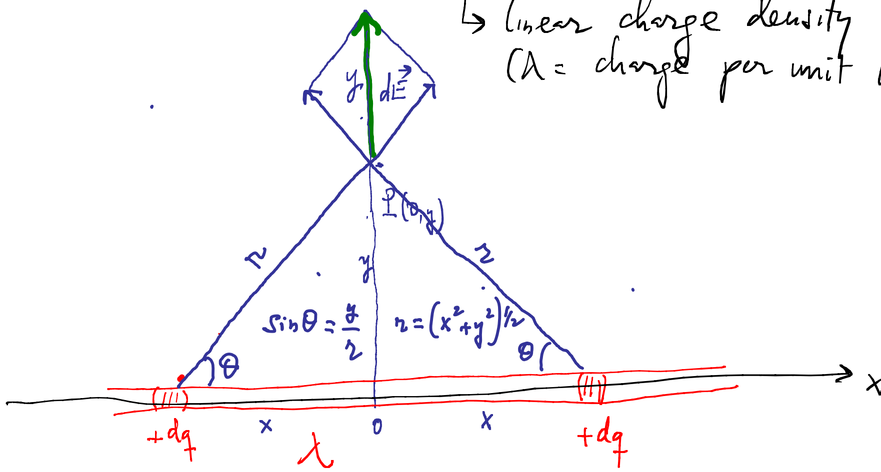
↳ $\vec{E} = \frac{kQx}{(x^2)^{3/2}}\hat{\imath} = \frac{kQx}{x^3}\hat{\imath} = \frac{kQ}{x^2}\hat{\imath}$

Makes sense

Very far away from ring, it looks like a single charge of value Q

4) Electric field due to ∞-long line of charge

↳ linear charge density $\lambda = \frac{dq}{dx}$
(λ = charge per unit length)



Set up $+dq \xleftrightarrow[2x]{} +dq$ so to use results 2) $\vec{E}$ due to two positive charges at a point P along their midline: $d\vec{E} = \frac{2k\,dq}{r^2}\sin\theta\,\hat{j} = \frac{2k\,dq\,y}{r^3}\hat{j} = \frac{2k\lambda dx \cdot y}{(x^2+y^2)^{3/2}}\hat{j}$

$\lambda = \frac{dq}{dx}$ or $dq = \lambda dx$

Total field → vector superposition : integral over half line:

$$\vec{E} = \int_{\text{Half line}} d\vec{E} = 2k\lambda y\,\hat{j} \underbrace{\int_{\text{Half line}} \frac{dx}{(x^2+y^2)^{3/2}}}_{} = 2k\lambda y\,\hat{j}\underbrace{\left[\frac{x}{y^2(x^2+y^2)^{1/2}}\right]_0^\infty}_{\text{Half-line}}$$

Table: $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$

$x \to \infty$ $\quad (x^2+y^2)^{1/2} \to (x^2)^{1/2} = x$

$$\vec{E} = 2k\lambda y\,\hat{j}\,\frac{\cancel{x}}{y^2\cdot\cancel{x}} = \frac{2k\lambda}{y}\hat{j}$$

Note: here field is inverse-linear law not inverse-squared law!

Ch 21 Gauss Law:

↳ $\phi_{G\text{-surface}} = \dfrac{q_{enclosed}}{\varepsilon_0}$; ε_0 = dielectric constant in vacuum $= \dfrac{1}{4\pi k} = 8.85 \cdot 10^{-12} \dfrac{C^2}{Nm^2}$

$\phi_{G\text{-surface}}$: electric flux through Gaussian surface.

$$\phi = \oint_{\substack{\text{closed}\\ \text{surface}}} \vec{E} \cdot d\vec{A}$$

↑ dot product or scalar product

$\vec{A} \cdot \vec{B} = AB \cos\theta$
θ : angle b/w $\vec{A}$ & $\vec{B}$
only if $\vec{A} \parallel \vec{B} \Rightarrow \theta = 0°$
$\Rightarrow \cos\theta = 1 \Rightarrow \vec{A} \cdot \vec{B} = A \cdot B$
(scalar product is reduced to arithmetic product)

