Ch 23  Electrostatic Energy & Capacitors.

Store electrostatic energy by separating charges of opposite types or bringing charges of the same types together.
(Similarly, water tends to fall to lower height due to gravitational attraction of the Earth, bringing water to certain height = storing hydrostatic energy.)

\[ U_{AB} = \frac{\Delta U_{AB}}{q_{AB}} \]

\[ U = \text{electric potential energy (J)} \]

\[ V = \text{electric potential (J/C)} \]

or \( V \) for Volt

\[ \Delta U_{AB} = -W_{AB} = -\int_{A}^{B} \vec{F} \cdot d\vec{l} \]

change in potential energy is minus the work done
(Relating this to the 1st Law of T.D.: \( Q = 0 \))

To separate charges of opposite types we need to apply a force \( \vec{F} \)
\( \left( \frac{kq_{1}q_{2}}{r^{2}} \right) \) b/w \( A \& B \). By doing this we are storing certain amount of energy. Similar thing happens when we bring charges of same type together:

1

\[ +q_{1} \]

\[ +q_{2} \]

2

\[ +q_{1} \]

\[ +q_{2} \]
- Charge $+q_1$ creates a field pointing in all directions away from it since $q_1$ is positive.
- While bringing $+q_2$ from a to A, we are going against the field created by $+q_1$, need work: $\Delta U_{voA}$

$$\Delta U_{voA} = q_2 \Delta V_{voA} = q_2 \cdot k \left( \frac{1}{r} - \frac{1}{\sqrt{A}} \right) = k \frac{q_1 q_2}{r}$$

**Fact:**

- Smaller final separation $r \rightarrow$ more energy stored
- Larger $q_1 \rightarrow$ more energy stored
- Larger $q_2 \rightarrow$ more energy stored.

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**Electrostatic Energy Storage Device: Capacitors - Parallel Plates**

**Symbol:** [Electricity symbol]

In a circuit:

- Battery
- RC circuit (low-pass filter) $\frac{1}{RC}$ (Capacitor)
- RC circuit: used to charge the capacitor.

**Notes:**

- Equal amount of each type of charge in either plates.
- Assumption: $d \ll \sqrt{A}$
  - $A$ is the area, section of each plate.
  - What we see here is just a front view.

Use $E$ as that created by an $\alpha$ plate.
\[ W_2 > W_1 \] (to move 2nd e\textsuperscript{-} we work against the field created by the 1st e\textsuperscript{-} that was moved to the left plate)

Energy stored gets larger & larger in the charging process.

What is the electric field at W plate? (assumption \( d \ll \sqrt{A} \))

\[ \rightarrow \text{plates look } \infty \text{ in this small d approximation:} \]

\[ \text{Gaussian} \]
\[ \text{Surface} \]
\[ \text{Cylinder as shown} \]

\[ 2 \) \( \varphi = \begin{cases} \text{Top:} & EA \\ \text{Bottom:} & EA \\ \text{Body:} & \varphi = 0 \end{cases} = 2EA \]

\[ 3 \) \( \varphi = \text{Flux value} = \frac{\sigma A}{\varepsilon_0} \]

charge on a plane sitting within the intersection of Gaussian cylinder with the plane \( \rightarrow \text{area A} \)

\[ 2EA = \frac{\sigma A}{\varepsilon_0} \rightarrow \frac{E}{A} = \frac{\sigma}{2\varepsilon_0} \]

Electric field due to an \( \infty \) plane of charge (of charge density, \( \sigma \))

\[ \text{Applies to the parallel plate capacitor (for one plate!)} \]
Electric field b/w plates of a capacitor:

\[ E = \frac{\sigma}{\varepsilon_0} \]

Capacitance: \( C = \frac{Q}{V} \) (charge on either plate over the electric potential V b/w plates)

Parallel plate: \( C = \frac{Q}{Ed} = \frac{Q}{\varepsilon_0 d} = \frac{q}{A \varepsilon_0} = \frac{AE_0}{d} \)

How do we write \( V \) in terms of \( E \)?

