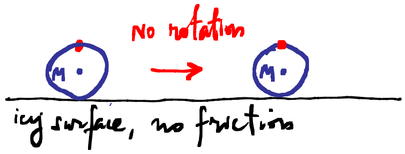
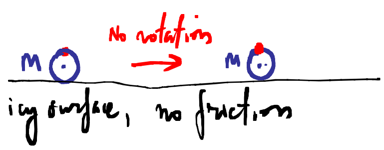
Ch 10 Rotational Motion

Translational motion

• Bowling ball $\boxed{\text{sliding}}$ on icy frictionless surface



Translation: distance travelled doesn't depend on size

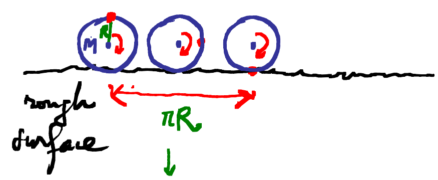


(i) Sliding balls of equal masses but different radii $(R_2 = \frac{R}{2})$ follow same translational motion, since here they can be considered as point-like objects of mass M.. Size doesn't matter

(ii) in this translational motion there is no rotation, orientation of ball (red dot) stays the same

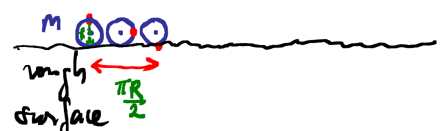
Rotational motion

• Bowling ball $\boxed{\text{rolling}}$ on rough surface



Rolling motion: distance travelled depends on size R

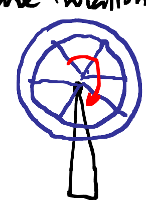
• Bowling ball of half radius: $R_2 = \frac{R}{2}$



(i) In rolling motion – with rotation around its CM, size does matter!

(ii) $\boxed{\text{Rolling}}$ motion
- → Rotation w.r.t. CM (red dot position changes)
- → As a consequence of this rotation: translation of CM (πR: in the diagram above)

(iii) Pure rotation motion without any translation of CM



Examples: 1) Car wheels
- icy road: translational motion
- road under normal condition: rolling motion
- in sands: pure rotational motion

ABS in cars : Antiblocking Braking Systems: allows wheels to rotate as we apply brakes

No ABS

Apply brakes → wheels are blocked
→ car slide forward, no wheel rotation
& wheels

KE can only go into translational motion

ABS

Apply brakes → wheels still rotate in rolling motion to a stop.

KE goes into { Translation + Rotation of 4 wheels } Shorten stopping distance!

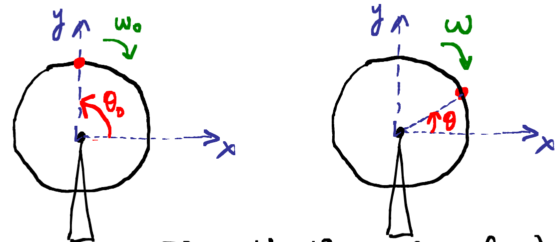
Quantitative Descriptions

Translational motion

- Change of position
- Variables: position x, velocity v, acceleration a

Rotational motion

- Change of orientation or angle
- Variables: angle θ (theta), angular velocity ω (omega), angular acceleration α (alpha)



$\bar{v} = \frac{v_0 + v}{2}$ (linear) — average velocity — $\bar{\omega} = \frac{\omega_0 + \omega}{2}$ (angular)

$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{\Delta t}$ — $\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta - \theta_0}{\Delta t}$

$v = \frac{dx}{dt}$ $\left(\frac{m}{s}\right)$ SI unit — instantaneous velocity — $\omega = \frac{d\theta}{dt}$ $\left(\frac{rad}{s}\right)$ SI unit

$a = \frac{dv}{dt}$ $\left(\frac{m}{s^2}\right)$ — instantaneous acceleration — $\alpha = \frac{d\omega}{dt}$ $\left(\frac{rad}{s^2}\right)$

Equations of motion
(constant acceleration)

1) $v = v_0 + a \cdot t$ — 1) $\omega = \omega_0 + \alpha \cdot t$

2) $x = x_0 + v_0 t + \frac{1}{2} a t^2$ — 2) $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

3) $\frac{v^2 - v_0^2}{x - x_0} = 2a$ — 3) $\frac{\omega^2 - \omega_0^2}{\theta - \theta_0} = 2\alpha$

2nd Newton's Law

$\vec{F}_{net} = \frac{d\vec{p}}{dt}$

$\vec{p} = m\vec{v}$ linear momentum

$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$

$\vec{\tau}$: torque (tau) (includes radius in addition to $\vec{F}$)

$\vec{L} = I\vec{\omega}$ angular momentum (includes radius in addition to $\vec{p}$)

I: moment of inertia (includes radius in addition to mass m)

$\vec{\omega}$: angular velocity vector

m is constant

$\vec{F}_{net} = m\vec{a}$ $\longleftrightarrow$ $\vec{\tau}_{net} = I\vec{\alpha}$

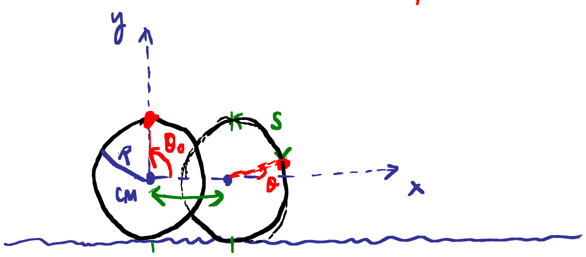
$\vec{\alpha}$ = angular acceleration vector

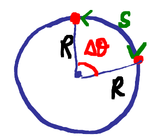
Rotational Motion Topics:

1) Rolling motion (rotation of object simultaneously with translation of its CM)
2) Angular acceleration α
3) Torque $\vec{\tau}$ (includes radius in addition to force $\vec{F}$)
4) Moment of inertia I (includes radius in addition to mass m)
5) KE for rotational motion

1) Rolling motion:

there is a quantitative connection b/w translation & rotation
linear vel. & angular vel.





