Refrigerators: reversed heat engine

\[ \begin{array}{c}
\text{Th} \\
\uparrow \\
\text{(kitchen)}
\end{array} \quad \begin{array}{c}
\text{TC} \\
\uparrow \\
\text{(inside fridge)}
\end{array} \quad \begin{array}{c}
\text{Qe} \\
\uparrow \\
\text{(electrical work on fridge)}
\end{array} \quad \begin{array}{c}
\text{W} \\
\end{array} \]

C.O.P. (Coefficient of Performance): \[ \text{C.O.P.} = \frac{Q_e}{W} \]

2nd Law of T.D.: it is impossible to transfer heat from a cold reservoir to a hot reservoir without requiring any work.

3rd Law of T.D.:

\[ \Delta S = \int \frac{dQ}{T} \]

Charge of entropy b/w states 1 & 2

Entropy of a closed system can never decrease \((\Delta S \geq 0)\)
3rd law of T.D.: disorder in the universe gets increased.

Heat engine: operate in cycle (2nd half of cycle is to bring engine in system back to original state) \(\rightarrow\) representation in PV diagram is a closed loop!

Type 1: Carnot engine
Type 2: Otto cycle

\(4\) reversible process (2 isothermal, 2 adiabatic)

\(4\) reversible process (2 adiabatic, 2 isovolumic)
Carnot Engine: efficiency by a Carnot engine is the maximum achievable so far. \( \eta_{\text{Carnot}} = \eta_{\text{max}} \)

\[
\eta_{\text{Carnot}} = \eta_{\text{max}} = 1 - \frac{|Q_c|}{|Q_h|} = \frac{W}{Q_h}
\]

(definition of efficiency: \( \eta = \frac{W}{Q_h} \))

1st Law of TD:

\[
|Q_c| = |Q_{cd}| = nRT_c \ln \left( \frac{V_c}{V_d} \right)
\]

 Isothermal process:

\[
DU_{cd} = 0 \rightarrow Q_{cd} = W_{cd} = nRT_c \ln \left( \frac{V_b}{V_c} \right)
\]

\[
|Q_c| = |Q_{cd}| = nRT_c \ln \left( \frac{V_c}{V_d} \right)
\]

\[
\Delta U_{AB} = \text{change of total energy in B & P, ideal gas} \rightarrow \Delta U_{AB} = 0 \rightarrow Q_{AB} = W_{AB}
\]

(4th Law of TD)

\[
Q_{AB} = nRT_h \ln \left( \frac{V_b}{V_a} \right)
\]

\[
|Q_h| = |Q_{AB}| = nRT_h \left(\ln \left( \frac{V_b}{V_a} \right) \right)
\]
Before plugging these, $|Q_h|, |Q_c|$ into the Carnot, we will derive the relationships $V_A, V_B, V_C, V_D$: they are related because $B \rightarrow C \rightarrow D \rightarrow A$

\[
\begin{align*}
B \rightarrow C: & \quad \text{adiabatic expansion:} & TV^{3-1} &= \text{constant} \\
& & T_B V_B^{3-1} &= T_C V_C^{3-1} \\
& & \left(\frac{V_B}{V_C}\right)^{3-1} &= \frac{T_C}{T_B} = \frac{T_C}{T_B} \\
D \rightarrow A: & \quad \text{adiabatic compression:} & T_D V_D^{3-1} &= T_A V_A^{3-1} \\
& & \left(\frac{V_D}{V_A}\right)^{3-1} &= \frac{T_A}{T_D} = \frac{T_D}{T_C} \\
& & \frac{V_B}{V_C} &= \frac{V_A}{V_D} \\
& & \frac{V_B}{V_A} &= \frac{V_C}{V_D}
\end{align*}
\]

\[
\eta_{\text{max}} = \eta_{\text{Carnot}} = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{\frac{mRT_c \ln \left(\frac{V_C}{V_D}\right)}{\frac{mRT_h \ln \left(\frac{V_B}{V_A}\right)}}}{T_C}{T_H} \\
\text{Max efficiency for any heat engine}
\]
**Otto Cycle Engines**

\[ \epsilon_{\text{Otto}} < \epsilon_{\text{Comet}} = \epsilon_{\text{max}} \]

**Entropy**: 

\[ \Delta S_{12} = \int_{1}^{2} \frac{dQ}{T} \]

1) Isothermal:

\[ \Delta S_{12} = \frac{1}{T} \int_{1}^{2} dQ = \frac{\Delta Q}{Q_2 - Q_1} \]

2) Isovolumetric:

\[ C_v = \frac{1}{n} \frac{dQ}{dT} \]

\[ -dQ = nC_v dT \]

\[ \Delta S_{12} = \int_{1}^{2} \frac{dQ}{T} = nC_v \int_{1}^{2} \frac{dT}{T} = nC_v \ln \left( \frac{T_2}{T_1} \right) \]

---

**End of block of T.O.**

**Ch 16**: Temp. & heat

**Ch 17**: 1st Law of T.O.: \[ \Delta U = Q - W \]

**Ch 19**: Thermal Behavior of Matter: \[ Q \rightarrow \{1) DT 2) \text{Phase change}\} \]

**Ch 29**: 2nd (heat engines, refrigeration) \[ \text{Laws of T.O.} \]

---

\[ \alpha, \beta \]
### Ch 20 Electric Charge, Force, Field

- **Charge:** is a multiple of $e$ or $e^+$ (Coulomb $\rightarrow C$)
- **Charge distribution:** a discrete or continuous group of charge
- **Fields (electric):** charges interact through their electric field $E$.
- **Force (electrice):** $F = qE$ (N): force felt by a test charge of value $q$ in the presence of $E$

**Electron is the elementary charge:** $e^- = 1.6 \times 10^{-19}$ C

**SI unit for a charge (Coulomb):**

We have electrons, we are neutral or not electric, there are also positive charges.