Unit (SI): \( F \) (Farad) \( \left\{ \begin{array}{c} \varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2} \\ A \ (\text{m}^2) \\ d \ (\text{m}) \end{array} \right. \)

\( k = \frac{1}{4\pi\varepsilon_0} \)

Total energy stored in a fully charged capacitor: \( \sigma = \frac{q}{A} \)

\( dU = -dW = \frac{d}{dV} q \rightarrow \quad dW = dq \cdot V = dq \cdot Ed \overset{\text{interm.}}{\rightarrow} \quad V = -\int E \cdot dl = -Ed \)

\( dU = \frac{d}{A \varepsilon_0} \quad q \, dq \rightarrow \quad U = \int dU = \frac{d}{A \varepsilon_0} \int_0^q q \, dq = \frac{1}{2} \frac{d}{A \varepsilon_0} \frac{q^2}{2} \)

\( \left[ \frac{q^2}{2} \right]_0^Q = \frac{1}{2} \frac{d}{A \varepsilon_0} \left( \frac{Q^2}{2} \right) \)
Total energy stored in a parallel plate capacitor:

\[ U = \frac{1}{2} \frac{\varepsilon_0}{A} Q^2 = \frac{1}{2} \frac{\varepsilon_0 d}{(\varepsilon_0)^2} Q^2 = \frac{1}{2} \varepsilon_0 d \frac{Q^2}{A^2} \]

\[ = \frac{1}{2} \varepsilon_0 d \left( \frac{E}{\varepsilon_0} \right)^2 = \frac{1}{2} \varepsilon_0 E^2 \left( \frac{Ad}{V} \right) \]

\[ \frac{U}{\text{vol}} = \frac{1}{2} \varepsilon_0 E^2 \rightarrow \text{Total energy stored per unit volume (b/w plates) is} \]

\[ \frac{1}{2} \varepsilon_0 E^2 \rightarrow \left( \frac{J}{m^3} \right) \]

(recall: \( \frac{1}{2} mv^2 \); \( \frac{1}{2} kx^2 \))

Alternative expression:

\[ U = \frac{1}{2} \varepsilon_0 E^2 \text{ vol} = \frac{1}{2} \varepsilon_0 E^2 \frac{Ad}{d} = \frac{1}{2} \frac{A \varepsilon_0}{d} E^2 \frac{d^2}{V^2} \]

\[ U = \frac{1}{2} GV^2 \]
How to connect capacitors:

Parallel:

\[ V = \frac{Q}{C} \]
\[ C = \frac{Q}{V} \]
\[ V = \frac{Q}{C} \]
\[ C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} \]
\[ C = C_1 + C_2 \]

Series:

\[ V_1 = \frac{Q}{C_1} \]
\[ V_2 = \frac{Q}{C_2} \]
\[ V = V_1 + V_2 \]
\[ C = \frac{Q}{V} = \frac{Q}{V_1 + V_2} = \frac{Q}{C_1} + \frac{Q}{C_2} \]
\[ C = \frac{1}{C_1} + \frac{1}{C_2} \]
\[ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \]
How to increase the capacitance?

1) Connecting 2 or more capacitors in parallel

2) \[ C = \frac{A\varepsilon_0}{d} \] (parallel plate capacitors)

2c) Decrease \( d \): inserting a conducting slab by plates:

\[ V = E \cdot d \]

The field now inside the conducting slab (width \( \frac{d}{2} \)) is 0

we have effectively reduced the separation in half

by inserting a conducting slab of width \( \frac{d}{2} \) → the capacitance is doubled \[ C' = \frac{A\varepsilon_0}{\frac{d}{2}} = 2C \]

2d) Notice \( \varepsilon_0 \) : dielectric constant in vacuum (air)

\[ \varepsilon_{\text{Vacuum}} \]

In a medium \( \varepsilon = K\varepsilon_0 \); \( K > 1 \) → \( K \) can help increase capacitance: if we insert a dielectric of \( \varepsilon = K\varepsilon_0 \) between the plates
Dielectric: not many free $\epsilon$- as in a conductor -> we don't get a perfect reorientation of dipoles, as in a conductor. So the field within the dielectric insert is only reduced, not 0 (by a factor of $K$).