↳ $\vec{E}$: electric field
$d\vec{A}$: element area of surface we integrate over. always points away from surface

$\vec{E} \cdot d\vec{A}$: electric flux element → $\int \vec{E} \cdot d\vec{A}$ → total electric flux through closed surface

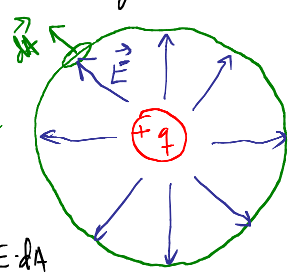
Gaussian surface: a closed surface such that $\oint \vec{E} \cdot d\vec{A} = E \cdot A$ { ↳ E is constant over Gaussian surface → $\vec{E} \parallel d\vec{A}$

↳ Goals { (i) Make $\phi = \oint \vec{E} \cdot d\vec{A}$ simply $E \cdot A$
(ii) This allows us to use Gauss Law to calculate E : $\phi_{G\text{-surface}} = \dfrac{q_{enclosed}}{\varepsilon_0}$

→ Requirement: symmetric charge distribution! $E \cdot A = \dfrac{q_{enclosed}}{\varepsilon_0} \Rightarrow E = \dfrac{q_{enclosed}}{\varepsilon_0 \cdot A}$

$\vec{E}$ due to a single charge:

Spherical Gaussian surface



$A = 4\pi r^2$ (surface of a sphere of radius r)

$\varepsilon_0 = \dfrac{1}{4\pi k} \rightarrow 4\pi \varepsilon_0 = \dfrac{1}{k}$

↳ { $\vec{E} \parallel d\vec{A}$
$\vec{E} \cdot d\vec{A} = E \cdot dA$
inverse-squared law: same separation r, same field strength → E is constant over Gaussian surface : $\phi = \oint \vec{E} \cdot d\vec{A} = \oint E \cdot dA = E \oint dA = E \cdot A$

→ Gauss Law: $\phi = \dfrac{q_{enclosed}}{\varepsilon_0}$

$E \cdot A = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{\varepsilon_0 A} = \dfrac{q}{4\pi \varepsilon_0 r^2} = \dfrac{q}{\frac{1}{k} r^2}$

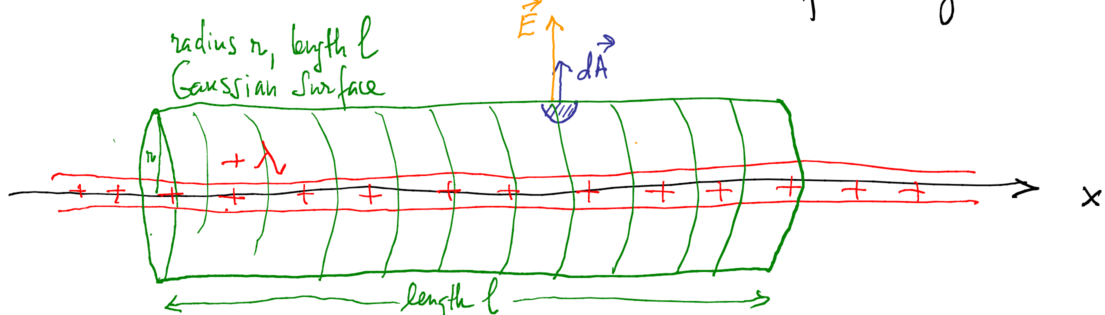
$= \dfrac{kq}{r^2}$ ✓

Calculation of $\vec{E}$ by an ∞-long line of charge (linear charge density $\lambda = \frac{dq}{dx}$)

↳ Using Gauss Law

↳ Symmetry of charge distribution

↳ Cylindrical Gaussian surface whose axis is the line of charge



With this Gaussian surface:

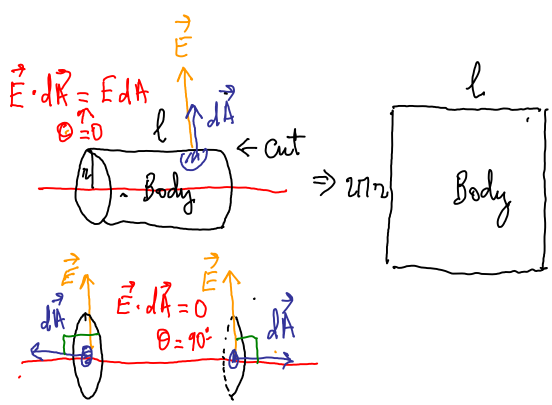
(i) $d\vec{A} \parallel \vec{E} \Rightarrow \vec{E}\cdot d\vec{A} = E\,dA$ (both $\vec{E}$ & $d\vec{A}$ point in the radial direction & away from line of charge)

(ii) $\vec{E}$ has same strength on surface (same separation r to line of charge)

$$\Rightarrow \phi = \oint \vec{E}\cdot d\vec{A} = E \oint dA = EA$$

(iii) Total surface area of the cylinder of radius r & length ℓ is: $2\pi r\cdot \ell + \underline{\pi r^2 + \pi r^2}$

don't count for ϕ as $\vec{E}\cdot d\vec{A} = 0$ on left & right sides (see diagrams on left)



(iv) $q_{enclosed}$ by G-surface: $\lambda\cdot\ell$

Gauss Law:

$$\phi = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\downarrow$$

$$EA$$

$$\downarrow$$

$$E\,2\pi r\,\cancel{\ell} = \frac{\lambda\cdot\cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{\lambda}{\cancel{2\pi}\cdot\frac{1}{\cancel{4\pi}_{2} k} r} = \frac{2k\lambda}{r}$$

$$\varepsilon_0 = \frac{1}{4\pi k}$$

(same result as using Vector superposition)

Ch 22 Electric Potential

(3rd method to calculate $\vec{E}$)

Mechanics: potential energy (difference) b/w two points A & B:

$$\Delta U_{AB} = -\int_A^B \underbrace{\vec{F} \cdot d\vec{l}}_{\text{scalar product}}$$

(i) $\vec{F} = m\vec{g}$ or $G\frac{Mm}{r^2}\hat{r}$ → ΔU is grav. potential energy

(ii) $\vec{F} = q'\vec{E}$ → ΔU is electric potential energy

q': probe charge

$$\rightarrow \Delta U_{AB} = -q'\int_A^B \vec{E} \cdot d\vec{l}$$

$$\rightarrow \Delta V_{AB} \equiv \frac{\Delta U_{AB}}{q'}$$: electric potential (difference) b/w two points A & B

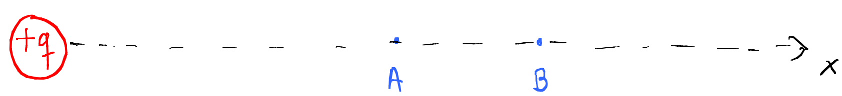
unit is $\frac{J}{C}$

$$\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l} \Rightarrow \vec{E} = -\vec{\nabla} V_{AB}$$

$\vec{\nabla}$ gradient: vector derivative

Calculate $\vec{E}$ due to a point charge using electric potential (difference):

$d\vec{l} = dx\,\hat{i}$



(i) $$\Delta V_{AB} = -\int_A^B \frac{kq}{x^2}\underbrace{\hat{i} \cdot \hat{i}}_{1 \cdot 1 \cdot \cos 0 = 1} dx = -kq\underbrace{\int_A^B \frac{dx}{x^2}}_{-\left[\frac{1}{x}\right]_A^B} = kq\left[\frac{1}{x}\right]_A^B = kq\left[\frac{1}{x_B} - \frac{1}{x_A}\right]$$