$\Delta\theta = \theta - \theta_0$

$\Delta\theta = \frac{S}{R}$

i) → Red dot travelled arc s

ii) → If ball was dipped in ink, it would have traced a distance s on surface

iii) → $\Delta\theta = \frac{S}{R}$; $\Delta\theta = \theta - \theta_0$ $\begin{cases} \theta_0: \text{initial angle (1st red dot @ top)} \\ \theta: \text{final angle (2nd red dot)} \end{cases}$

S: distance travelled by CM of ball along x-axis which is a linear motion. → The CM has some linear velocity $= v$

Note: motion of red dot along perimeter of ball, the arc, dictates the motion of the CM

iv) $\frac{d}{dt}\left[\Delta\theta = \frac{S}{R}\right]$

$$\omega = \frac{1}{R}\underbrace{\frac{ds}{dt}}_{\text{linear velocity of CM}} = \frac{1}{R}v \rightarrow$$

Rolling motion

$$\boxed{\omega = \frac{v}{R}}$$

rotation

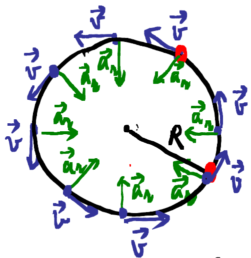
linear motion

$$\underbrace{\frac{\text{rad}}{s}}_{\text{or } s^{-1}} = \frac{\frac{\cancel{m}}{s}}{\cancel{m}}$$

2) Angular acceleration α

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \quad ; \quad \alpha = \frac{d\omega}{dt} \qquad \left(\frac{\text{rad}}{s^2}\right)$$

UCM



→ linear speed v along circle is constant

non-UCM

→ linear speed v along circle is NOT constant → $a_t = \frac{dv}{dt} \neq 0$

$\vec{v}$ { i) same magnitude ~ blue vectors with same length (Uniform C.M.)
ii) direction is tangential → there is a need of a a_r (radial acceleration toward center of curvature)

$\vec{a}$ in UCM $\begin{cases} a_r = \frac{v^2}{R} & (v = |\vec{v}| \text{ or linear speed around circle}) \\ a_t = \frac{dv}{dt} = 0 \end{cases}$

$\vec{a}$ $\begin{cases} a_r = \frac{v^2}{R} & \text{(now with changing values since it depends on changing } v) \\ a_t = \frac{dv}{dt} \neq 0 \end{cases}$

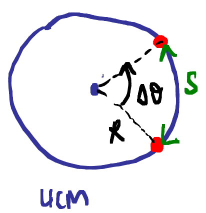
↳ $\alpha = \frac{a_t}{R} \neq 0$

What is α in a UCM?

$$\alpha = \frac{d\omega}{dt} = \frac{d\left(\frac{v}{R}\right)}{dt} = \frac{1}{R}\frac{dv}{dt} = \frac{a_t}{R} = 0 \quad \longleftrightarrow \quad \text{also: } \left[a_t = \frac{dv}{dt} = \frac{d(\omega R)}{dt} = R\,\frac{d\omega}{dt} = R\cdot\alpha\right]$$

$$\boxed{\alpha = \frac{a_t}{R}}$$

$\omega = \frac{v}{R}$: Proof: $\Delta\theta = \frac{s}{R}$



$\frac{d}{dt}\left[\Delta\theta = \frac{s}{R}\right]$

$\omega = \frac{1}{R}\frac{ds}{dt} = \frac{v}{R}$

• Similar result to rolling motion but here v is linear speed along circular trajectory, not of the CM of bowling ball.

3) Torque $\vec{\tau}$ (vector tau)

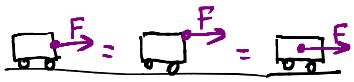
i) Cross product or X between two vectors: $\vec{\tau} = \vec{r} \times \vec{F}$

("tau is a cross product b/w r and F")

ii) Force application point matters → need to define clearly { pivot or center of rotation
↳ need to know where is the force applied: position of force application point with respect to pivot is described by vector $\vec{r}$

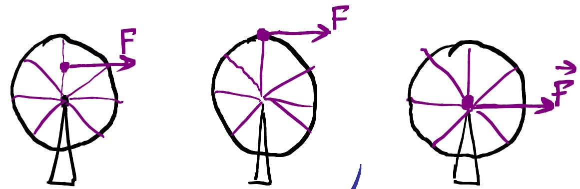
iii) Radial vector $\vec{r}$ (from pivot to force application point)

Linear motion



Force application point doesn't matter here

Rotation



a) 0 b) 13 ✓ c) 1

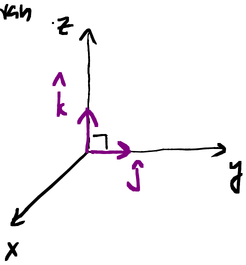
Which force application point gives best speed ω (best α to go from $\omega_0 = 0$ to ω)

iv) Cross-product :
1) b/w two vectors (e.g.: $\vec{r}$ & $\vec{F}$), that produces another vector (e.g.: $\vec{\tau}$)
2) direction of the cross-product vector is perpendicular to both vectors ($\vec{\tau} = \vec{r} \times \vec{F}$ is perpendicular to BOTH $\vec{r}$ & $\vec{F}$), + or − as given by the RHR (right-hand rule)
3) magnitude of the cross-product is the magnitude of 1st vector times magnitude of 2nd vector times $\sin\theta$ where θ is the angle b/w the two vectors :

$$|\vec{\tau}| = \tau = rF\sin\theta$$

v) RHR : directions of cross-products of Cartesian unit vectors :