\[ E = \frac{\sigma}{\varepsilon_0} \]
\[ \varepsilon_i \rightarrow \varepsilon = \varepsilon_0 \]

\[ C' = \frac{A \varepsilon_0}{d} \]
\[ C'' = \frac{A K \varepsilon_0}{d} = K C \]

\[ V = Ed \]
\[ C = \frac{Q}{V} \]

\[ V' = \frac{E d}{K} = \frac{V}{K} \]
\[ C' = \frac{Q}{V'} = \frac{Q}{\frac{V}{K}} = K \frac{Q}{V} = KC \]
From Example 23.4: Energy in storm is $140 \text{ GJ}$ ($G=10^9$)

Data: Lightning flashes

- $Q = 30 \text{ C}$
- $V = 30 \text{ MV}$ ($M=10^6$)

Question: How long will lightning last?

Answer:

1) How much energy per lightning flash:

$U = q \cdot V$ since lightning moves $q_{\text{flash}} = 30 \text{ C}$ across a potential of $30 \text{ MV} \rightarrow$ lightning has energy $J = 30 \times 30 \times 10^6 = 9 \times 10^8 \text{ J}$

2) Number of flashes $N$:

$N = \frac{\text{Total energy in storm}}{\text{Energy in each lightning flash}} = \frac{140 \times 10^9 \text{ J}}{9 \times 10^8 \text{ J}} = 156$ flashes

3) Total time = Number of flashes $\times 5 \text{ s}$

$= 156 \times 5 \text{ s} \times \frac{1 \text{ min}}{60 \text{ s}} = 13 \text{ min}$
Clarification: for an isolated capacitor

\[ C_0 = \frac{Q}{V_0} \]

\[ C = \frac{Q}{V} = \frac{Q}{V_0/K} = K C_0 \]

In this problem:

\[ C = C_1 + C_2 \]

Total capacitance when dielectric is inserted a distance \( x \):

\[ C = C_1 + C_2 = \frac{Q_1}{V} + \frac{Q_2}{V} = \frac{Q_1 + Q_2}{V} \]

Find \( Q_1 \) in term of \( Q_2 \) by looking \( E_1 \) & \( E_2 \):

\[ E_1 = \frac{Q_1}{K \varepsilon_0} = \frac{Q_1}{K \varepsilon_0} = \frac{Q_1}{x \varepsilon_0} \]

\[ E_2 = \frac{Q_2}{\varepsilon_0} = \frac{Q_2}{\varepsilon_0} = \frac{Q_2}{(L-x) \varepsilon_0} \]

There is no discontinuity \( \varepsilon \rightarrow \text{field should be continuous} \rightarrow E_1 = E_2 \)

\[ \frac{Q_1}{x \varepsilon_0} = \frac{Q_2}{(L-x) \varepsilon_0} \rightarrow \left[ Q_1 = Q_2 \frac{x \varepsilon_0}{(L-x) \varepsilon_0} = Q_2 \frac{K x}{L-x} \right] \]

\[ C = \frac{Q_1 + Q_2}{V} = \frac{Q_2}{V} \left( \frac{K x}{L-x} + 1 \right) \]
What is \( \frac{Q_x}{V} \) in terms of \( L, x, \varepsilon, l \) ...

\[
\frac{Q_x}{V} = \frac{Q_e}{\varepsilon_0 l} = \frac{\sigma_x}{\varepsilon_0 l} = \frac{\Phi_x}{A_2 \varepsilon_0} = \frac{A_2 \varepsilon_0}{l} = \frac{(L-x)w \varepsilon_0}{l}
\]

\[
C(x) = \frac{(L-x)w \varepsilon_0}{l} \left( \frac{K x}{L-x} + 1 \right) = \frac{w \varepsilon_0}{l} \left( K x + L - x \right)
\]

\[
C(x) = \frac{w \varepsilon_0}{l} \left[ x(K-1) + L \right]
\]

Positive!