The proton has a positive charge $e^+ = 1.6 \times 10^{-19}$ C

2 types of charges (electrical): $+$ and $-$

Charge distributions interact through their electric fields:

- **Electric field $E_1$:** pointing away from the distribution
- **Electric field $E_2$:** also away from the distribution ($+$ charge)

These two charge distributions through their electric fields $E_1$ & $E_2$ will repel each other.
\[ \mathbf{E}_1 \text{ pointing towards the charge distribution (negative)} \]

These two distributions through their electric fields \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) will repel each other.

\[ \rightarrow \text{ same type of charge } (+ \delta + \text{ or } -\delta -) \text{ repel each other.} \]

\[ \rightarrow \text{ opposite type of charge } (+\delta - \text{ or } -\delta +) \text{ attract each other.} \]

Field lines can go from \(+Q_1\) to \(-Q_2\).

\[ \text{If any problem } \rightarrow \text{ can stay as close as possible} \]

\[ \rightarrow \text{ opposite charges attract each other through their electric fields.} \]
How much work on gas in this cycle: $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$?

$$W_{1231} = W_{12} + W_{23} + W_{31}$$

**W$_{12}$**: adiabatic

$$W_{12} = nRT \ln\left(\frac{V_1}{V_2}\right) = nRT \ln\left(\frac{V_1}{V_2}\right) = \frac{P_1 V_1 \ln\left(\frac{V_1}{V_2}\right)}{R T_1}$$

**1$\rightarrow2$: adiabatic**

$$P_1 V_1^\gamma = P_2 V_2^\gamma \Rightarrow \left(\frac{V_1}{V_2}\right)^\gamma = \frac{P_2}{P_1} \Rightarrow \frac{V_1}{V_2} = \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}}$$

$$\Rightarrow W_{12} = 50 \times 10^3 \times \frac{25}{1000} \ln\left(3^{\frac{1}{1.67}}\right) = \frac{50 \times 25}{1.67} \ln 3 = 822 J$$

**1000L in 1 m$^3$**

$$W_{12} = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1} = \frac{1 - 3 \times 3^{\frac{1}{1.67}}}{0.67} = \frac{1 - \left(\frac{3}{3^{\frac{1}{1.67}}}\right)}{0.67} \times 50 \times 16 \times \frac{25}{10}$$

$$= \left[1 - \frac{3^{\frac{1}{1.67}}}{3}\right] \times 100 = 1033 J$$

$$W_{1231} = 822 J - 1033 J = -211 J$$

Work done by gas is: $-211 J$
Quantitative description of the electric field $\mathbf{E}$

Electric field due to a charge $Q$ at a point $r$ from $Q$ has intensity $kQ/r^2$, direction along the radial direction from $Q$ and point $r$: $\hat{n}$ (unit vector).

$$\mathbf{E} = \frac{kQ}{r^2} \hat{n}$$

$K =$ electric constant \(9 \times 10^9 \text{ Nm}^2\text{/C}^2\)

$Q =$ net charge creating the field

$r =$ separation from charge to point where field is measured

$\hat{n} =$ radial unit vector (always points away from the charge)

Electric field around a $+Q$:

Higher line density @ 1 (compared to 2) indicate stronger electric field @ smaller separation $r$ to the charge

Note: radial unit vector $\hat{n}$ always points away from the charge or center where the charge is located. Direction of $\mathbf{E}$ is parallel to $\hat{n}$ if $Q$ is positive. Opposite direction if $Q$ is negative.
Electric field

\[ \mathbf{E} = k \frac{Q}{r^2} \hat{\mathbf{r}} \]

Gravitational field

\[ \mathbf{g} = \frac{G M}{r^2} \hat{\mathbf{r}} \]

**Similarities:**
- Inverse square law
- Radial direction (\( \hat{\mathbf{r}} \))
- Electric constant
- Proportional to charge
- In this field

**Differences**
- Charge can be + or -
  - Field can be attractive (\( Q < 0 \))
  - or repulsive (\( Q > 0 \))
- \( k = 9 \times 10^9 \text{Nm}^2\text{C}^{-2} \)
- Mass has no sign.
  - Grav. field is always attractive.
- \( G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \)
Calculation of the Electric Field (Direct method):

1. Due to one charge
2. Due to two charges (diplde)
3. Continuous ring of charge
4. Infinite line of charge

Electric field by one charge, $q_1$

\[ \hat{E}_a = k \frac{q_1}{r_a^2} \hat{r}_a \]

Can calculate $\hat{E}$ at any point around $q_1$

\[ \hat{E}_b = k \frac{q_1}{r_b^2} \hat{r}_b \]

For a second charge (test charge) $q_{test}$ comes into the picture (field created by $q_1$) it will feel a force:

\[ \vec{F} = q_{test} \hat{E} \]

- Attractive if $q_{test} > 0$
- Repulsive if $q_{test} < 0$

In a field created by $-q_1$, the test charge $q_{test}$ would feel a force:

\[ \vec{F} = q_{test} \hat{E} \]

- Attractive if $q_{test} > 0$
- Repulsive if $q_{test} < 0$
F = \vec{q}_1 \vec{E}_a = \frac{k \vec{q}_1 \vec{q}_2}{r^2} \hat{\mathbf{r}}$

This is the force applied by \( q_1 \) on \( q_2 \).