3D Cartesian

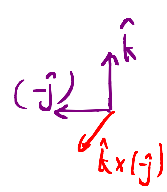


(out of page)

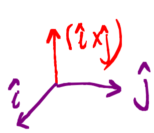
a) $\hat{k} \times \hat{j} =$ { magnitude : $1 \cdot 1 \cdot \sin 90 = 1$; direction : by RHR : { 1) Align RH fingers along 1st vector 2) Close RH fingers towards 2nd vector 3) Thumb indicates direction of the cross-product } $\rightarrow -\hat{i}$ }

$\rightarrow \hat{k} \times \hat{j} = -\hat{i}$

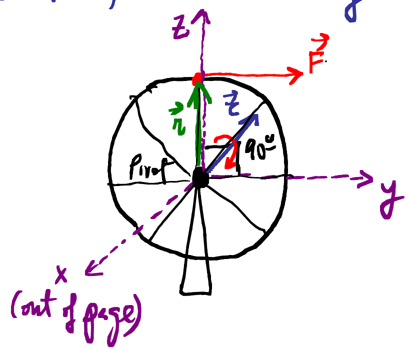
b) $\hat{k} \times (-\hat{j}) = \hat{i}$



c) $\hat{i} \times \hat{j} = \hat{k}$



RHR to find direction of a torque: $\vec{\tau} = \vec{r} \times \vec{F}$

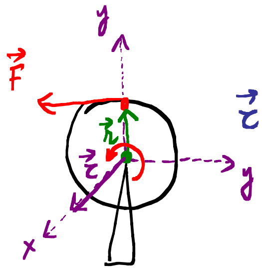


$\vec{r} = r\hat{k}$

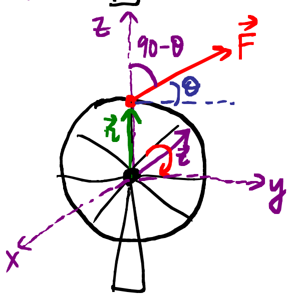
$\vec{F} = F\hat{j}$

$\vec{\tau} = rF \underbrace{\hat{k} \times \hat{j}}_{RHR\ \ -\hat{i}} = rF(-\hat{i})$ (into page)

For a torque that points into page, direction of rotation is also given by a RHR: if thumbs is along direction of torque, right-hand fingers close in direction of rotation. In this example, a torque along negative x means a CW rotation.

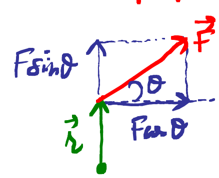


$\vec{\tau} = \vec{r} \times \vec{F} = rF \underbrace{\hat{k} \times (-\hat{j})}_{RHR:\ \hat{i}} = rF\ \hat{i}$ → a torque out of page corresponds to a CCW rotation



$\vec{\tau} = rF \underbrace{\sin(90-\theta)}_{\cos\theta} \underbrace{(-\hat{i})}_{RHR} = -r\boxed{F\cos\theta}\ \hat{i}$ → a torque into page gives a CW rotation

Component of the force applied that is perpendicular to the radial vector $\vec{r}$

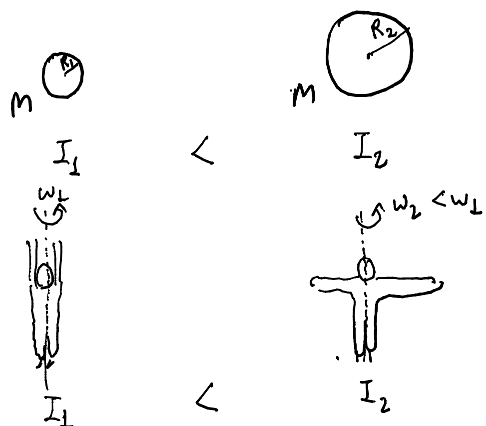


(i) $F\sin\theta$ is $\parallel \vec{r}$ → no torque

(ii) $F\cos\theta$ is $\perp \vec{r}$ → only component that produces torque

If $\vec{F}$ is applied @ angle θ, only part of it ($F\cos\theta$) produces a torque leading to rotational motion.

4) **Moment of inertia I** : (i) Rotational counterpart of mass m
(ii) Includes radius (size does matter)



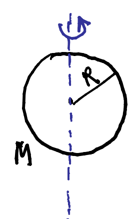
Quantitative formula for I :

- discrete system : $I = \sum_i m_i r_i^2$ { m_i = mass of component i; r_i = position of i w.r.t. axis of rotation
- continuous system $I = \int r^2 \, dm$ { dm : infinitesimal mass; r : position of dm wrt. axis of rotation

Simple geometrical shapes : sphere, cylinder, ring, disk :

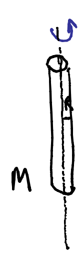
$I = cMR^2$ { M: total mass of object; R: radius of mass distribution; C: coefficient depending on the actual shape

1) Sphere wrt its center axis



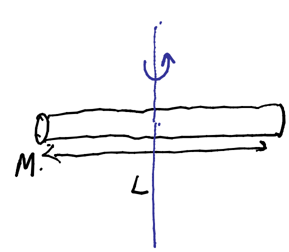
$c = \frac{2}{5} \rightarrow I = \frac{2}{5}MR^2$

2) Vertical cylinder wrt. its center axis



$c = \frac{1}{2} \rightarrow I = \frac{1}{2}MR^2$

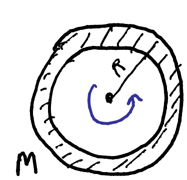
3) Horizontal cylinder wrt. its vertical center axis