\[
a) \quad C(x = \frac{L}{2}) = \frac{w \varepsilon_0}{l} \left[ \frac{L}{2} (K-1) + L \right] = \frac{w \varepsilon_0 L}{2l} \left[ \frac{K-1}{2} + 1 \right] = \frac{w \varepsilon_0 L}{2l} (K+1)
\]

\[
b) \quad U(x = \frac{L}{2}) = ?
\]

\[
U(x) = \frac{1}{2} C V^2 = \frac{1}{2} \varepsilon \frac{Q^2}{C^2} = \frac{Q^2}{2C}
\]

\[
C = \frac{Q}{V} \rightarrow V = \frac{Q}{C}
\]

\[
U(x) = \frac{Q^2}{2} \frac{\varepsilon_0}{l} \left[ x(K-1) + L \right] = \frac{Q^2}{2w \varepsilon_0} \left[ x(K-1) + L \right]
\]

\[
U_0 = \frac{L}{x(K-1) + L}
\]

\[
U(x) = U_0 \left[ \frac{L}{x(K-1) + L} \right]
\]

\[
U(x = \frac{L}{2}) = U_0 \left[ \frac{L}{2(K-1) + K} \right]
\]

\[
U(x = \frac{L}{2}) = U_0 \frac{2}{K+1} = \frac{CV_0^2}{K+1}
\]
c) Force on dielectric slab?

Why there is a force? Capacitors will not suck in the dielectric slab, a force needs to apply:

\[ F = -\frac{dU}{dx} \]

(since \( E = -\frac{dV}{dx} \) \rightarrow \( \frac{Q}{\epsilon} \) \( E = -\frac{dV}{dx} \)

\[ F = -\frac{dU}{dx} \]

This is why we needed to work out \( C(x) \rightarrow U(x) + \ldots \).

\[ \vec{F} = -\frac{d}{dx} \left( \frac{U_0 L}{x(k-1)+L} \right) = U_0 L \frac{k-1}{x(k-1)+L} \]

\[ \rightarrow \text{gradient operator} \quad \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \]

\[ \rightarrow U_0 \text{ pushing in along } +x \cdot \]

\[ F(x = \frac{L}{2}) = U_0 L \frac{k-1}{\left( \frac{L}{2} (k-1)+L \right)^2} \hat{i} = \frac{U_0 (k-1)}{L \left[ \frac{k-1}{2} + 1 \right]^2} \hat{i} \]

\[ = \frac{4U_0 (k-1)}{L (k+1)^2} \hat{i} = \frac{2\epsilon_0 U_0^2 (k-1)}{L (k+1)^2} \hat{i} \]

\[ U_0 = \frac{1}{2} \epsilon_0 V_0^2 \]

(Total force required to hold dielectric slab inserted halfway in the spacing)
Ch. 24  Electric Current:

Motion of charges \( \frac{dq}{dt} = I \) (average).

Electric current \( I = \frac{dq}{dt} \) (SI: \( \frac{C}{s} = A \) for Amp).

Macroscopic (measurable with simple instruments: ammeter)

Minuscule motion of charges:

Drift velocity \( v_d \) average velocity along a wire (in the x-direction)

\( v \) of individual charges \( \{ \text{random (collisions)} \} \)

\( v_d \) \{ forward (along x) \} net motion of charge along x-axis (in the forward direction).

Number of charges per unit volume: \( n \)

Individual charge: \( q \)

\( I = \frac{dq}{dt} = n \times A \times \frac{q}{v_d} = nq A v_d \)

minuscule

\( \) microscopically

Very large

What would be \( v_d \) in a copper wire \( A = 1 \text{mm}^2 \); \( I \geq 5 \text{A} \);

each atom of copper contributes 1.3e of charge ?