By 2nd Newton's Law (action & reaction):

\( q_2 \) applies a same force on \( q_1 \), in the opposite direction:

\[
\vec{F} = -k \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}
\]

Electric field by two positive charges: along the mid line by the two charges.

Electric field @ \( P \) due to charge \( \#1 \):

\[
\vec{E}_1 = k \frac{q_1}{r^2} \hat{\mathbf{r}}_1 = E_{1x} \hat{\mathbf{i}} + E_{1y} \hat{\mathbf{j}} = E_1 \cos \alpha \hat{\mathbf{i}} - E_1 \sin \alpha \hat{\mathbf{j}}
\]

Electric field @ \( P \) due to charge \( \#2 \):

\[
\vec{E}_2 = k \frac{q_2}{r^2} \hat{\mathbf{r}}_2 = E_{2x} \hat{\mathbf{i}} + E_{2y} \hat{\mathbf{j}} = E_2 \cos \alpha \hat{\mathbf{i}} + E_2 \sin \alpha \hat{\mathbf{j}}
\]

Same magnitudes \( \Rightarrow E_1 = E_2 \)

Total electric field @ \( P \):

\[
\vec{E} = \vec{E}_1 + \vec{E}_2 = 2E_1 \cos \alpha \hat{\mathbf{i}} \quad \text{(only x-component)}
\]
Back to polar form: (using separation \( r \) instead of cartesian coordinates, \( x \) & \( y \))

\[
\vec{E} = 2E_0 \cos \alpha \ \hat{\alpha} = \frac{2kq}{r^2} \frac{x}{r} \ \hat{i} = \frac{2kq x}{r^3} \ \hat{i}
\]

\[
\cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}
\]

\[
E = \frac{2kq x}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \ \hat{\alpha}
\]

Electric field by 2 charges of value \( \pm q \)

@ \( \pm \frac{d}{2} \) along the y-axis

Unit: \( \frac{N}{C} \) (SI)

Electric field by a dipole: along mid-line \( y = 0 \) of the 2 charges.

\[
\vec{E}_1 = \frac{kq}{r^2} \ \hat{n}_1 = (E_1 \cos \alpha \ \hat{i} - E_1 \sin \alpha \ \hat{j})
\]

\[
\vec{E}_2 = -\frac{kq}{r^2} \ \hat{n}_2 = (E_2 \cos \alpha \ \hat{i} - E_2 \sin \alpha \ \hat{j})
\]

\[
\vec{E} = \vec{E}_1 + \vec{E}_2 = -2E_1 \sin \alpha \ \hat{j}
\]

\[
E = -\frac{kq d}{r^3} \ \frac{d}{\sin \alpha} \ \hat{j} = \frac{-kq d}{\sqrt{x^2 + \left(\frac{d}{2}\right)^2}} \ \frac{d}{\sin \alpha} \ \hat{j}
\]

\[
\sin \alpha = \frac{d}{2r}
\]

(if I switch \( +q \) \( \rightarrow \) \( -q \) points along \( +y \))
Electric field due to a continuous ring of charge: at a point along its axis

\[ E = \frac{kQx}{(x^2 + a^2)^{3/2}} \hat{y} \] (N/C)

Electric field by an infinite (very long) line of charge:

\[ E = \frac{2k\lambda}{y} \hat{j} \] (perpendicular to the line of charge)

Linear density of charge is \( \lambda \) (charge per unit length)
Ideal, diatomic: \( C_V = \frac{5}{2}R \)
\[
\begin{align*}
\text{a) constant vol} & \quad \Delta S_{12} = \int_1^2 \frac{dQ}{T} \\
\text{b) constant } P & \quad \Delta S_{12} = \int_1^2 \frac{dQ}{T} \\
\text{c) adiabatically} & \quad \Delta S_{12} = 0
\end{align*}
\]
\[
\begin{align*}
\text{constant volume} & \quad C_V = \frac{1}{n} \frac{dQ}{dT} \rightarrow dQ = C_V n dT \\
\text{constant pressure} & \quad C_P = \frac{1}{n} \frac{dQ}{dT} \rightarrow dQ = C_P n dT \\
\text{adiabatic} & \quad dQ = 0 \rightarrow \Delta S_{12} = 0
\end{align*}
\]

a) Isothermal:
\[
\begin{align*}
\Delta S_{12} &= \int_1^2 \frac{C_V n dT}{T} \\
&= nC_V \int_1^2 \frac{dT}{T} \\
&= nC_V \ln \left( \frac{T_2}{T_1} \right)
\end{align*}
\]
\[
\begin{align*}
&= 5 \times \frac{5}{2} \times 8.314 \ln \left( \frac{500}{300} \right) = 53.1 \text{ J/°K}
\end{align*}
\]

b) Isobaric:
\[
\begin{align*}
\Delta S_{12} &= nC_P \ln \left( \frac{T_2}{T_1} \right) \\
&= 53.1 \times \frac{7}{5} \text{ J/°K} = 74.3 \text{ J/°K}
\end{align*}
\]
\[
C_P = C_V + R = \frac{7}{2}R
\]
Given in the problem