$c = \frac{1}{12}$

$I = \frac{1}{12}ML^2$

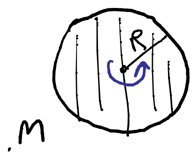
4) Ring wrt. its center axis



$c = 1$

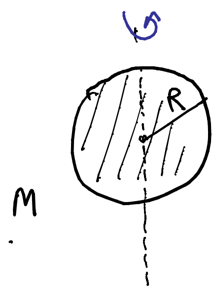
$I = MR^2$

5) Uniform disk wrt. its center axis



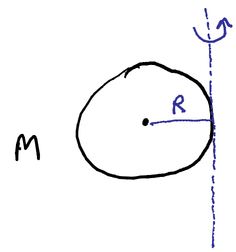
$c = \frac{1}{2} \rightarrow I = \frac{1}{2} M R^2$

6) Sphere wrt. its center axis



$c = \frac{2}{5} \rightarrow I = \frac{2}{5} M R^2$

Sphere wrt its tangential axis



$I_{\text{tangential axis}} = \frac{2}{5} M R^2 + M R^2 = \frac{7}{5} M R^2$

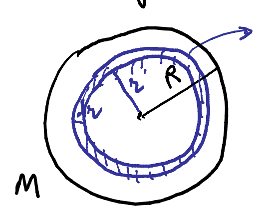
Parallel axis theorem :

$I_{\text{tangential axis}} = I_{\text{center axis}} + M R^2$

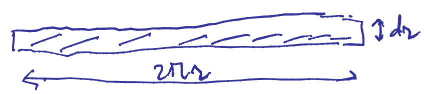
R: separation b/w the two axes

Calculation of moment of inertia for a uniform disk of mass M, radius R, wrt. its center axis :

$I = \int r^2 \, dm$



Tiny ring { @ r from center of disk, thickness dr, mass : dm



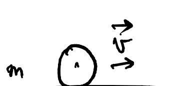
Surface areas { (i) tiny ring of radius r & thickness dr is $2\pi r dr$; (ii) disk of radius R is πR^2 } $\frac{dm}{M} = \frac{2\pi r \, dr}{\pi R^2}$

$I = \frac{2M}{R^2} \int_0^R r^3 \, dr = \frac{2M}{4R^2} \left[r^4 \right]_0^R = \frac{2M}{4R^2} R^{4} = \frac{1}{2} M R^2$

($I = c M R^2$, $c \equiv \frac{1}{2}$ for uniform disk wrt. its center axis)

5) Kinetic energy in rotational motion :

Linear motion



→ disk sliding on frictionless surface

$KE = \frac{1}{2} m v^2$

m: inertia for linear motion

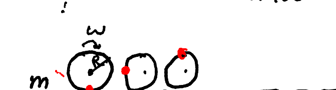
Pure rotation

→ rotating disk of mass m, radius R wrt its center axis (no friction @ pivot)

→ no translation of CM

$KE = \frac{1}{2} I \omega^2$, I : inertia in rotations

Rolling motion { Rotation wrt center axis simultaneous with translation of CM



→ sufficient friction for rolling motion

→ { rotation wrt center axis, $I = \frac{1}{2} m R^2$; translation of CM $v_{CM} = \omega \cdot R$

$KE = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} \left(\frac{1}{2} m R^2 \right) \omega^2$

$= \frac{1}{2} m v_{CM}^2 + \frac{1}{2} \left(\frac{1}{2} m \right) v_{CM}^2 = \frac{1}{2} \left(\frac{3}{2} m \right) v_{CM}^2$

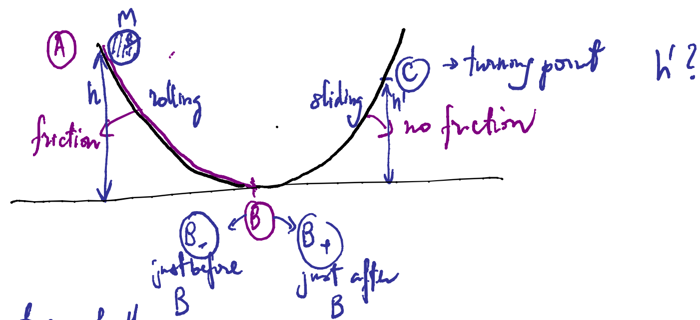
Rolling motion: $KE = \frac{1}{2}\left(\frac{3}{2}m\right)v_{CM}^2$

ABS braking: model each wheel of a car as a disk $\begin{cases} \text{w/o ABS: sliding wheel: } KE = \frac{1}{2}m v_{CM}^2 \\ \text{w/ ABS: rolling wheel: } KE = \frac{1}{2}\left(\frac{3}{2}m\right)v_{CM}^2 \end{cases}$

↓ with rolling motion: each wheel has an effective mass increase of 50% → shorter stopping distance

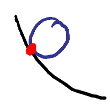
10.64 | sphere rolls down left side of a parabolic potential well from height h, then slides up the right side of well where there is no friction, to what height h'

Step 1:



Focus on description of motion of the sphere: (i) left side: friction: rolling sphere only one contact point with surface

$\underbrace{\text{friction}}$ ↓ just enough to allow rotation wrt center axis in addition to translation of CM of sphere



(ii) $ME_A = ME_{B_-}$ & $ME_{B_+} = ME_C$

$ME_{B_+} = ME_{B_-} - \frac{1}{2}I\omega^2$ (rotational part of KE is on the left side only)

↓

$h' < h$

Step 2:

1) $\underset{(A)}{Mgh} = \underset{(B_-)}{\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2}$

2) $I_{\substack{\text{sphere} \\ \text{wrt} \\ \text{center axis}}} = \frac{2}{5}MR^2$

3) Rolling motion b/w (A) & (B_-)

$v_{CM} = \omega \cdot R$ or $\omega = \frac{v_{CM}}{R}$

4) $\underset{(B_+)}{\frac{1}{2}Mv_{CM}^2} = \underset{(C)}{Mgh'}$ (turning point)

Step 3: Solve for h':