\( \rho_{\text{Copper}} = 8920 \text{kg/m}^3 \) (\( \rho_w = 1000 \text{kg/m}^3 \))

data from periodic table: mass for one atom of copper: 63.55 a.u. (atomic unit)

Converting factor: \[ \text{1 atm} = 1.60 \times 10^{-27} \text{ kg} \]

\[ I = 5A; \ A = 1 \text{ mm}^2 \]

\[ \nu_A = \frac{I}{n_q A} \]

\[ q = \text{charge of one electron} = 1.6 \times 10^{-19} \text{ C} \]

\[ n_e = \text{number of electrons per unit volume} \]

Need \( n_e \) or \# of atoms of copper per unit volume:

\[ n_e = n_a = 1.3 \]

\[ n_a = \frac{p_A}{m_a} = \frac{8.920}{63.55 \times 1.66 \times 10^{-27}} = 8.5 \times 10^{28} \text{ atoms/m}^3 \]

\[ n_e = 1.3 n_a = 1.3 \times 1.28 \times 10^{28} \text{ electrons/m}^3 \]

\[ \nu_A = \frac{5A}{11.05 \times 10^{28} \frac{1}{n_e^3} \times 1.6 \times 10^{-19} \times 10^{-6}} = \frac{5}{11.05 \times 10^{-3} \times 1.6} = 0.283 \text{ mm/s} \]

(Walking speed \( 0.3 \frac{\text{m}}{\text{s}} = 300 \frac{\text{mm}}{\text{s}} \))

Can you estimate actual speed of electrons (random)?

Tough estimate: using equipartition theorem:

\[ \text{KE per molecule} = \frac{3}{2} kT \]

\[ \text{average } \nu = \left( \frac{3}{2} kT \right)^{1/2} \]

\[ T = 298.16 \text{ K} \]

\[ \frac{3 \times 1.38 \times 10^{-16} \times 298.16}{63.55 \times 1.66 \times 10^{-27}} = 9.0 \times 10^{-3} \text{ m/s} \]

\[ \nu = \left( \frac{3}{2} \times 10^{-27} \right)^{1/2} = \left( \frac{3 \times 1.38 \times 10^{-16} \times 298.16}{63.55 \times 1.66 \times 10^{-27}} \right)^{1/2} = 9.0 \times 10^{-3} \text{ m/s} \]

\[ \frac{1500}{10} = 12 \times 10^4 \text{ m/s} = 12000 \text{ m/s} \]
Ohm's Law & Electric Power:

\[ I = \frac{V}{R} \]

- **electrical potential**
- **Resistance** (will find difficulty pushing through atoms in the electrical wire)

**SI:** \( \text{Ohm} = \Omega \)

For a material: **resistivity** \( \rho \) (rho)

\[ R = \frac{\rho l}{A} \]

- wire for dryer is normally short

\[ P = I \cdot V \]

\[ \frac{\partial V}{\partial t} = \frac{\partial V}{\partial t} \]

\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \]
Resistors

Parallel $\leftrightarrow$ Current division

\[ I = I_1 + I_2 \]

\[ I = \frac{V}{R} \]

\[ V = \frac{V}{R_1} + \frac{V}{R_2} = V \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

Series $\leftrightarrow$ Voltage division

\[ V = V_1 + V_2 \]

\[ V = V_1 + V_2 = IR_1 + IR_2 = I(R_1 + R_2) \]

\[ V = IR \]

\[ R = R_1 + R_2 \]

\[ V_1 = \frac{V}{R_1 + R_2} \]

\[ V_2 = \frac{VR_2}{R_1 + R_2} \]

\[ V_1 + V_2 = \left( \frac{R_1}{R_1 + R_2} + \frac{R_2}{R_1 + R_2} \right) V = V \]

Power Consumption

\[ P_i = I_i V_i \]

\[ P_i = \frac{V_i^2}{R_i} \]

\[ P_i = \frac{V^2}{R_i} \]
There is a downhill force of $mg \sin 10^\circ$.

→ For the car to go uphill @ constant speed $v$ (a = 0)
  engine needs to apply at least $F = mg \sin 10^\circ$.

→ Power consumed by engine $Fv$
  (Why? $Fv = \frac{F \cdot Ax}{dt} = \frac{\text{energy}}{\text{time}} = \text{power}$)
  or $Fv$ is the rate of energy consumption per unit time.