Define: $V \equiv \frac{kq}{x} \Rightarrow \Delta V_{AB} = V_B - V_A$

(ii) Set reference for electric potential $V = 0$ @ $x = \infty$ or $V_\infty = \frac{kq}{\infty} = 0$

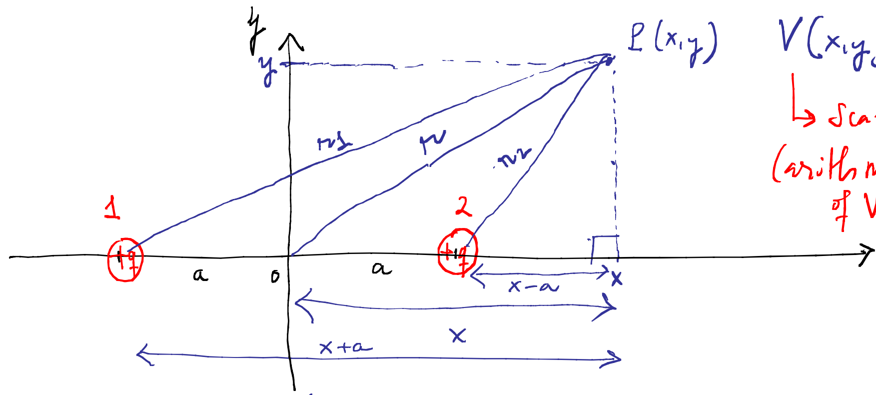
$$\Delta V_{\infty B} = V_B - V_\infty = V_B \Rightarrow V_r \equiv \Delta V_{\infty r}$$

$$\boxed{V(r) = \frac{kq}{r}}$$

Electric potential due to a point charge q at separation r from charge.

(iii) $E(r) = -\nabla V = -\frac{d}{dr}V = -\frac{d}{dr}\left(\frac{kq}{r}\right) = -kq \underbrace{\frac{d}{dr}\left(\frac{1}{r}\right)}_{-\frac{1}{r^2}} = \frac{kq}{r^2}$ ✓

22.52 Electric potential for 2 identical charges at $x = \pm a$ at any point $P(x,y)$



$V(x,y)$ or $V(r)$?
↳ scalar superposition (arithmetic addition) of $V_1(r_1) + V_2(r_2)$

a) $V(r) = V_1(r_1) + V_2(r_2)$

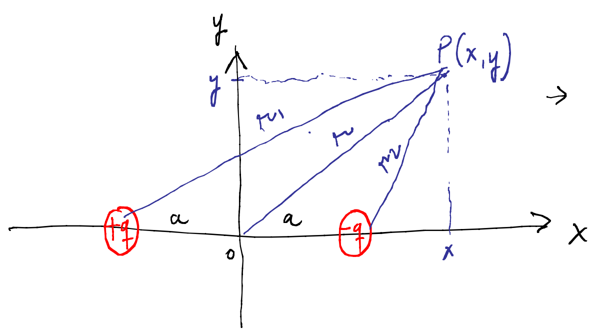
$$= \frac{kq}{r_1} + \frac{kq}{r_2} = kq\left[\frac{1}{\sqrt{(x+a)^2+y^2}} + \frac{1}{\sqrt{(x-a)^2+y^2}}\right]$$

b) Very far away from the two charges: $x \gg a$ or $x+a \simeq x$, $x-a \simeq x$
→ the two denominators are identical:

$$V(r) = \frac{2kq}{\sqrt{x^2+y^2}} = \frac{2kq}{r} = \frac{k(2q)}{r}$$

22.53 Electric potential for a dipole
A dipole consists of two identical charges of opposite signs separated by a distance $2a$