4) $h' = \frac{v_{CM}^2}{2g}$

v_{CM}: plug 2) & 3) into 1) $Mgh = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\cdot\left(\frac{v_{CM}}{R}\right)^2$

$= \frac{1}{2}\left(M + \frac{2}{5}M\right)v_{CM}^2 = \frac{1}{2}\left(\frac{7}{5}M\right)v_{CM}^2$

$Mgh = \frac{1}{2}\left(\frac{7}{5}M\right)v_{CM}^2 \rightarrow \frac{5}{7}h = \frac{v_{CM}^2}{2g}$

4) $\boxed{h' = \frac{5}{7}h}$

Linear Motion	Rotational Motion
2nd Newton's law, $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
$\vec{p} \equiv m\vec{v}$ linear momentum	$\vec{L} \equiv \vec{r} \times \vec{p}$ angular momentum
	$\vec{r}$: position vector w.r.t. axis of rotation
	$\vec{L} = I\vec{\omega}$ angular momentum
m = constant: $\vec{F}_{net} = m\dfrac{d\vec{v}}{dt} = m\vec{a}$	I: constant (mass distribution is not changing wrt time)
	$\vec{\tau}_{net} = I\dfrac{d\vec{\omega}}{dt} = I\vec{\alpha}$

$m \longrightarrow I$

$\vec{p} \longrightarrow \vec{L}$

$\vec{v} \longrightarrow \vec{\omega}$

$\vec{a} \longrightarrow \vec{\alpha}$

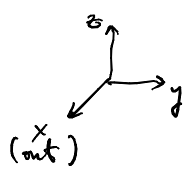
i) So far 2 cross-products in rotational motion $\begin{cases} \vec{\tau} = \vec{r} \times \vec{F} \\ \vec{L} = \vec{r} \times \vec{p} \end{cases}$

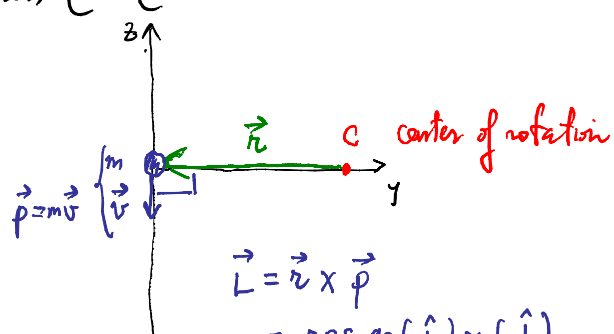
$$\vec{\tau} = \frac{d\vec{L}}{dt} \underset{\substack{\downarrow \\ \vec{r}\text{ constant}}}{=} \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$$

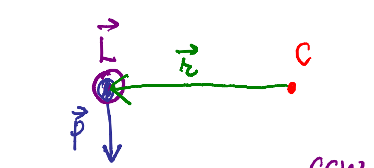
Calculations of angular momentum vector: $\rightarrow$ importance:
(magnitude & direction)

Linear	Rotational
$\vec{F}_{net} = 0 = \dfrac{d\vec{p}}{dt}$	$\vec{\tau}_{net} = 0 = \dfrac{d\vec{L}}{dt}$
$\vec{p}_i = \vec{p}_f$	$\vec{L}_i = \vec{L}_f$

1) Object of mass m moving along $-z$ axis, center of rotation @ C







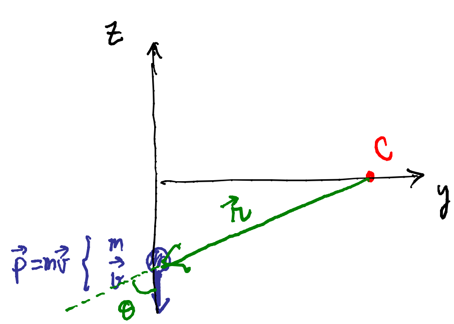
$\vec{L} = \vec{r} \times \vec{p}$

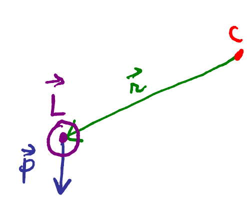
$= rp\sin 90\,\underbrace{(-\hat{j}) \times (-\hat{k})}_{\text{RHR}\ \hat{i}}$

$= rp\,\hat{i}$

The object "rotates" CCW wrt center of rotation C

2)

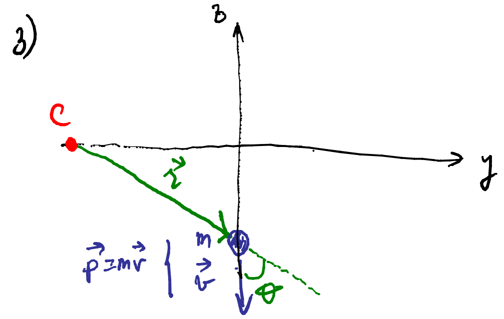




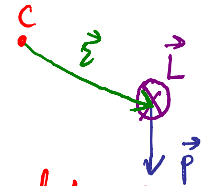
$\vec{L} = \vec{r} \times \vec{p}$

$= rp\sin\theta\ \hat{i}$ (smaller than when object was at origin by factor $\sin\theta < 1$)

3)

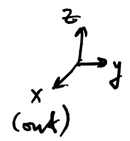


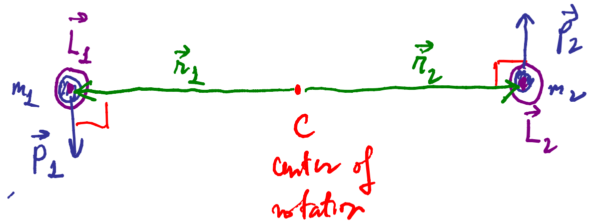
$\vec{L}=\vec{r}\times\vec{p}=rp\sin\theta\ \underbrace{(-\hat{i})}_{RHR}$