→ How long it will last @ speed $v$?

$t = \frac{\text{total mechanical energy available (from batteries)}}{\text{rate of energy consumed by engine}}$

\[ t = \frac{0.85 \times \text{total electrical energy available (from batteries)}}{0.85 \times V \times Q} = \frac{0.85 \times 312 \times 3.6 \times 10^5}{1500 \times 9.81 \times \sin 10^\circ \times 0.85 \times 3.6} \]

\[ = 2989 s = 49.8 \text{ min} \]
\[ I = 20 \text{A} \]

\[ d = 2.1 \text{mm} \]

\[ A = \pi \times (1.05 \times 10^{-3})^2 \]

\[ \nu_d = \text{(if each Al atom contribute 3.5 electrons)} \]

\[ \nu_d = \frac{I}{n \cdot A} \] charge of one electron
non-sectional area of the wire

\[ \# \text{electrons per unit volume} \]

\[ \# \text{of Al atoms per unit volume} \times 3.5 \]

\[ \rho_{\text{Al}} = \frac{\text{mass of Al}}{\text{mass of one atom of Al}} \]

\[ \frac{m_{\text{Al}}}{n} = \frac{2702}{26.48 \times 1.66 \times 10^{-27}} \times 3.5 \]

\[ n = \frac{2702}{26.48 \times 1.66 \times 10^{-27}} \times 3.5 \]

\[ \nu_d = \frac{20 \text{A}}{2702 \times 3.5 \times 1.66 \times 10^{-17} \times \pi \times (1.05 \times 10^{-3})^2} \]

\[ \nu_d = \frac{20 \text{A}}{2702 \times 3.5 \times 1.66 \times 10^{-17} \times \pi \times (1.05 \times 10^{-3})^2} = \frac{0.171 \text{mm}}{s} \] (Cov !)
Next find $I_2$ & $I_3$ as fractions of $I$ ($I = I_2 + I_3$)

1. Use 
   - Current div.
   - Voltage div.
   - Ohm's Law

\[
I_2 = \frac{V_{23}}{R_2} = \frac{I R_{23}}{R_2} = \frac{R_3}{R_2 + R_3} I
\]

\[
I_3 = \frac{V_{23}}{R_3} = \frac{R_2}{R_2 + R_3} I
\]

**Note:**

- In a current division, $I_2$ is proportional to the resistance $R_2$ in the other branch.

a) $R_1$ & $R_2$ are not in series, neither in parallel.
   - $I_1$ thru $R_1$
   - $V_1$ across $R_1$
   - $I_2$ thru $R_2$
   - $V_2$ across $R_2$

**Node analysis:**

- Total current at any node is zero (conservation of charge).
- Node: where 3 branches converge
  - (There are 2 nodes: a & b)

**Loop analysis:**

- Total voltage difference across elements in a closed loop is zero (conservation of energy).
- There are 2 independent loops in this circuit.

- Introduce directions for your variables (currents & voltages).
  - Actual directions will be given by the equation via signs for numeric answers.

Note: loop analysis → assume directions for currents.
Chapter 25: Electrical Circuits

Linear circuits (i.e., those involving elements with linear relationships, e.g., voltage $V$ and current $I$)

$R$, $L$, $C$

- Resistors
- Inductors
- Capacitors

2 types of linear circuits

1a) Those circuits that are reducible to

Diagram of circuit reducing to one voltage & one resistor using series/parallel connection

1b) Series/parallel are not useful, but loop or node analysis

1) Resistors only

2) Resistors & capacitors

Circuit analysis:

- Solve for unknowns

$I_1$, $I_2$, $I_3$
Loop Analysis

Once direction for currents are assumed:

Sign convention for voltage:
1) $V$ in a battery will have + sign if a current goes from $-$ to $+$ through the battery.
2) $V$ will have - sign if current that you assumed goes from $+$ to $-$
3) Across a resistor is negative

$\sum V_i = 0$

Node Analysis

Sign convention for current:
1) Current into node $\rightarrow +$
2) Current leaving node $\rightarrow -$