(+q) —a— | —a— (−q) ⇒ electric dipole moment $p \equiv q \cdot 2a$

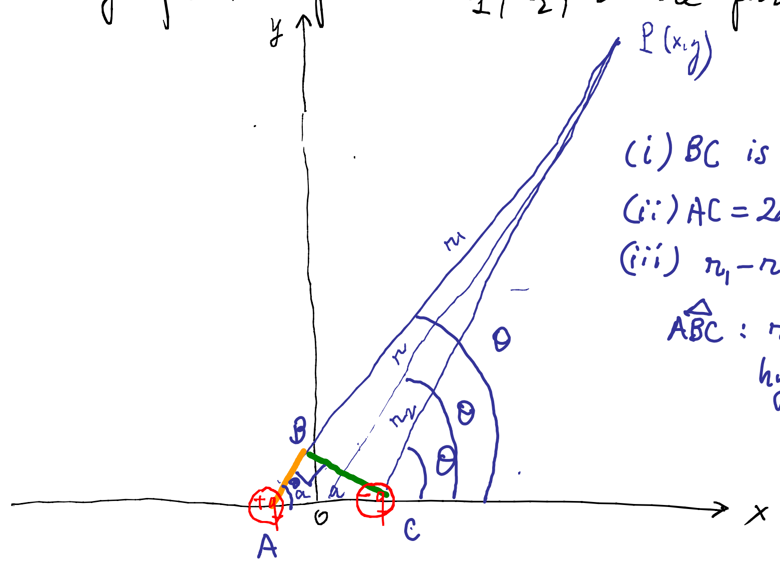


→ $V(x,y)$ or $V(r) = V_1(r_1) + V_2(r_2)$

$$= \frac{kq}{r_1} - \frac{kq}{r_2}$$

$$V(r) = kq\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = kq\,\frac{r_2 - r_1}{r_1 r_2}$$

Very far away from dipole: r_1, r_2, r are parallel to each other:



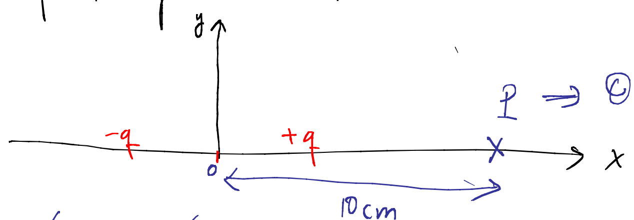
(i) BC is ⊥ to all three lines
(ii) AC = 2a
(iii) $r_1 - r_2 = AB = AC \cdot \cos\theta = 2a\cos\theta$
$\hat{ABC}$: right Δ
hypothenuse is AC = 2a

$$V(r) = kq \frac{r_2 - r_1}{r_1 r_2} \cong -kq \frac{2a\cos\theta}{r^2} = -\frac{kp\cos\theta}{r^2} \ *$$

very far away from dipole $p = q \cdot 2a$

$*$ If $-q$ is on left & $+q$ on right $V(r) = kq\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = kq \frac{r_1 - r_2}{r_1 r_2} \approx kq \frac{2a\cos\theta}{r^2}$
$= +\frac{kp\cos\theta}{r^2}$

If $p = 2.9$ nC·m ($2a \ll 10$ cm → very far away approx.!)
a) V? 10 cm from dipole on its axis



$P \Rightarrow \theta = 0°$ $V = \frac{kp}{r^2} = \frac{kp}{x^2}$
on axis

$$V = \frac{9 \cdot 10^{9} \cdot 2.9 \cdot 10^{-9}}{0.1^2} = 26.1 \cdot 10^2 \ V = 2.61 \cdot 10^3 \ V = 2.61 \ kV$$

(Volt, SI unit for electric potential)

b) $V(\overset{10\,cm}{\theta = 45°}) = \frac{9 \cdot 10^9 \cdot 2.9 \cdot 10^{-9} \cdot \cos 45°}{0.1^2} \ V$

c) $V\left(\overset{10\,cm}{\text{on} \perp \text{bisector}}\right) = 0$
$\theta = 90°$