\[ P \]

\[ \begin{align*}
V_3 &= \frac{V_1}{5} \\
V_1 &= V_4
\end{align*} \]

\[ P \]

\[ \begin{align*}
\text{adiabatic expansion } Q=0 \\
\text{combustion} \\
\text{adiabatic } Q=0 \text{ compression}
\end{align*} \]

\[ P \]

\[ e = 1 - \frac{T_4}{T_3} \text{ only for a Carnot engine! max.} \]

\[ Q_c = Q_{41} = n c_v \Delta T = n c_v (T_i - T_f) \]

\[ Q_h = Q_{23} = n c_v \Delta T = n c_v (T_3 - T_2) \]

\[ e = 1 - \frac{|T_i - T_f|}{|T_3 - T_f|} \]

\[ TV \text{ const.} \]

\[ 1 \to 2 \quad \frac{T_1 V_1^{\gamma-1}}{T_2 V_2^{\gamma-1}} \]

\[ 3 \to 4 \quad \frac{T_3 V_3^{\gamma-1}}{T_4 V_4^{\gamma-1}} \]

\[ V_i = V_4, \quad V_e = V_3 \]

\[ e = 1 - \frac{|T_4 (\frac{T_i}{T_4} - 1)|}{|T_3 (1 - \frac{T_i}{T_3})|} \]

\[ \Rightarrow e = 1 - \frac{T_4}{T_3} \]
\[ \frac{\gamma - 1}{\gamma - 1} = \frac{T_4}{T_3} = \frac{V_3}{V_2} \rightarrow \left[ \frac{T_4}{T_3} = \left( \frac{V_3}{V_2} \right)^{\gamma - 1} = \left( \frac{\frac{V_1}{5}}{V_4} \right)^{\gamma - 1} = \frac{1}{5^{\gamma - 1}} \right] = 5^{1 - \gamma} \]

\[ e = 1 - \left( \frac{T_4}{T_3} \right) = 1 - 5^{1 - \gamma} \rightarrow \text{Otto cycle} \]

b) Find \( T_{\text{max}} \) in terms of \( T_{\text{min}} \)

\[ \begin{align*}
&\text{ideal gas: } PV = nRT \\
&\text{solve for } P_2 V_2 \\
&\left\{ \begin{array}{ll}
2 & P_2 V_2 = nR T_2 \\
3 & P_3 V_3 = nR T_3 \\
& 3V_2 V_2
\end{array} \right.
\]

\[ \frac{T_4}{T_3} = 5^{1 - \gamma} \rightarrow T_3 = \frac{T_4}{5^{1 - \gamma}} = 3T_1 \]

\[ \text{can we relate } T_4 \text{ and } T_1? \]

\[ T_3 = 3x5^{1 - \gamma} T_1 \]

\[ \begin{align*}
\text{previous page:} & \quad \text{adab} \quad \text{122} \\
\text{334} & \quad \left\{ \begin{array}{ll}
& \frac{T_1}{T_4} = \frac{T_2}{T_3} = \frac{1}{3} \\
& \frac{T_4}{T_3} = 3T_1
\end{array} \right.
\end{align*} \]

c) For a constant engine \( \left\{ \frac{T_h}{T_c} = T_3 \right\} \rightarrow e_{\text{max}} = 1 - \frac{T_c}{T_h} = 1 - \frac{T_1}{T_3} \]

\[ e_{\text{max}} = 1 - \frac{1}{3\times5^{1 - \gamma}} = 1 - \frac{5^{1 - \gamma}}{3} \]

\[ e_{\text{stot}} = 1 - 5^{1 - \gamma} < e_{\text{max}} \]
\( \eta = 0.2 \) 

\[ P \] 
\[ Q \]

isothermal
\( \Delta U = 0 \rightarrow Q = W \)

\[ P_1 V_1 \ln \left( \frac{V_3}{V_1} \right) = 8 \times 1.013 \times 10^5 \times \frac{1}{10^2} \ln 2 = 561.7 \text{ J} \]

\[ Q_4 = Q_{34} = P_3 V_3 \ln \left( \frac{V_4}{V_3} \right) = 2.05 \times 1.013 \times 10^5 \times \frac{3.224}{10^2} \ln \left( \frac{1.612}{3.224} \right) \]

\[ = 464.1 \text{ J} \]

heat rejected.

c) Work done: \( W \)
\[ \text{Cycle: } \Delta U = 0 \rightarrow W = Q_{\text{net}} = |Q_4 - 1| = 561.7 - 464.1 = 97.6 \text{ J} \]

\( \alpha = Q_3 + Q_4 \)

d) \( \eta = \frac{W}{Q_4} = \frac{97.66}{561.7} = 0.1739 \) or \(17.39\%\)

e) Compare with: \( \epsilon = 1 - \frac{T_c}{T_h} = 1 - \frac{P_3V_3}{nR} = 1 - \frac{P_2V_2}{nR} \)

\[ T_h = \frac{P_2V_2}{nR} = \frac{2.05 \times 1.013 \times 10^5 \times 3.224}{10^5} = 4897.4 \text{ K} \]

\[ T_c = \frac{P_3V_3}{nR} = \frac{2.05 \times 3.224}{4 \times 2} = 549.7 \text{ K} \]

