Ch. 20: Electric Charge, Force, and Field

Interaction b/w these two distributions of charges: via the electric fields $\vec{E}_1$ (by distribution #1) & $\vec{E}_2$ (by distribution #2). To calculate the electric force of Q_1 on Q_2, we first calculate the $\vec{E}_1$, then $\underbrace{\vec{F}_{12}}_{\text{force of 1 on 2}} = Q_2 \underbrace{\vec{E}_1}_{\text{electric field by 1}}$

We will calculate the electric fields for simple charge distributions.

Charges: • types: 2 $\begin{cases} + & \text{(a proton has } 1e^+) \\ - & \text{(that of an electron } 1e^- \end{cases}$

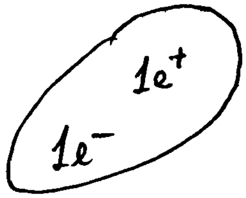
$e = 1.6 \times 10^{-19}\,C$

C ↓ Coulomb, SI unit for charge

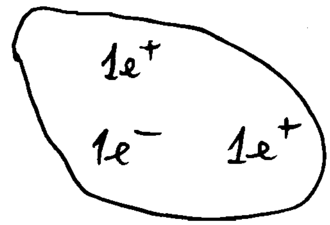
$1e^- = -1.6 \times 10^{-19}\,C$

$1e^+ = +1.6 \times 10^{-19}\,C$

- Superposition



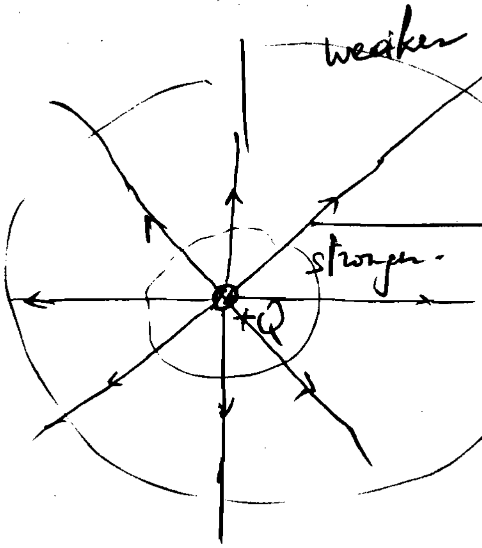
$Q_{net} = 0$



$Q_{net} = 1e^+ = +1.6 \times 10^{-19}\, C$

- Source of electric field:

$$\longrightarrow \vec{E} = k\frac{Q}{r^2}\hat{r} \quad \left\{ \begin{array}{l} k: \text{ electric constant: } 9\times10^{9}\,\frac{Nm^2}{C^2} \\ Q: \text{ net charge} \\ r: \text{ separation from the charge} \\ \hat{r}: \text{ unit radial vector, pointing away from the charge} \end{array} \right.$$



Field lines

Strength ~ density of field lines.

Electric field

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

+ or −

Field: attractive (−) or repulsive (+)

$k = 9\times10^{9}\,\frac{Nm^2}{C^2}$

Gravitational field

$$\vec{g} = -G\frac{M}{r^2}\hat{r}$$

always +

Field: always attractive

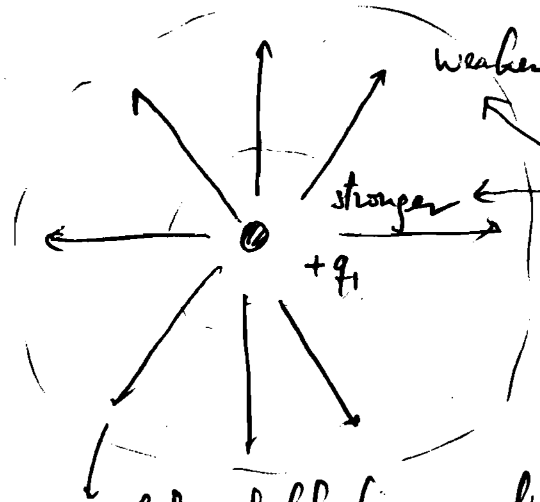
$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$$\frac{\text{Electric force}}{\text{Gravitat. force}} = ? \simeq 10^{40}$$

b/w a proton & an electron;

$m_e = 9.11 \times 10^{-31}$ kg ; $m_p \simeq 2000\, m_e$

<u>Electric field due to one charge</u>: (from the field we can calculate forces on test charge)



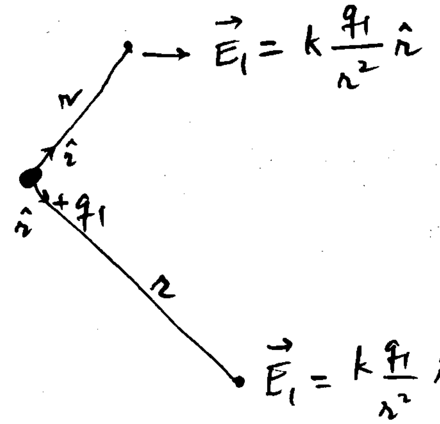
$$\vec{E}_1 = k\frac{q_1}{r^2}\hat{r}$$

$\hat{r}$: unit vector, pointing along radial directions away from the charge

r : separation from the source charge

Electric field lines : directions & strengths of electric fields

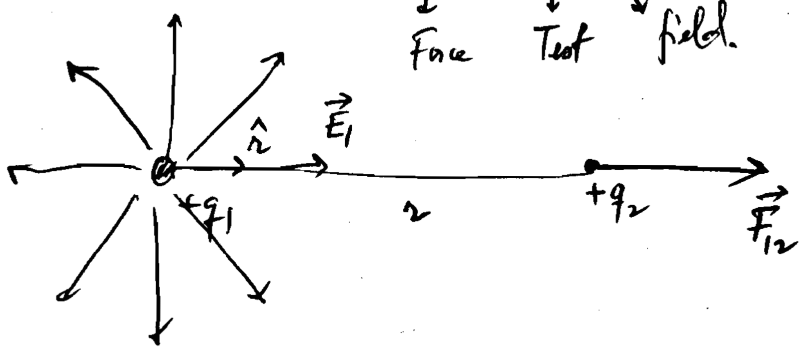
(higher density of line, stronger fields)



The field created by this positive charge q_1 is repulsive.
(lines point away from the charge)