Note: $\vec{L}$ depends on selection of center of rotation

4) $\vec{L}_{Total}$ for 2 objects of masses m_1 & m_2 moving in opposite directions wrt. middle center of rotation





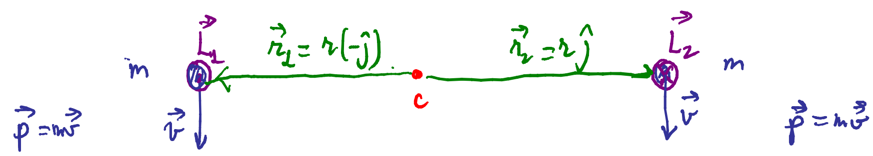
$$\text{Total } \vec{L} = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2 = r_1p_1\underbrace{\hat{i}}_{RHR} + r_2p_2\underbrace{\hat{i}}_{RHR} = (r_1p_1+r_2p_2)\,\hat{i}$$

If $m_1=m_2=m$; $r_1=r_2\equiv r \rightarrow \vec{L}_{Total} = 2rp\,\hat{i}$

$v_1=v_2\equiv v$

$\rightarrow p_1=p_2\equiv p$

5) Similar to 4) but both masses going in same direction



$\vec{L}_{Total} = (rp - rp)\,\hat{i} = 0$ $\begin{cases} m_1 \text{ rotates about C in CCW} \\ m_2 \text{ rotates about C in CC} \end{cases}$

Applications: Conservation of Angular Momentum:

$$\vec{\tau}_{net}=0 \rightarrow \frac{d\vec{L}}{dt}=0 \rightarrow \vec{L}_i = \vec{L}_f$$

Example:

Initial

Disk w/ mouse at outer edge

System rotates wrt. center axis @ ω_1

$L_i = I_m\cdot\omega_i + \frac{1}{2}M_d\cdot R^2\cdot\omega_i$

Final

Mouse walked to axis of rotation

$I_{m_f}=0$

$L_f = \frac{1}{2}M_dR^2\omega_f$

Since there is no net external torque on system of disk & mouse

$\vec{\tau}_{net} = 0 \rightarrow \frac{d\vec{L}}{dt} = 0 \rightarrow L_i = L_f$

$$I_m \cdot \omega_i + \frac{1}{2} M_d \cdot R^2 \cdot \omega_i = \frac{1}{2} M_d R^2 \omega_f$$

→ Can solve for $\omega_f > \omega_i$ (less rotational inertia when mouse stands @ center of rotation)

11.53 Turntable, moment of inertia I, rotating @ ω about vertical axis, no friction w/ axis (78)
Wad of clay of mass m, tossed on, sticks @ d from rotation axis, hits horizontally with velocity $\vec{v}$ $\perp$ radius of turntable, same direction as turntable's rotation.

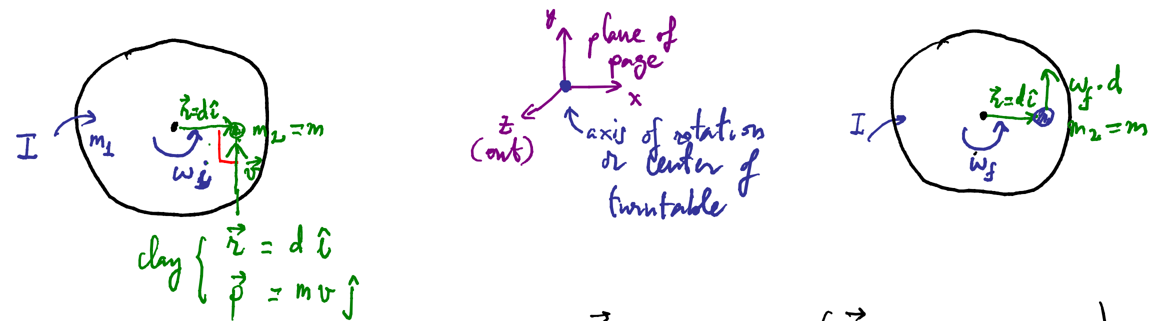
v? a) $\omega_f = \frac{\omega}{2}$
b) $\omega_f = \omega$
c) $\omega_f = 2\omega$

Step 1: Initial $\leftarrow$ Conservation of angular momentum $\rightarrow$ Final

clay about to hit turntable

clay has just landed of turntable

View from above



Step 2:

$$\vec{L}_i = \vec{L}_f \qquad (\vec{\tau}_{net,\,external} = 0)$$

$$\vec{L}_{1_i} + \vec{L}_{2_i} = \vec{L}_{1_f} + \vec{L}_{2_f}$$

$$I\vec{\omega}_i + \vec{r}\times\vec{p} = I\vec{\omega}_f + md^2\,\vec{\omega}_f$$

$$I\omega\hat{k} + dmv\sin 90\,\underbrace{(\hat{i}\times\hat{j})}_{\hat{k}} = (I + md^2)\,\omega_f\hat{k}$$

Note: CCW rotation, by RHR, angular velocity vector points in direction of the thumb when right-hand fingers close in CCW direction $\rightarrow$ $\vec{\omega}_i = \omega\,\hat{k}$

$$\rightarrow I\omega + dmv = (I + md^2)\,\omega_f \rightarrow v = \frac{(I+md^2)\omega_f - I\omega}{m \cdot d}$$

Step 3: a) v? $\omega_f = \frac{\omega}{2}$ $\rightarrow$ $v = \dfrac{\left[(I+md^2)\frac{1}{2} - I\right]\omega}{md} = \dfrac{(md^2 - I)\,\omega}{2md}$

b) v? $\omega_f = \omega$ $\rightarrow$ $v = \dfrac{(I + md^2 - I)\,\omega}{md} = d\cdot\omega$