\( T_c - T_h = 402.6 \text{ K} \)
\[ \vec{F} = q \vec{E} \rightarrow \text{can find interactions (electric) b/w objects by knowing their electric fields.} \]

\[ \Rightarrow \text{How to calculate the electric field?} \]

1) **Direct method:** Vector superposition (e.g. \( \vec{E} \) by 2 charges:
\[ \vec{E} = \vec{E}_1 + \vec{E}_2 \) \) (Ch 20)

2) **Using Gauss Law** (symmetry) (Ch 21)

3) **Using Electric Potential** (using derivative, similar to mechanics: \( \vec{F} = -\frac{dU}{dx} \)) (Ch 22)

1) **Direct Method:** (vector superposition)

Electric field due to a continuous ring of charge, at a point away its axis (perpendicular to the ring)

From results for \( \vec{E} \) due to 2 positive charges:
\[ d\vec{E} = \frac{2k dq}{(x^2 + a^2)^{3/2}} \hat{c} \]
To get $\vec{E}$ by the whole ring: 

$$ \vec{E} = \int_{\text{Half Ring}} \frac{2kQ}{(x^2+a^2)^{3/2}} \, dq $$ 

(Uniformly distributed charge on the ring)

$$ \vec{E} = \frac{kQx}{(x^2+a^2)^{3/2}} \hat{z} \quad (\frac{N}{C}) $$

Electric field due to a very long line of charge (with linear charge density $\lambda = \frac{dq}{dx}$) 

$$ dq = \lambda \, dx $$

$$ d\vec{E} = \frac{2k\lambda dx \hat{y}}{(y^2+x^2)^{3/2}} \hat{j} $$

$$ \vec{E} = \int_{\text{Half line}} \frac{dq}{\text{Line}} $$

Table for integrals:

$$ \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}} $$
\[ E_{\text{axial}} = 2k \lambda y \int \left[ \frac{x}{y^2} \left( \frac{x^2}{x^2 + y^2} \right)^{1/2} \right]_{x=0}^{x=\infty} = \frac{2k \lambda \hat{j}}{y} \]

(Unlike a finite charge distribution, the field decreases as \( \frac{1}{y} \) not \( \frac{1}{y^2} \).)

**Method #2: Using Gauss Law**

Electric flux: \( \Phi = \oint E \cdot d\mathbf{A} \)

Closed surface integral

Electric field \( d\mathbf{A} \) is perpendicular to the element of area. For a spherical surface, \( d\mathbf{A} \) points along the radial direction: \( d\mathbf{A} = dA \hat{r} \)

Element of surface area: \( d\mathbf{A} = dA \hat{r} \)

Scalar product of two vectors: \( \hat{A} \cdot \hat{B} = AB \cos \theta \)

Examples: \( W = \hat{P} \cdot d\mathbf{A} \)

- Top: \( d\mathbf{A} = dA \hat{j} \)
- Bottom: \( d\mathbf{A} = -dA \hat{j} \)
- Left: \( d\mathbf{A} = -dA \hat{i} \)
- Right: \( d\mathbf{A} = dA \hat{i} \)
- Front: \( d\mathbf{A} = dA \hat{k} \)
- Back: \( d\mathbf{A} = -dA \hat{k} \)
Electric flux: \[ \Phi = \oint \mathbf{E} \cdot d\mathbf{A} = \Phi_{\text{closed surface}} = \int_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{A} = E_l \oint dA \]

If there is symmetry so that \( E_l \) is constant over the surface, \[ E_l \oint dA = E_l A \]

\[ E \cdot dA = E \cdot dA \cdot \cos \theta = \frac{E_l}{\cos \theta} \cdot dA \]

Component of \( E \) that is perpendicular to the surface.

We will use Gauss Law to calculate electric fields in these simple symmetry situations.

Gauss Law:

\[ \Phi_{\text{closed surface}} = \frac{\text{charge enclosed by surface}}{\varepsilon_0} \]

\( \varepsilon_0 = \text{dielectric constant in vacuum} \)

\[ \varepsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12} \, \text{C}^2 \text{Nm}^{-2} \]

\[ k = \frac{1}{4\pi \varepsilon_0} \]

\( k = 9 \times 10^9 \, \text{Nm}^2 \text{C}^{-2} \)
Meaning of Gauss Law:

\[ \Phi_{\text{closed surface}} = \frac{q}{\varepsilon_0} \]

However, to calculate \( \vec{E} \) using Gauss law, our Gaussian surface exhibits high symmetry.

1) Using Gauss law to calculate \( \vec{E} \) due to a point charge

First of all: determine the Gaussian surface (with high symmetry so
\[ \phi = E_1 A \]
otherwise it will take additional efforts to calculate \( \vec{E} \))

Gaussian surface: sphere centered @ the charge.

Also for this Gaussian surface
\( E_1 = E \) (\( E \) is radial so it is perpendicular to the surface)

\[ E_1 A = \frac{Q}{\varepsilon_0} \]
\[ E_1 = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{kQ}{r^2} \]

off-centered sphere will not allow
\[ \phi = E_1 \int dA \]
Using Gauss law and a highly symmetrical Gaussian surface (sphere centered @ charge) we have derived an expression for the electric field due to a point charge \( E = \frac{kQ}{r^2} \) that agrees with what we know from Chapter 20.
21.49

E = 26 kN/C

Charge stay on surface of balloon at R = 0.7m from center.