22.31 Given $V(x,y,z) = 2xy - 3zx + 5y^2$ (V)

a) $V(1,1,1) = 2 - 3 + 5 = 4V$

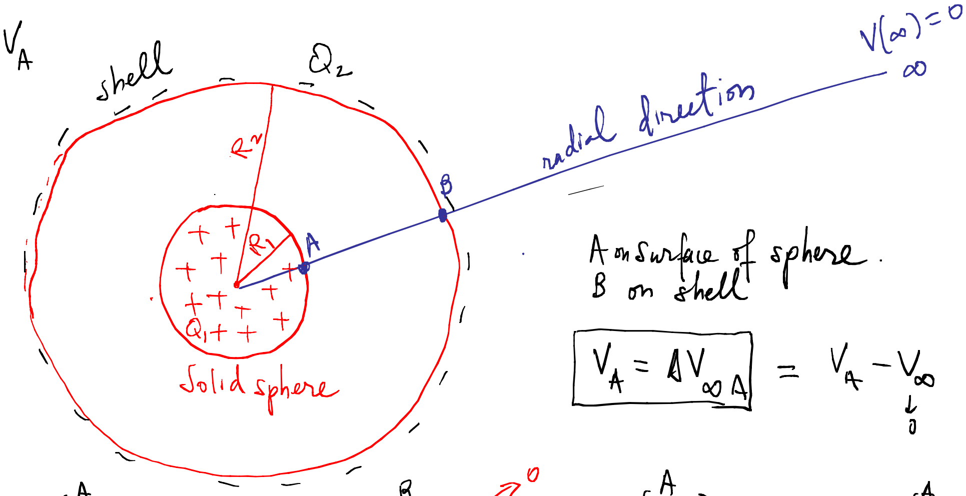
b) $\vec{E}(x,y,z) = -\vec{\nabla}V = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k} = -(2y - 3z)\hat{\imath} - (2x + 10y)\hat{\jmath} + 3x\hat{k}$

$\vec{E}(1m, 1m, 1m) = -(-1)\hat{\imath} - (12)\hat{\jmath} + 3\hat{k} = \hat{\imath} - 12\hat{\jmath} + 3\hat{k}$ $\left(\frac{N}{C}\right)$

22.67 a) Find V_A

$Q_1 = 60\,nC$
$R_1 = 0.05\,m$
$Q_2 = -60\,nC$
$R_2 = 0.15\,m$

Reference potential
$V=0$ @ ∞



A on surface of sphere.
B on shell

$$\boxed{V_A = \Delta V_{\infty A}} = V_A - \underset{0}{V_\infty} = V_A$$

$$V_A = \Delta V_{\infty A} = -\int_\infty^A \vec{E}\cdot\underbrace{d\vec{l}}_{\text{radial}} = \underbrace{\cancel{-\int_\infty^B \vec{E}\cdot d\vec{r}}}_{\text{outside shell}} \underbrace{- \int_B^A \vec{E}\cdot d\vec{r}}_{\text{b/w sphere \& shell}} = -\int_B^A \frac{kQ_1}{r^2}\underbrace{\hat{r}\cdot\hat{r}}_{1}\,dr$$

outside shell
$Q = Q_1 + Q_2 = 0$
$\vec{E} = 0$

b/w sphere & shell
$$\vec{E} = \frac{kQ_1}{r^2}\hat{r}$$

Notes: Gauss Law: $EA = \frac{q_{enclosed}}{\varepsilon_0}$
(only charge enclosed creates E)
$d\vec{r} = dr\,\hat{r}$

$$V_A = -kQ_1\underbrace{\int_B^A \frac{dr}{r^2}}_{-\left[\frac{1}{r}\right]_B^A} = kQ_1\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = 9\cdot\cancel{10^9}\cdot 60\cdot\cancel{10^{-9}}\left[\frac{1}{0.05} - \frac{1}{0.15}\right] = 7.2\ kV$$