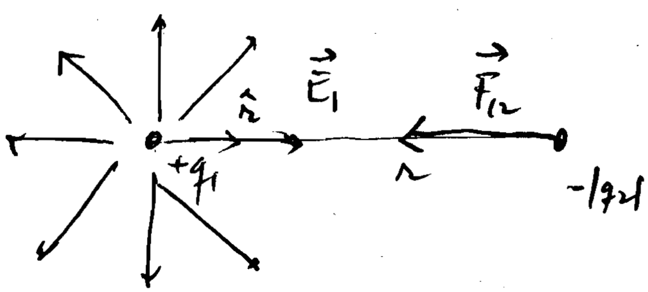
A positive test charge in this field will feel a repulsive force:

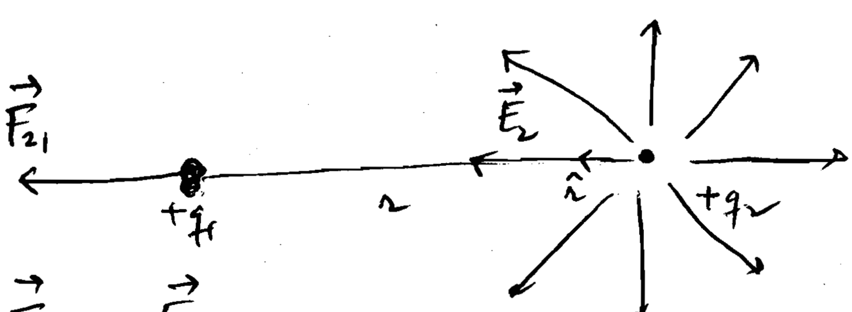
$$\underbrace{\vec{F}_{12}}_{\text{Force}} = \underbrace{q_2}_{\text{Test}} \underbrace{\vec{E}_1}_{\text{field.}} = k \frac{q_1 q_2}{r^2} \hat{r}$$



A negative test charge in this field will feel an attractive force:

$$\vec{F}_{12} = -|q_2| \vec{E}_1 = -k \frac{|q_1||q_2|}{r^2} \hat{r}$$



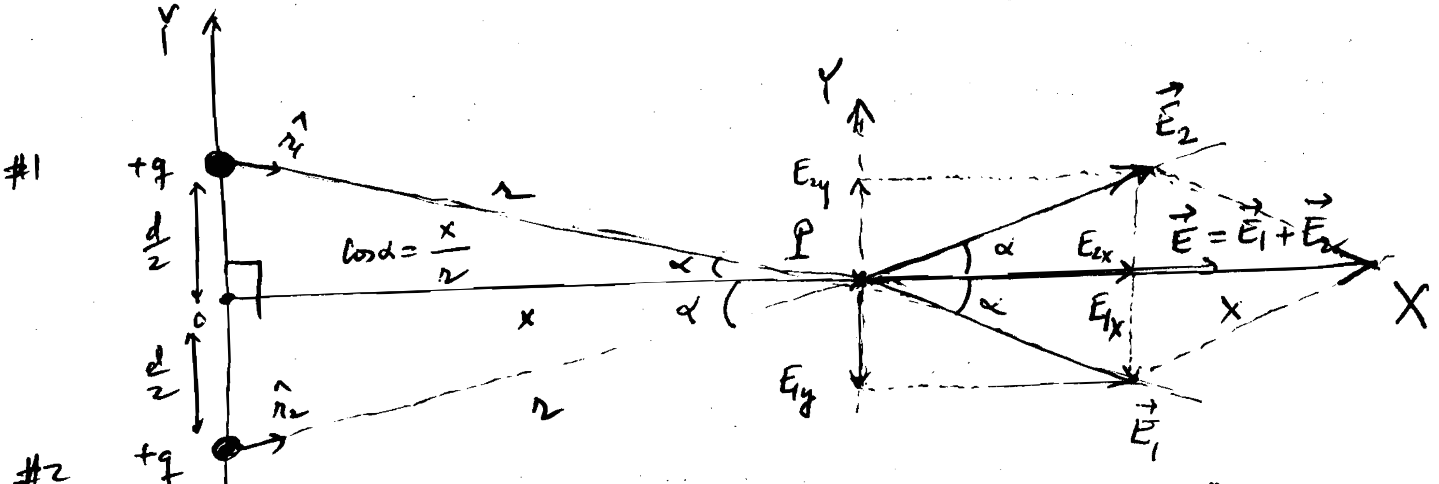


$$\vec{E}_2 = k \frac{q_2}{r^2} \hat{r}$$

$$\underbrace{\vec{F}_{21}}_{\text{force}} = \underbrace{q_1}_{\text{test}} \underbrace{\vec{E}_2}_{\text{field}}$$

→ Electric forces b/w 2 charges of the same type or sign are repulsive ; b/w 2 charge of opposite types or signs are attractive.

<u>Electric field due to two positive charges</u>: by superposition of fields created by each charge. What is the electric field along the centerline ?



$$\vec{E}_1 = k\frac{q}{r^2}\hat{r}_1 \; ; \qquad \vec{E}_2 = k\frac{q}{r^2}\hat{r}_2$$

$$E_1 = |\vec{E}_1| = \frac{kq}{r^2} = E_2 = |\vec{E}_2| = \frac{kq}{r^2}$$

$\vec{E}_1 = E_{1x}\hat{i} + E_{1y}\hat{j}$ — unit vectors along x or y direction.

$\vec{E}_2 = E_{2x}\hat{i} + E_{2y}\hat{j}$

$$\begin{cases} \vec{E}_1 = E_{1x}\hat{i} + E_{1y}\hat{j} = E_1\cos\alpha\,\hat{i} - E_1\sin\alpha\,\hat{j} \\ \vec{E}_2 = E_2\cos\alpha\,\hat{i} + E_2\sin\alpha\,\hat{j} \overset{E_2=E_1}{=\!=} E_1\cos\alpha\,\hat{i} + E_1\sin\alpha\,\hat{j} \end{cases}$$

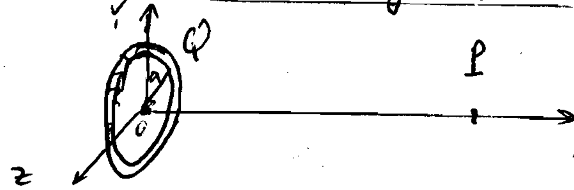
$$\vec{E} = \vec{E}_1 + \vec{E}_2 = E_1\cos\alpha\,\hat{i} - \cancel{E_1\sin\alpha\,\hat{j}} + E_1\cos\alpha\,\hat{i} + \cancel{E_1\sin\alpha\,\hat{j}}$$

$$= \underline{2E_1\cos\alpha}\,\hat{i}$$ → Total field points along x-axis.