4) E (r = 0.5m or inside balloon)

Using Gauss Law:
1) Det: Gaussian surface = sphere centered at center of balloon

2) \[ \Phi_{\text{Gaussian surface}} = \oint E \cdot dA = EA \]

- \( E \): electric field on Gaussian surface
- \( A \): area of Gaussian surface = \( 4\pi r^2 \)
3) **Gauss Law:** \[ \phi = \frac{q_{\text{enclosed}}}{\varepsilon_0} \]
\[ E_{4\pi r^2} = \frac{Q}{\varepsilon_0} \rightarrow \frac{E(r = 0.5 m)}{\varepsilon_0} = 0 \]

1) **$E(r = 1.9 m, \alpha$ outside balloon)**

2) Determine Gaussian surface $\to$ sphere centered at center of balloon

3) \[ \phi = E_{4\pi r^2} \]

4) \[ \phi = \frac{9q_{\text{enclosed}}}{\varepsilon_0} \hspace{1cm} (\text{Gauss Law}) \]

\[ E_{4\pi r^2} = \frac{Q}{\varepsilon_0} \rightarrow E(r > R) = \frac{Q}{4\pi \varepsilon_0 r^2} = \frac{kQ}{r^2} \]

(like that of a point charge!)

Alternatives:

- **Find $Q$**
- **Then $E(r = 1.9 m)$**

**Observations:**

\[ E(r = R) = \frac{kQ}{0.7^2} = 26 \text{ kN/C} \]

\[ E(r = 1.9 m) = \frac{kQ}{1.9^2} \]

\[ \frac{E(r = 1.9 m)}{E(r = 0.9 m)} = \frac{0.9^2}{1.9^2} \rightarrow E(r = 1.9 m) = \frac{0.9^2}{1.9^2} \times 26 \text{ kN/C} \]

\[ Q = \frac{1.9^2 \times 3.53 \times 10^3}{9 \times 10^9} = 1.62 \mu C \]

\[ = 3.33 \text{ kN/C} \]
Ink jet printer:

Ink drop while carrying field region, feels a downward force $F = qE/m$, downward acceleration: $a_y = \frac{F}{m} = \frac{qE}{m}$, constant downward acceleration.

Min $v$ for ink drop to make it through field region:

During time it takes to go $A \rightarrow B$ ($x$ direction), it should be going not more than $AC$ ($y$ direction).

$x$ direction: $t_{AB} = \frac{L}{v}$.

Motion in $x$ direction:

is NOT affected by $E$ → uniform motion.

$y$ direction: constant acceleration motion: $y = \frac{1}{2}a_y t^2$.

$y = \frac{1}{2}a_y t_{AB}^2 < \frac{d}{2}$.

$\frac{1}{2} \cdot \frac{qE}{m} \cdot \frac{L^2}{v^2} < \frac{d}{2}$ → $\frac{qEL^2}{mv^2} < \frac{d}{2}$.

$\frac{d}{L} \sqrt{\frac{qE}{md}} < v$.

$\nu_{\text{min}} = L \sqrt{\frac{qE}{md}}$. 
\[
E_1 = \frac{380}{160} = \frac{x_1}{x_2} \left( \frac{x_2^2 + a^2}{x_1^2 + a^2} \right)^{3/2}
\]

\[
\left[ \frac{380}{160} \right]^{2/3} = \left( \frac{1}{3} \right)^{2/3} \frac{0.15^2 + a^2}{0.05^2 + a^2} \quad \Rightarrow \quad a = 0.07 m
\]

\[
E_1 = \frac{kQx_1}{(x_1^2 + a^2)^{3/2}} \quad \Rightarrow \quad Q = \frac{E_1 (x_1^2 + a^2)^{3/2}}{kx_1}
\]

\[
= \frac{380 \times 10^3 (0.05^2 + 0.07^2)^{3/2}}{9 \times 10^9 \times 0.05}
\]

\[
Q = 538 \text{ nC}
\]
Method 2: Calculation of \( E \) using Gauss Law.

Example 2: Very long line of charge (linear charge density \( \lambda \))

\[
\lambda = \frac{dq}{dx}
\]

Using Gauss Law to find electric field:

1) Gaussian surface: such that \( E \) is constant on the surface:

\[
\phi = \oint E \cdot dA = E \cdot A
\]

A cylinder of radius \( r \) with its axis along the line of charge.

2) Gaussian surface:

- **Body**: \( E_1 = E \)
- **Left side**: \( E_1 = 0 \) (\( E \) perpendicular to the left side has to point along \(-x\) since all electric fields are perpendicular to \( x \))
- **Right side**: \( E = \)

Similarly \( E_1 = 0 \)

\[
\phi = E_1 A = E_1 A_{\text{Body}} + E_1 A_{\text{Left}} + E_1 A_{\text{Right}} = E_1 A_{\text{Body}} = E A_{\text{Body}}
\]

\[
\phi = E, \text{ net}
\]

3) Gauss Law:

\[
E \cdot \Delta x = \left( \frac{AR}{\varepsilon_0} \right) \implies E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{2\lambda}{4\pi\varepsilon_0 r} = \frac{2kA}{r}
\]
Method #3 Electric Potential (Ch. 22)

Electric Potential

Potential energy difference by points A & B in mechanics:

\[ \Delta U_{AB} = -q \int_{A}^{B} \mathbf{F} \cdot d\mathbf{l} \]

Electric interaction:

\[ \mathbf{F} = q \mathbf{E} \]

Electric potential energy difference by points A & B:

\[ \Delta U_{AB} = -q \int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} \] (unit SI: J)

Electric potential difference by points A & B:

\[ \Delta V_{AB} = \frac{\Delta U_{AB}}{q'} = -\int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} \] (unit SI: V/C)

\[ \mathbf{E} = -\nabla \Delta V_{AB} \]

\[ \nabla : \text{gradient vector} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]
Electric field can be calculated by differentiating the electric potential with a minus sign.