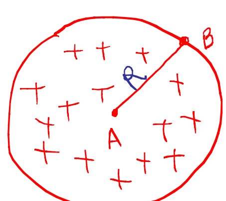
b) Now shell carries $+60\,nC$

$$V_A = -\int_\infty^B \frac{k(Q_1+Q_2)}{r^2}\,dr + V_A(a) = +k(Q_1+Q_2)\left[\frac{1}{r_B} - \overset{0}{\cancel{\frac{1}{r_\infty}}}\right] + 7.2\,kV$$

$$= \underbrace{9\cdot\cancel{10^9}\cdot 120\cdot\cancel{10^{-9}}\ \frac{1}{0.15}}_{7.2\,kV} + 7.2\ kV = 14.4\,kV$$

22.65 Sphere of radius R & $\vec{E} = E_0\left(\frac{r}{R}\right)^2\hat{r}$ $(0<r<R)$

inside sphere



A @ center $r_A = 0$
B @ surface $r_B = R$

ΔV_{BA}? Potential surface to center

$$\Delta V_{BA} = -\int_B^A \vec{E}\cdot d\vec{r} = -\frac{E_0}{R^2}\underbrace{\int_B^A r^2\,dr}_{\left[\frac{r^3}{3}\right]_B^A}\ \underbrace{\hat{r}\cdot\hat{r}}_{1} = -\frac{E_0}{R^2}\left[0 - \frac{R^3}{3}\right] = \frac{E_0 R}{3}$$

21.70

Gauss Law

Electric

$$\phi = \oint_{\text{Gaussian surface}} \vec{E}\cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

q_{enclosed} → by G-surface

Grav.

$$\phi_g = \oint_{\text{Gaussian surface}} \vec{g}\cdot d\vec{A} = -4\pi G \cdot M_{\text{enclosed}}$$

M_{enclosed} → by G-surface

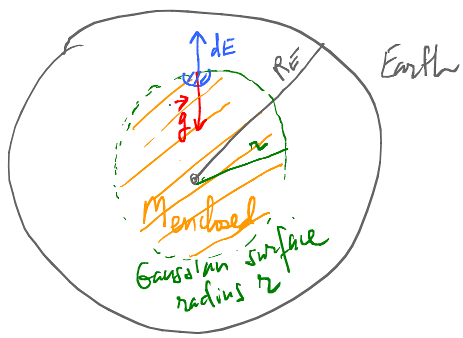
G → Univ. grav. constant

Prove $\boxed{g(r) = g_0 \frac{r}{R_E}}$ $(0<r<R_E)$ inside Earth.

Use Gauss Law to calculate field (grav. or electric)

(i) Choose Gaussian surface : symmetry of charge or mass distribution

↳ Earth → spherical



$$\frac{M_{\text{enclosed}}}{M_E} = \frac{\cancel{\frac{4\pi}{3}} r^3}{\cancel{\frac{4\pi}{3}} R_E^3} \Rightarrow M_{\text{enclosed}} = M_E \frac{r^3}{R_E^3}$$

(ii) $d\vec{A}$ points in radial direction away from G-surface
$\vec{g}$ " " " " towards center of G-surface } $\theta = 180°$, $\cos 180° = -1$

$\vec{g}\cdot d\vec{A} = -g\,dA$

(iii) Due to spherical symmetry g is constant on G-surface

$$\phi_g = \oint \vec{g}\cdot d\vec{A} = -g \underbrace{\oint dA}_{4\pi r^2} = -g\,4\pi r^2$$

(iv) Gauss Law for grav. field : $\phi_g = -4\pi G\, M_{\text{enclosed}}$

$$\cancel{-}g\,\cancel{4\pi} r^{\cancel{2}} = \cancel{-}\cancel{4\pi} G\cdot M_E \frac{r^{\cancel{3}}}{R_E^3}$$

$$\boxed{g = \frac{G M_E}{R_E^3}\, r}$$ any $0<r<R_E$

(v) On surface $r = R_E \Rightarrow g(R_E) = g_0 = \frac{GM_E}{R_E^3} R_E = \frac{GM_E}{R_E^2} = 9.81\ \frac{m}{s^2}$

$$\Rightarrow \quad g(r) = \frac{GM_E}{R_E^2} \, \frac{r}{R_E} = g_0 \, \frac{r}{R_E} \quad \checkmark$$