↳ $$\vec{E} = 2E_1\cos\alpha\,\hat{i} = 2\frac{kq}{r^2}\cos\alpha\,\hat{i} = 2\frac{kq}{x^2+\frac{d^2}{4}}\frac{x}{\left(x^2+\frac{d^2}{4}\right)^{1/2}}\hat{i}$$

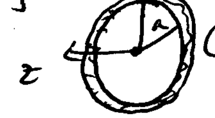
$$\boxed{\vec{E} = \frac{2kq\,x}{\left(x^2+\frac{d^2}{4}\right)^{3/2}}\hat{i}}$$ Unit in S.I. : $\frac{N}{C}$

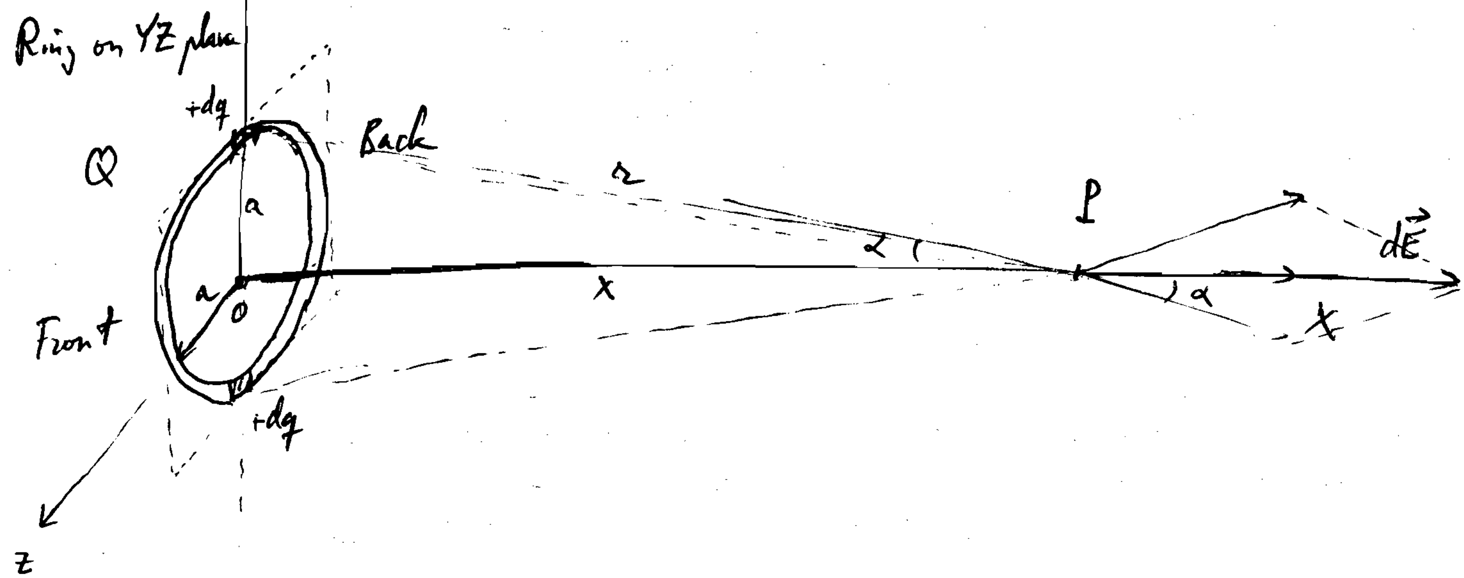
Electric field due to a continuous ring of charge, at a point along its axis:



(axis along x)

Ring on YZ plane, origin of coordinates at center of the ring. From the right:





To use previous results: focus on +dq top of ring & +dq bottom of ring. $\rightarrow$ $d\vec{E} = \dfrac{2k\,dq\,x}{(x^2+a^2)^{3/2}}\,\hat{\imath}$

For total Field due to whole ring: $\vec{E} = \displaystyle\int_{\text{half ring}} d\vec{E} = \frac{\cancel{2}kx}{(x^2+a^2)^{3/2}}\,\hat{\imath}\underbrace{\int_{\text{half ring}} dq}_{\frac{Q}{\cancel{2}}}$

bring out what is constant along ring

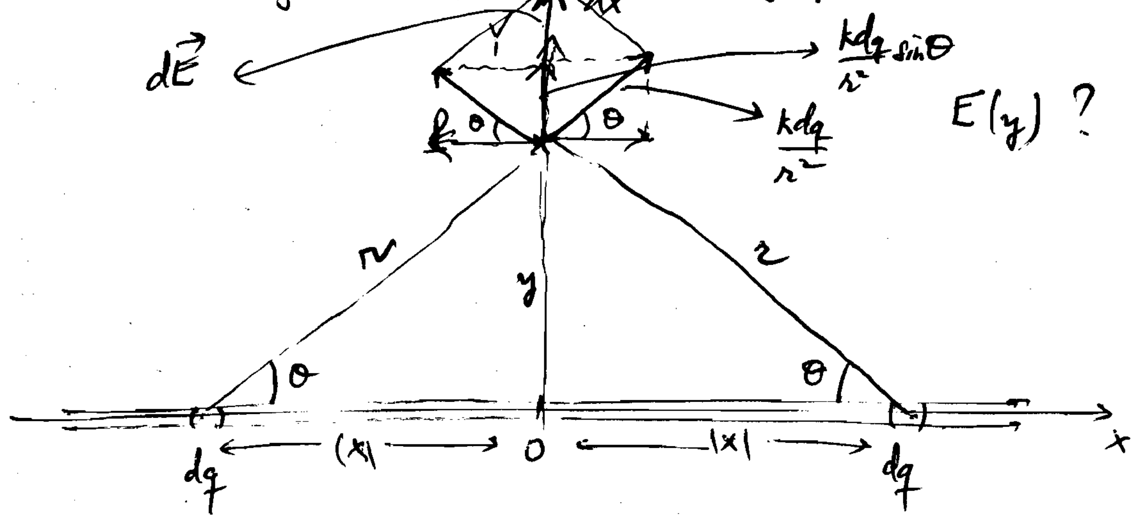
$$\boxed{\vec{E} = \frac{kQx}{(x^2+a^2)^{3/2}}\,\hat{\imath}}$$