**Example #1: Calculation of \( \vec{E} \) for a point charge using the Electric Potential**

For a point charge \( q \): \( V(r) = \frac{kq}{r} \) \( \rightarrow \) \( \vec{E} = -\nabla V = -\left( \frac{\partial V}{\partial r} \right) \hat{r} = -\frac{1}{r^2} \cdot \hat{r} \)

\[
\vec{E} = \frac{kq}{r^2} \hat{r}
\]

First time contact with electric potential:

\[
\Delta V_{AB} = -\int_{A}^{B} \vec{E} \cdot d\vec{l} = -\int_{A}^{B} \frac{kq}{x^2} \cdot \hat{r} \cdot dx = -kq \int_{A}^{B} \frac{dx}{x^2}
\]

Use a reference point (zero potential: \( x_A \rightarrow \infty \))

\[
\Delta V_{AB} = kq \left( \frac{1}{x_B} - \frac{1}{x_A} \right) \]

Always same reference point \( 0 \rightarrow \infty \) \( \rightarrow \) Convention is Electric potential due to a point charge, is a scalar (unit \( \frac{V}{m} \) or \( V \))
Example #2: Calculation of $\vec{E}$ due to 2 point charges at a point $p$ along the midline between the 2 charges.

$V_1 = \frac{kq}{r}$

$V_2 = \frac{kq}{r}$

What is $V$ at $P$, due to 2 point charges?

$V(P) = \frac{2kq}{r} \Rightarrow V = V_1 + V_2$

Electric field due to charge #1

Electric field due to charge #2

Strength for Method #3

Adding numbers instead of vectors

$\vec{E}(P) = -\nabla V = -\frac{\partial V}{\partial x} \vec{i}$

$= -2kq \frac{\partial}{\partial x} \frac{1}{[x^2 + \frac{d^2}{4}]} \vec{i} = -2kq \frac{2}{\partial x} [x^2 + \frac{d^2}{4}]^{-\frac{1}{2}} \vec{i}$

$= kq [x^2 + \frac{d^2}{4}]^{(-\frac{1}{2})} 2x \vec{i} = 2kq x \left[ x^2 + \frac{d^2}{4} \right]^{-\frac{3}{2}} \vec{i}$

$\vec{E} = \frac{2kq x}{\left[ x^2 + \frac{d^2}{4} \right]^{3/2}} \vec{i}$
a) \[ V(x, y) = V_1(x, y) + V_2(x, y) \]
   due to charge #1
   due to charge #2
   \[ = \frac{kq}{r_1} + \frac{kq}{r_2} = \frac{kq}{(x-a)^2 + y^2}^{1/2} + \frac{kq}{(x-a)^2 + y^2}^{1/2} \]

b) What is \( V(x, y) \) approximately if \( P \) is very far away from the two charges: \( x \gg a \) & \( y \gg a \)

   \[ V(x, y) \approx \frac{kq}{(x^2 + y^2)^{1/2}} + \frac{kq}{(x^2 + y^2)^{1/2}} = \frac{2kq}{r} \]
   \( r \to \text{Far away the electric potential is that of one point charge of value } 2q \)
V(x,y) = kq \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) = kq \left[ \frac{1}{(x-a)^2 + y^2} - \frac{1}{(x+a)^2 + y^2} \right]

a) \parallel dipole axis \parallel x-axis \rightarrow y=0

V(x,0) = kq \left[ \frac{1}{x-a} - \frac{1}{x+a} \right] = kq \left[ \frac{x-a - x+a}{(x-a)(x+a)} \right]

\Rightarrow \frac{V}{x^2 - a^2} = \frac{kq 2a}{x^2 - a^2} = \frac{kq p}{x^2 - a^2} \approx \frac{kq p}{x^2}

\Rightarrow \begin{cases} 0^\circ \text{ to axis} \\ \theta = 0.1 \text{m.} \\ n \gg a \end{cases}

dipole sep. \ll (2a)

\Rightarrow x = 10 \text{cm} (\text{data})

= 2.61 \times 10^3 \text{ V}

b) \phi \in \{45^\circ \text{ to axis} \}

\begin{cases} \phi = 0.1 \text{m.} \\ n \gg a \end{cases}

V(\phi y) = kq \left[ \frac{1}{(x-\phi y)^2} - \frac{1}{(x+\phi y)^2} \right]

31 \text{ July, 67}
Here polar coordinates are more useful. 