Approx: far from ring: $x \gg a \rightarrow x^2+a^2 \simeq x^2 \rightarrow \boxed{\vec{E}_{far} = \dfrac{kQ}{x^2}\,\hat{\imath}}$

$\rightarrow$ Makes sense: it's field of a point charge Q

Electric field due to a ∞ long line of charge:

– linear charge density $\lambda = \frac{dq}{dx}$ (charge per unit length)



$$d\vec{E} = 2 \, k \frac{dq}{r^2} \sin\theta \; \hat{j} \qquad \text{(no component along } x\text{)}$$

from both element of charge.

$$= 2 \, \frac{k\,dq}{r^2} \frac{y}{r} \, \hat{j} \quad = \quad 2k \, \frac{\lambda\, dx \; y}{r^3} \, \hat{j}$$

$$\vec{E} = \int_{\text{half line}} d\vec{E} = 2k\lambda y \, \hat{j} \underbrace{\int_0^\infty \frac{dx}{(x^2+y^2)^{3/2}}}$$

$$\boxed{\text{From table: } \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2 (x^2+a^2)^{1/2}}}$$

$$\vec{E} = 2k\lambda y \, \hat{j} \left[\frac{x}{y^2 (x^2+y^2)^{1/2}} \right]_{0=x}^{\infty=x} = 2k\lambda y \, \hat{j} \left[\frac{1}{y^2} - 0 \right]$$

$$\boxed{\vec{E}(y) = \frac{2k\lambda}{y} \, \hat{j}}$$

<u>L'Hopital Rule of limits</u>: $\frac{\infty}{\infty}$ or $\frac{0}{0} = \frac{f(x)}{g(x)}$

$$\rightarrow \frac{\frac{df}{dx}}{\frac{dg}{dx}}$$

Ch. 21 Gauss Law

How to calculate the Electric field?

1) Vector addition (Ch. 20)
2) <u>Gauss Law</u> (Ch. 21) using symmetry
3) Electric potential (Ch. 22) using scalars and derivatives.

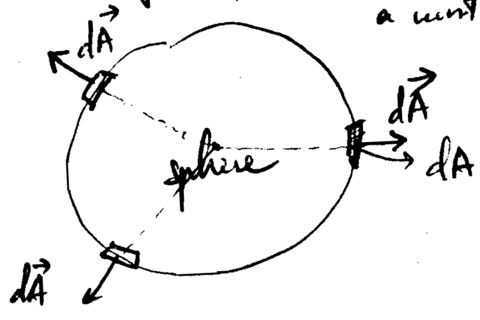
<u>Electric flux</u>: ϕ "Phi" $= \oint_{\text{closed surface}} \vec{E} \cdot d\vec{A}$

$\downarrow$ scalar product

$\vec{A} \cdot \vec{B} = AB \cos\theta$

$A = |\vec{A}|$, $B = |\vec{B}|$, θ = angle b/w $\vec{A}$ & $\vec{B}$

$d\vec{A}$: element of area vector of the closed surface, direction is given by a unit vector perpendicular to the element of area.



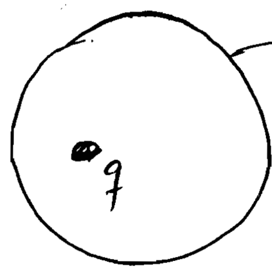
* ϕ can be calculated easily for simple surfaces sphere, cylinder, rectangular box, etc.

Gauss Law:

$$\boxed{\Phi_{\substack{\text{closed}\\ \text{surface}}} = \frac{q_{\text{enclosed by surface}}}{\varepsilon_0}}$$

$$\varepsilon_0 = \text{dielectric constant in vacuum} = \frac{1}{4\pi k} = 8.85\times10^{-12}\,\frac{C^2}{Nm^2}$$

E.g.:



$$\Phi = \frac{q}{\varepsilon_0}$$

* If we can write the flux Φ easily in term of the electric field, then we can use Gauss Law to obtain the electric field.

Simple geometries (sphere, cylinder, boxes) & highly symmetric situations.

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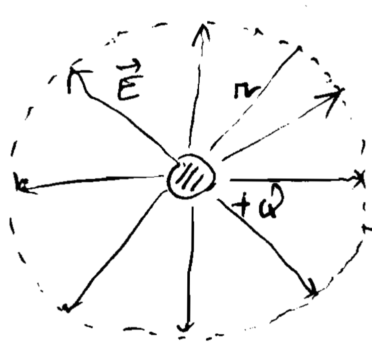
1) Let's use Gauss Law to calculate the field from a point charge.



Gauss law → Flux → Surface (Gaussian surface)

→ spherical Gaussian surface with charge Q at center.

$$\Phi = \frac{Q}{\varepsilon_0} \text{ (Gauss Law)} \rightarrow \Phi = \int \vec{E}\cdot d\vec{A} = \int \frac{kQ}{r^2}\hat{r}\cdot dA\,\hat{r} = \frac{kQ}{r^2}\int dA$$

$$\hat{r}\cdot\hat{r} = 1$$

$$\left.\begin{array}{l}\Phi = \dfrac{kQ}{\cancel{r^2}}\, 4\pi \cancel{r^2} \\[2ex] \Phi = \dfrac{Q}{\varepsilon_0}\end{array}\right\} \quad k\cancel{Q}4\pi = \dfrac{\cancel{Q}}{\varepsilon_0} \rightarrow \quad \varepsilon_0 = \dfrac{1}{4\pi k}$$

But I wanted to calculate the field! Should not plug in $\vec{E} = \dfrac{kQ}{r^2}\hat{r}$ but only $\vec{E} = E\hat{r}$

$$\Phi = \int \vec{E}\cdot d\vec{A} = \int E\hat{r}\cdot dA\hat{r} \underset{\hat{r}\cdot\hat{r}=1}{=} \int E\,dA \underset{\substack{E \text{ is constant} \\ \text{for a fixed separation} \\ r \text{ (over the surface)}}}{=} E\int dA = E4\pi r^2$$

$$\Phi = \frac{Q}{\varepsilon_0}$$

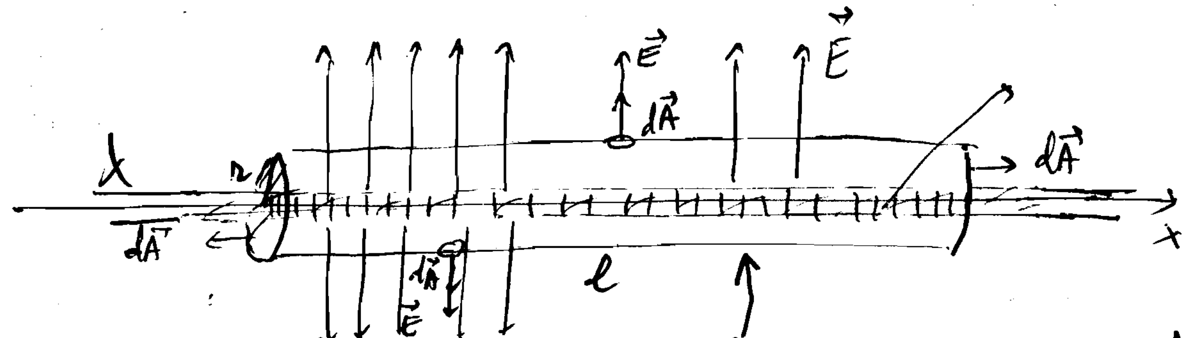
$$E4\pi r^2 = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{\underbrace{4\pi\varepsilon_0}_{\frac{1}{4\pi k}} r^2} = \frac{kQ}{r^2}$$

(as expected from ch. 20)

Gauss Law is useful!

2) Electric field due to a ∞ long line of charge using Gauss law:

Linear density of charge is $\lambda = \frac{dq}{dx}$



Gauss Law → Flux → Gaussian surface: Cylinder with the line at center.

$$\phi = \frac{q_{enclosed}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$$

$$\phi = \oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint_{\text{Body of cylinder}} dA$$

(not the left & right circles!)

$\vec{E}$ points away from the line along perpendicular direction. always parallel to $d\vec{A}$ on this Gaussian cylindrical surface centered at the line.

$$= E \cdot \underbrace{2\pi r \ell}_{\text{area of body of cylinder}}$$

$$\frac{\lambda \cancel{\ell}}{\varepsilon_0} = E\, 2\pi r \cancel{\ell} \rightarrow \boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{2k\lambda}{r}}$$

$$\varepsilon_0 = \frac{1}{4\pi k}$$

Same as if using vector addition!

Ch. 22 Electric Potential

$$\Delta U_{AB} = -\int_A^B \vec{F} \cdot d\vec{l}$$

Potential energy difference b/w A & B

For electric interaction: $\vec{F} = q' \vec{E}$

q' → Test charge

$\vec{E}$ → field

Electric potential energy difference b/w A & B:

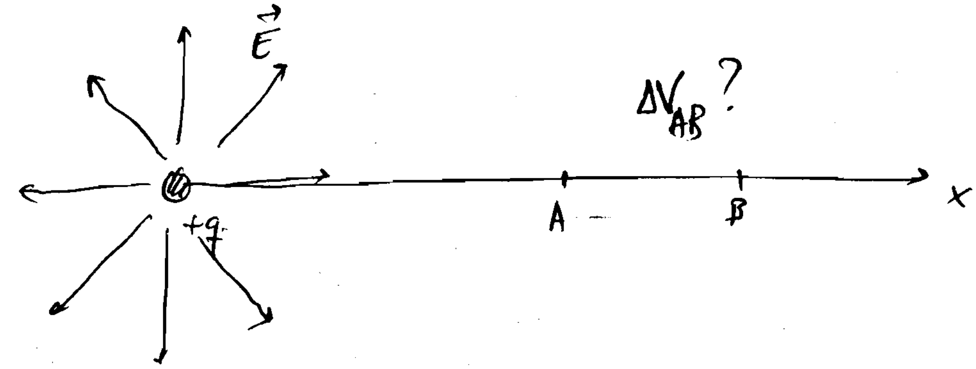
$$\Delta U_{AB} = -q' \int_A^B \vec{E} \cdot d\vec{l}$$

Electric potential difference: $\Delta V_{AB} = \dfrac{\Delta U_{AB}}{q'} = -\int_A^B \vec{E} \cdot d\vec{l}$

ΔV_{AB} → Unit: SI: $\dfrac{J}{C} = V$ for Volt

$\vec{E}$ → Field!

1) Point charge: what is the electric potential?

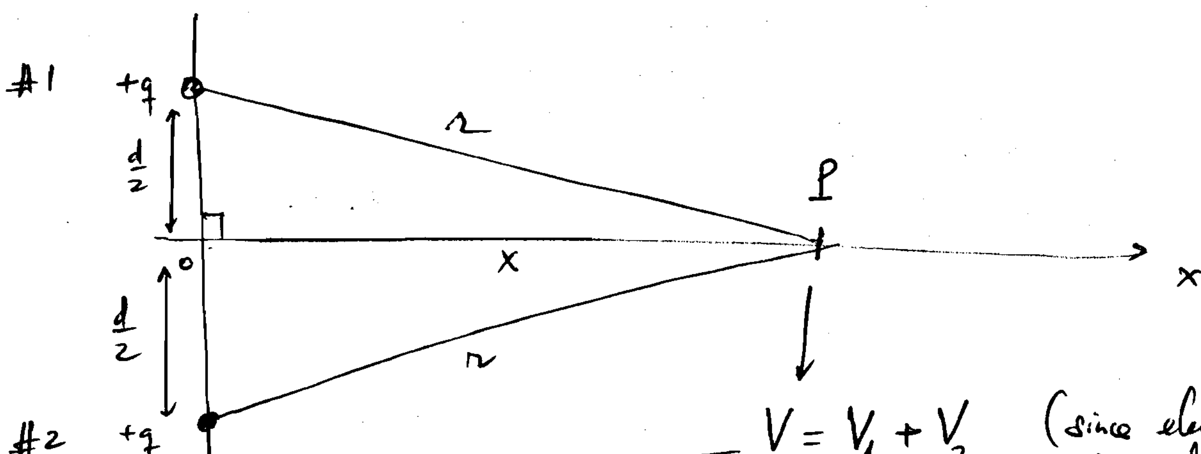


$$\Delta V_{AB} = -\int_A^B \frac{kq}{x^2}\hat{i}\cdot dx\,\hat{i} \underset{\hat{i}\cdot\hat{i}=1}{=} -kq\underbrace{\int_A^B \frac{dx}{x^2}}_{-\frac{1}{x}} = kq\left(\frac{1}{x_B}-\frac{1}{x_A}\right)$$

Ref. point $A \to \infty \rightarrow \Delta V_{\infty B} \equiv V_B = \frac{kq}{x_B}$ or $\frac{kq}{r_B}$

$$\rightarrow \boxed{V(r) = \frac{kq}{r}}$$

Electric potential due to 2 charges: at point P.