\[ V(x,y) = k \frac{q}{4 \pi \varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = k \frac{r_1 - r_2}{r_1 r_2} \approx k \frac{2 a \cos \theta}{r^2} \quad \text{P very far away} \]

\[ \begin{align*} \alpha &= 45^\circ \\ \rho &= 0.1 \, \text{m} \\
V(x,y) &= \frac{k_0 q}{4 \pi \varepsilon_0} \left( \frac{2 a \cos \theta}{r^2} \right) = \frac{9 \times 10^9 \times 2.9 \times 10^{-9} \cos 45^\circ}{0.1^2} = 1.85 \, \text{kV} \]

\[ c) \quad \alpha = 90^\circ: \quad V(x,y) = \frac{9 \times 10^9 \times 2.9 \times 10^{-9} \cos 90^\circ}{0.1^2} = 0 \]
(22.31) \[ V(x, y, z) = 2xy - 3xz + 5y \]

\[ P(x = 1m, y = 1m, z = 1m) \]

c) \[ V(1, 1, 1) = 2 - 3 + 5 = 4 \]

b) \[ \vec{E} = -\nabla V = -\frac{\partial V}{\partial x} \hat{i} - \frac{\partial V}{\partial y} \hat{j} - \frac{\partial V}{\partial z} \hat{k} \]

\[ \vec{E} = \hat{i}(2y - 3z) - \hat{j}(2x + 10y) - \hat{k}(-3x) \]

\[ \vec{E}(1, 1, 1) = \hat{i}(-1) - \hat{j}(12) - \hat{k}(-3) \]

\[ = \hat{i} - 12\hat{j} + 3\hat{k} \left( \frac{N}{C} \right) \]

---

(22.70)

---

\( r_a = 0.05 \text{ m} \)

\( Q = 60 \text{nC} \)

Concentric sphere

\( r_b = 0.15 \text{ m} \)

\( Q = -60 \text{nC} \)

Concentric shell

---

\( V(r = r_a) = \Delta V_{\text{qA}} = -\int_{\infty}^{A} \frac{kQ}{r^2} \, dr \)

\[ = \frac{kQ}{r_a} \]

(True if there was an outer shell)

\[ V(r = r_a) = \Delta V_{\text{qA}} = \Delta V_{\text{qB}} + \Delta V_{\text{qBA}} \]

\[ E \text{ outside } \begin{cases} \text{ Gaussian surface} & \text{ outer shell} \\ \text{conducting shell} & \text{ inner shell} \end{cases} \]

\[ \begin{aligned} \Delta V_{\text{qB}} &= -\int_{B}^{A} \frac{kQ}{r^2} \, dr = kQ \left[ \frac{1}{r_b} \right]^{A} \\ &= kQ \left[ \frac{1}{r_a} - \frac{1}{r_b} \right] = 9 \times 10^5 \times 60 \times 10^{-7} \\ &\quad \left[ \frac{1}{0.05} - \frac{1}{0.15} \right] \\ V(r = r_a) &= 7200 \text{ V} \end{aligned} \]
\[ V(r=r_a) = \Delta V_{aA} = \Delta V_{AB} + \Delta V_{BA} \]

\[ = \int_0^B \frac{kQ}{r^2} \, dr = kQ \left[ -\frac{1}{r} \right]_0^B = \frac{kQ}{r_B} = \frac{9 \times 10^8 \times 2 \times 60 \times 10^4}{6.15} \]

\[ = 7200 \text{ V} \]

\[ V(r=r_a) = 7200 + 7200 = 14400 \text{ V}. \]

22-67

\[ \lambda = -\frac{75\mu C}{m} \]

\[ \lambda = +\frac{75\mu C}{m} \]

b) Coaxial cable

\[ \Delta V_{AB} = -\int_{A}^{B} E \cdot dr \]

Field in inner & outer conductor

Gaussian Law with 6-surface of radius \( r \) \( r_a < r < r_b \)

Electric due to inner conductor (very long wire)

\[ E = \frac{2kQ}{r^2} \]
$$\Delta V_{AB} = -\int_{A}^{B} \frac{2k\lambda}{r} dr = -2k\lambda \int_{A}^{B} \frac{dr}{r} = -2k\lambda \ln\left(\frac{r_B}{r_A}\right)$$

$$= -2 \times 4 \times 10^9 \times 75 \times 10^{-9} \ln\left(\frac{10 \times 10^3}{2 \times 10^3}\right) = -2170 \text{V}$$

b) if a far outer conductor charged to $+150 \text{nC/m}$

$$\Delta V_{AB} \text{some} = -2170 \text{V} \quad \text{(since this would not change the electric field inside outer conductor, it only changes field outside outer conductor)}$$

c) \[ \vec{E}(x=0, y=0) = \vec{E}_1 + \vec{E}_2 = \frac{k_e}{x^2} \times 2 \int = \frac{9 \times 10^9 \times 1.6 \times 10^{-19} \times x^2}{0.6 \times 10^{-9}} \int \left(\frac{N}{C}\right) \]

$$= 8 \times 10^9 \int \left(\frac{N}{C}\right)$$

b) \[ \vec{E}(x=2 \text{mm}, y=0) = 2E_y \int = 2 \left(\frac{k_e}{x^2}\right) \sin \alpha \int = 2 \frac{k_e a}{x^2} \int \]

$$= 2 \frac{k_e a}{(x^2 + a^2)^{3/2}} \int = 2 \frac{9 \times 10^9 \times 1.6 \times 10^{-19} \times 0.6 \times 10^{-9}}{(2 \times 10^{-9})^2 + (0.6 \times 10^{-9})^2} \int$$

$$= 190 \times 10^6 \frac{N}{C}$$

c) \[ \vec{E}(x=-20 \text{mm}, y=0) = j \int \frac{9 \times 10^9 \times 1.6 \times 10^{-19} \times 0.6 \times 10^{-9}}{(20 \times 10^{-9})^2 + (0.6 \times 10^{-9})^2} \int = 216 \times 10^3 \frac{N}{C} \]