$V = V_1 + V_2$ (since electric potentials are numbers, we just add numbers)

$$V = \frac{2kq}{r} = \frac{2kq}{\left(x^2+\frac{d^2}{4}\right)^{1/2}} = V(x)$$

$$V = -\int \vec{E}\cdot d\vec{l} \longrightarrow \vec{E} = -\vec{\nabla} V$$

gradient (derivative)

a vector operator:

$$\vec{\nabla} \equiv \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right)$$

$$\vec{E} = -\vec{\nabla} V = -\left(\frac{dV}{dx}\,\hat{\imath} + \frac{dV}{dy}\,\hat{\jmath} + \frac{dV}{dz}\,\hat{k} \right)$$

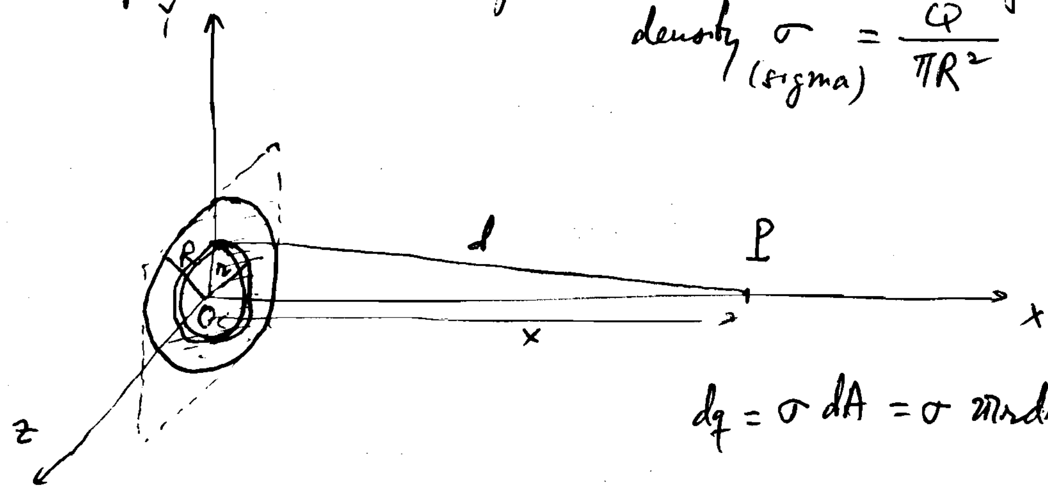
For 2 point charges: $V = V(x)$ (there is no dependence on y nor z)

$$\rightarrow \vec{E} = -\frac{dV}{dx}\hat{\imath} = -2kq \frac{d}{dx}\left(x^2 + \frac{d^2}{4}\right)^{-\frac{1}{2}} \hat{\imath}$$

$$= kq \left(x^2 + \frac{d^2}{4}\right)^{-\frac{3}{2}} 2x\,\hat{\imath} = \frac{2kqx}{\left(x^2 + \frac{d^2}{4}\right)^{3/2}}\hat{\imath}$$

Electric potential due to a uniformly charged circular disk at a point along its axis.:

Disk on YZ plane: total charge Q, radius R $\rightarrow$ charge density $\underset{(\text{sigma})}{\sigma} = \frac{Q}{\pi R^2}$



$dq = \sigma\, dA = \sigma\, 2\pi r\, dr$

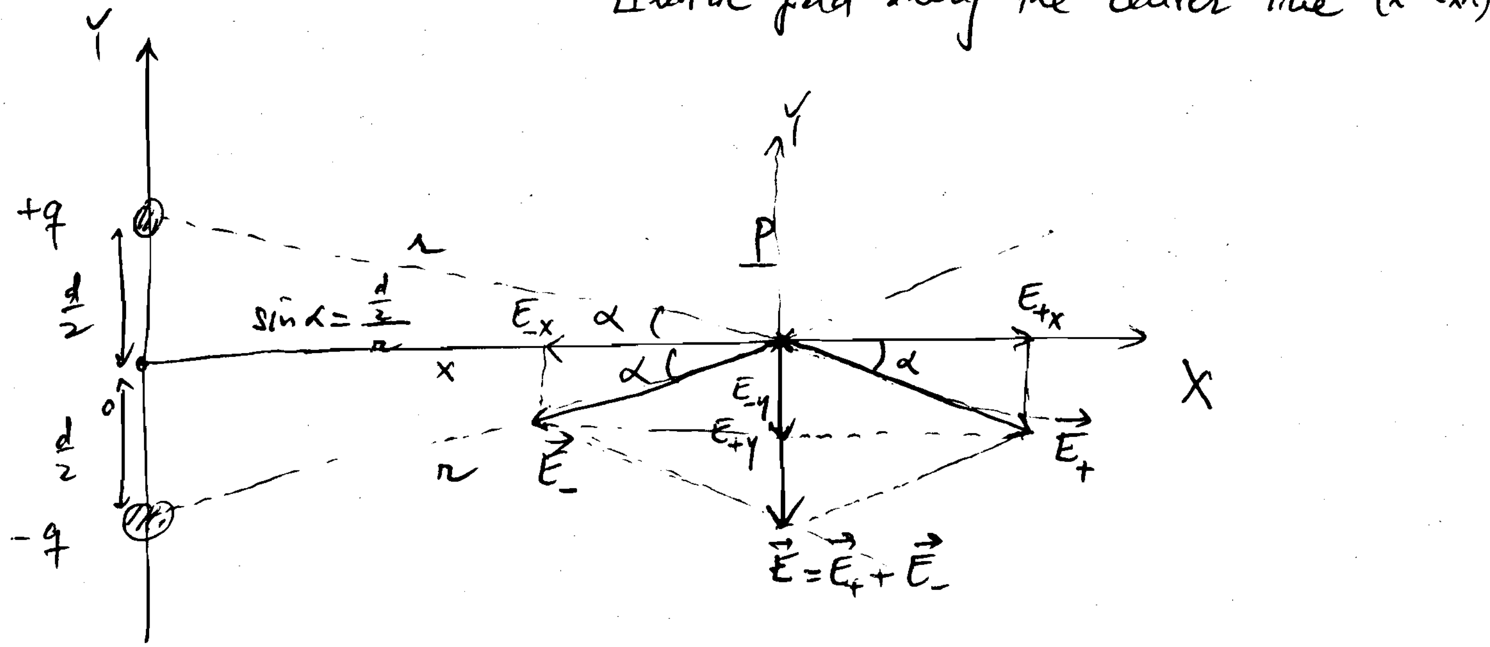
We will first start with dV from a ring of radius r, & thickness dr. Charge on this ring: dq, all point on this ring is at a fixed separation d to P: $\underbrace{dV}_{\text{infinitesimal electric potential due to ring}} = \frac{k\,dq}{d}$

$$dV = \frac{k\,dq}{(x^2 + r^2)^{1/2}}$$

$$\rightarrow V = \int_{0=r}^{R=r} dV = k\int_{r=0}^{r=R} \frac{dq}{(x^2+r^2)^{1/2}} = k\int_{r=0}^{r=R} \frac{\sigma 2\pi r\, dr}{(x^2+r^2)^{1/2}} = 2\pi\sigma k \int_0^R \frac{r\,dr}{(x^2+r^2)^{1/2}}$$

Electric field due to a dipole (one positive and one negative charge)

Electric field along the center line (x-axis)



$$\vec{E}_+ = k\frac{q}{r^2}\hat{r}_1 \; ; \quad \vec{E}_- = \frac{-kq}{r^2}\hat{r}_2 \; ; \quad |\vec{E}_+| = |\vec{E}_-| = E_+ = \frac{kq}{r^2}$$

$$\left.\begin{aligned} \vec{E}_+ &= E_+\cos\alpha\,\hat{i} - E_+\sin\alpha\,\hat{j} \\ \vec{E}_- &= -E_+\cos\alpha\,\hat{i} - E_+\sin\alpha\,\hat{j} \end{aligned}\right\} \quad \vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \cancel{E_+\cos\alpha\,\hat{i}} - E_+\sin\alpha\,\hat{j} - \cancel{E_+\cos\alpha\,\hat{i}} - E_+\sin\alpha\,\hat{j}$$

$$\boxed{\vec{E} = -2E_+\sin\alpha\,\hat{j}}$$

pointing along $-y$ direction.

$$\boxed{\vec{E} = -2\frac{kq}{r^2}\sin\alpha\,\hat{j} = -2\frac{kq}{r^2}\frac{\frac{d}{2}}{r}\hat{j} = -k\frac{qd}{r^3}\hat{j}}$$

top charge sign

$$\boxed{\vec{E} = -k\frac{qd}{\left(x^2+\frac{d^2}{4}\right)^{3/2}}\hat{j}}$$ → dipole moment.

Approximation: $x \gg \frac{d}{2} \rightarrow x^2 + \left(\frac{d}{2}\right)^2 \simeq x^2$

$$\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2} \simeq x^3$$

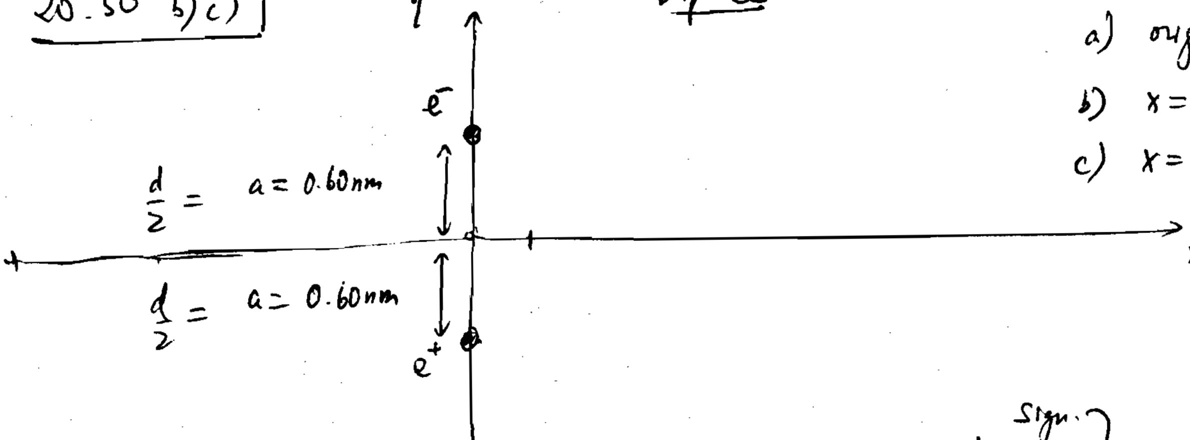
$$\rightarrow \vec{E}_{Dipole} \simeq -\frac{kqd}{x^3}\hat{j} \quad \text{(large distance)}$$

$$\int \frac{r\,dr}{(x^2+r^2)^{1/2}} = \frac{1}{2}\int \frac{du}{u^{1/2}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{\cancel{2}}\,\frac{u^{1/2}}{1/\cancel{2}} = \sqrt{u}$$

$$\underset{\substack{\downarrow \\ \text{const.}}}{r^2 + x^2} \equiv u \rightarrow 2r\,dr = du \rightarrow r\,dr = \frac{1}{2}\,du$$

$$V(x) = 2\pi\sigma k \underbrace{\int_{r=0}^{r=R} \frac{r\,dr}{(x^2+r^2)^{1/2}}} = 2\pi\sigma k\left(\sqrt{R^2+x^2} - \sqrt{x^2}\right) \leftarrow$$

20.50 b) c) | Dipole $\vec{E}$?



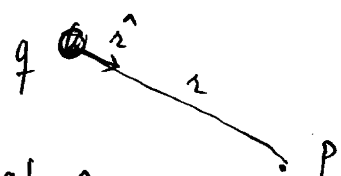
a) origin
b) x = 2nm
c) x = −20nm.

$$\text{a)}\quad \vec{E}(x=0, y=0) = \vec{E}_{e^-} + \vec{E}_{e^+} = \frac{k|(-e)|}{a^2}\overset{\text{sign.}}{(+\hat{j})} + \frac{k(e)}{a^2}(+\hat{j})$$

$$= \frac{2ke}{a^2}\,\hat{j} = \frac{2\times 9\times 10^{9}\times 1.6\times 10^{-19}}{(0.6\times 10^{-9})^2}\,\hat{j} = 8\times 10^{9}\,\frac{N}{C}$$

alternative from result derived previously $= \dfrac{-k(-e)\,d}{\left(0+\frac{d^2}{4}\right)^{3/2}}\,\hat{j} = \dfrac{9.10^{9}\times 1.6\times 10^{-19}\times 1.2\times 10^{-9}}{(0.6\times 10^{-9})^3}\,\hat{j}$

Field due to a point charge at P



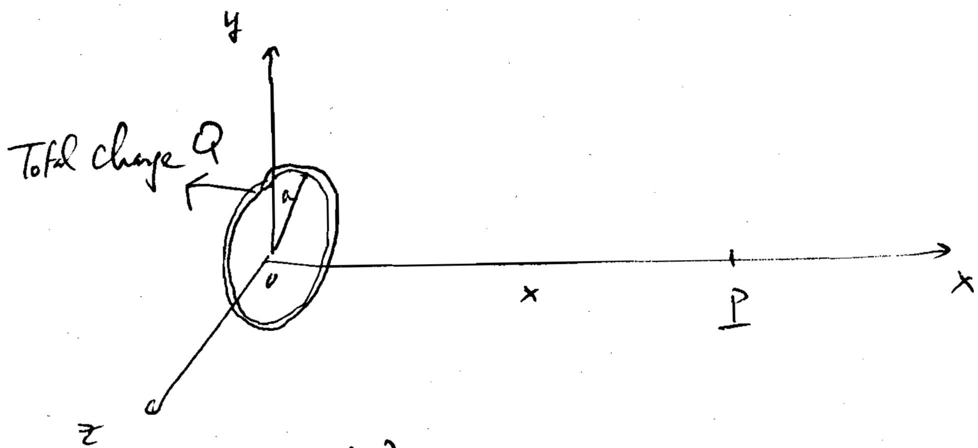
$$\vec{E} = \frac{kq}{r^2}\,\hat{r} \quad \text{or} \quad \frac{k|q|}{r^2}\,\hat{r}$$

↳ (away from charge for + charge
toward charge for − charge)

b) $\vec{E}\,(x = 2nm\,,\ y=0) = \dfrac{-k(-e)d}{\left[(2\times10^{-9})^2 + (0.6\times10^{-9})^2\right]^{3/2}}\,\hat{j} = 190\times10^{6}\ \dfrac{N}{C}\,\hat{j}$

c) $\vec{E}\left(x = -20nm\,,\ y=0\right) = \dfrac{-k(-e)d}{\left[(-20\times10^{-9})^2 + (0.6\times10^{-9})^2\right]^{3/2}}\,\hat{j} = 216\times10^{3}\ \dfrac{N}{C}\,\hat{j}$

20.65 Electric field due to a uniformly charged ring at a point along its axis: $\vec{E}(x) = \dfrac{kQx}{(x^2+a^2)^{3/2}}\,\hat{i}$



$\left\{\begin{array}{l} x_1 = 0.05m \rightarrow |\vec{E}| = 380\ \frac{kN}{C} \\ x_2 = 0.15m \rightarrow |\vec{E}| = 160\ \frac{kN}{C} \end{array}\right.$

Q? a?

$$\frac{E_1}{E_2} = \frac{\dfrac{\cancel{kQ}\,x_1}{(x_1^2+a^2)^{3/2}}}{\dfrac{\cancel{kQ}\,x_2}{(x_2^2+a^2)^{3/2}}} \rightarrow \left(\frac{380}{160}\right)^{2/3} = \left(\frac{0.05}{0.15}\right)^{2/3} \frac{(0.15^2+a^2)}{(0.05^2+a^2)}$$

$$\underbrace{\left(\frac{38}{16}\times3\right)^{2/3}}_{3.70}\left(0.05^2+a^2\right) = 0.15^2 + a^2$$

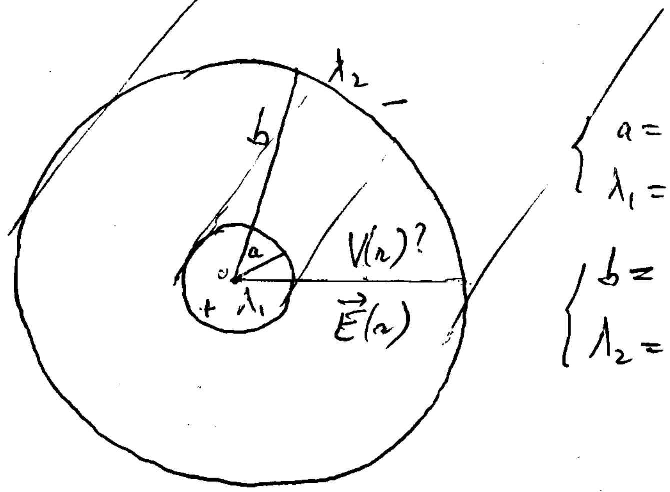
$$2.7\,a^2 = 0.15^2 - 3.7\times0.05^2$$

$$a = \sqrt{\frac{0.15^2 - 3.7\times0.05^2}{2.7}}\ m = 0.07m$$

$$E_1 = \frac{kQx_1}{(x_1^2+a^2)^{3/2}} \rightarrow Q = \frac{E_1\,(x_1^2+a^2)^{3/2}}{k\,x_1}$$

$$= \frac{380\times10^3\,(0.05^2+0.07^2)^{3/2}}{9\times10^9\times0.05} = 538\,nC$$

22-67 | Look at cross section:



$\begin{cases} a = 2\,mm \\ \lambda_1 = 75\,\frac{nC}{m} \end{cases}$ $\begin{cases} b = 10\,mm \\ \lambda_2 = -75\,\frac{nC}{m} \end{cases}$

a) Pot. difference b/w outer & inner conductors

$$\Delta V = -\int_2^1 \vec{E}\cdot d\vec{r} = -\int_2^1 \frac{2k\lambda_1}{r}\,dr = -2k\lambda\left[\ln r\right]_2^1$$

Field ∥ $d\vec{r}$

$\vec{E}(r)$: is due to the long inner wire : $\frac{2k\lambda_1}{r}\hat{r}$

$$\Delta V_{21} = -2k\lambda \ln\frac{2\,mm}{10\,mm} = 2k\lambda\ln\left(\frac{10}{2}\right)$$

$$= 2\times9\times10^9\times75\times10^{-9}\ \ln 5$$

$$= 1350\ \ln 5 = 2170\,V$$

b) if $\lambda_2 \rightarrow 150\,\frac{nC}{m}$ → same: $\Delta V = 2170\,V$

(no λ_2 involved in ΔV)