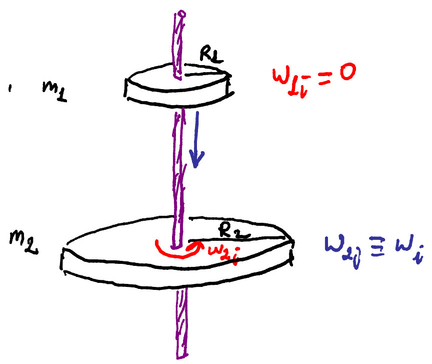
c) v? $\omega_f = 2\omega$ $\rightarrow$ $v = \dfrac{\left[(I+md^2)2 - I\right]\omega}{md} = \dfrac{I + 2md^2}{md}\,\omega$

11.57 | Two disks $\begin{cases} m_1 = 0.27\,kg ; & R_1 = 0.023\,m ; & \omega_{1i} = 0 \\ m_2 = 0.44\,kg ; & R_2 = 0.035\,m ; & \omega_{2i} = 180\,rpm\ \left(\frac{revs}{min}\right) \end{cases}$ } Top (1) disk is dropped onto rotating bottom (2) disk arriving @ common final angular velocity ω_f (79)

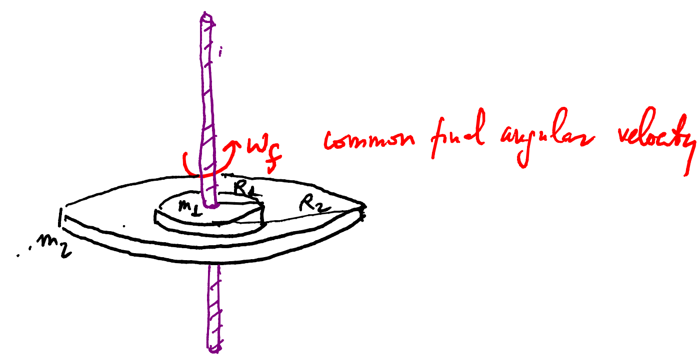
Step 1: $\vec{\tau}_{net,\,external} = 0$
$\downarrow$
$\vec{L}_i = \vec{L}_f$

{ System = disk 1 + disk 2
→ Shelf is external, but frictionless
→ Friction b/w ① & ② is internal!

Initial



Final



Step 2: Eq. of conservation of angular momentum: $(L = I\omega)$ $\left(I_{Disk} = \frac{1}{2} m R^2\right)$

$$\vec{L}_i = \vec{L}_f$$

Same axis of rotation → $L_i = L_f$

$$0 + I_2 \omega_i = (I_1 + I_2)\,\omega_f$$

$$\cancel{\tfrac{1}{2}} m_2 R_2^2 \omega_i = \left(\cancel{\tfrac{1}{2}} m_1 R_1^2 + \cancel{\tfrac{1}{2}} m_2 R_2^2\right) \omega_f$$

Step 3: a) Solve for ω_f:

$$\omega_f = \underbrace{\frac{m_2 R_2^2}{m_1 R_1^2 + m_2 R_2^2}}_{<1} \omega_i$$

$$\omega_f = \frac{0.44 \cdot 0.035^2}{0.27 \cdot 0.023^2 + 0.44 \cdot 0.035^2}\ 180\,rpm = 142\,rpm$$

b) Friction b/w ① & ② is what keep them rotating @ same ω_f: $KE_f = KE_i - KE_{lost}$

$$\text{lost fraction} = \frac{KE_{Lost}}{KE_i} = \frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i} = 1 - \frac{\cancel{\tfrac{1}{2}} I_1 \omega_f^2 + \cancel{\tfrac{1}{2}} I_2 \omega_f^2}{\cancel{\tfrac{1}{2}} I_2 \omega_i^2}$$

$$= 1 - \frac{(I_1 + I_2)}{I_2} \left(\frac{\omega_f}{\omega_i}\right)^2$$

$$= 1 - \frac{\cancel{\tfrac{1}{2}} m_1 R_1^2 + \cancel{\tfrac{1}{2}} m_2 R_2^2}{\cancel{\tfrac{1}{2}} m_2 R_2^2} \left(\frac{\omega_f}{\omega_i}\right)^2 = 1 - \frac{0.27 \cdot 0.023^2 + 0.44 \cdot 0.035^2}{0.44 \cdot 0.035^2} \left(\frac{142}{180}\right)^2 = 0.211$$

$\downarrow$

21.1%

Alternative #2 to calculate the lost fraction: $1 - \frac{KE_f}{KE_i} = 1 - \frac{\cancel{\frac{1}{2}} I_f \omega_f^2}{\cancel{\frac{1}{2}} I_i \omega_i^2}$

By noting from $L_i = L_f$

$I_i \omega_i = I_f \omega_f \rightarrow \frac{I_i}{I_f} = \frac{\omega_f}{\omega_i}$

Total $= I_2$ Total $= I_1 + I_2$

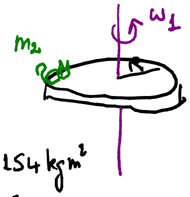
or $\frac{I_f}{I_i} = \frac{\omega_i}{\omega_f} \rightarrow 1 - \frac{\omega_i}{\cancel{\omega_f}} \cdot \frac{\omega_f^{\cancel{2}}}{\omega_i^{\cancel{2}}} = 1 - \frac{\omega_f}{\omega_i}$

$\rightarrow$ lost fraction $= 1 - \frac{\omega_f}{\omega_i} = 1 - \frac{142}{180} = 0.211$ or 21.1%

11.42

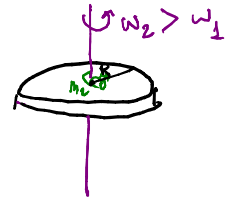
Step 1:

Initial



$$\vec{\tau}_{net,ext} = 0 \;\downarrow\; \vec{L}_i = \vec{L}_f$$

Final



$I_1 = 0.0154\,kgm^2$
$R = 0.25\,m$
$\omega_1 = 22\,rpm$
Rotates freely: no external torque on system
$m_2 = 19.5 \cdot 10^{-3}\,kg$

Step 2: a)

$$\vec{L}_{1_i} + \vec{L}_{2_i} = \vec{L}_{1_f} + \cancel{\vec{L}_{2_f}}^{\,0} \qquad \vec{L} = \begin{cases} I\vec{\omega} \\ \vec{R} \times \vec{p} \end{cases}$$

$$I_1\omega_1 + m_2R^2 \cdot \omega_1 = I_1\omega_2 \rightarrow \omega_2 = \frac{I_1\omega_1 + m_2R^2 \cdot \omega_1}{I_1} = \left(1 + \frac{m_2R^2}{I_1}\right)\omega_1$$

$$\omega_2 = \Bigg(\underbrace{1 + \frac{19.5 \cdot 10^{-3} \cdot 0.25^2}{0.0153}}_{>1}\Bigg) 22\,rpm = 23.74\,rpm > \omega_1$$

b) Work done by the mouse: $KE_f - KE_i = \frac{1}{2}(I_1 + 0)\omega_2^2 - \frac{1}{2}(I_1 + m_2R^2)\omega_1^2$

$$= \frac{1}{2}\,0.0154 \cdot \left(\frac{47.48\pi}{60}\right)^2 - \frac{1}{2}\left(0.0154 + 19.5 \cdot 10^{-3} \cdot 0.25^2\right)\left(\frac{44\pi}{60}\right)^2$$

$$= 0.0476\,J - 0.0441\,J = 0.0035\,J$$
$$= 3.5\,mJ.$$

$$\omega_2 = 23.74 \frac{\cancel{rev}}{\cancel{min}} \frac{2\pi\,rad}{1\,\cancel{rev}} \frac{1\,\cancel{min}}{60\,s} = \frac{47.48\pi}{60} \frac{rad}{s}$$

$$\omega_1 = 22 \frac{\cancel{rev}}{\cancel{min}} \frac{2\pi\,rad}{1\,\cancel{rev}} \frac{1\,\cancel{min}}{60\,s} = \frac{44\pi}{60} \frac{rad}{s}$$

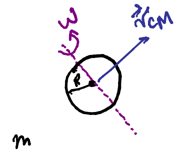
10.36

Flying baseball $\begin{cases} m = 0.15\,kg \\ R = 3.7\cdot 10^{-2}\,m \end{cases}$

linear speed $v_{CM} = 33\,\frac{m}{s}$ angular speed $\omega = 42\,\frac{rad}{s}$

What fraction of its KE comes from rotational motion

Step 1:



$$\frac{KE_{rotation}}{KE_{CM} + KE_{Rotation}}$$

Step 2:

Fraction of rotational KE is $\dfrac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv_{CM}^2 + \frac{1}{2}I\omega^2} = \dfrac{\frac{2}{5}\cancel{m}R^2\omega^2}{\cancel{m}v_{CM}^2 + \frac{2}{5}\cancel{m}R^2\omega^2} = \dfrac{\frac{2}{5}R^2\omega^2}{v_{CM}^2 + \frac{2}{5}R^2\omega^2}$

Baseball: solid sphere w.r.t. center axis: $I = \frac{2}{5}mR^2$

$$= \frac{\frac{2}{5}(3.7\cdot 10^{-2})^2\, 42^2}{33^2 + \frac{2}{5}(3.7\cdot 10^{-2})^2\, 42^2}$$

$= 8.86\cdot 10^{-4}$ or 0.0886%

very small fraction of total KE was rotational in a flying baseball

Flying baseball

$v_{CM} = 33\,\frac{m}{s}$

$\omega = 42\,\frac{rad}{s}$ → linear speed of a point on the outer edge of baseball

$v = \omega\cdot R = 42\cdot 0.0037 = 1.554\,\frac{m}{s}$

much smaller compared to v_{CM}!

Rolling baseball

- if same linear speed as flying baseball

↳ Rolling: $v_{CM} = \omega\cdot R \rightarrow \omega = \frac{v_{CM}}{R} = \frac{33}{0.0037} =$

$= 892\,\frac{rad}{s}$

(this is 21.23 times the rotational speed of the flying baseball!)

- $\dfrac{KE_{Rot}}{KE_{CM} + KE_{Rot}} = \dfrac{\alpha}{1+\alpha}$; $I = \alpha MR^2$ $\begin{cases} \text{sphere}: \alpha = \frac{2}{5} \\ \text{disk}: \alpha = \frac{1}{2} \end{cases}$

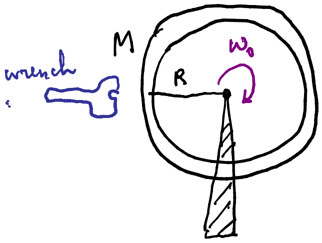
$= \dfrac{\frac{2}{5}}{1 + \frac{2}{5}} = \frac{2}{7} = 0.286 \sim 28.6\%$

↑ sphere

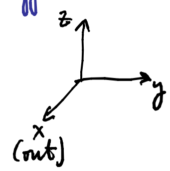
10.58

Step 1:

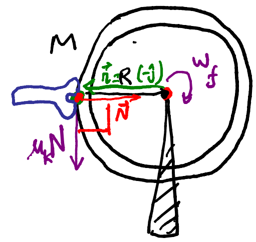
initial



* system = bike wheel has a nonzero net torque applied due to friction b/w wrench & wheel



final (after wrench is applied for $\Delta t = 3.1s$) (84)



$w_0 = 230$ rpm

$M = 1.9$ kg , mostly at rim → ring → $I = MR^2$

$R = 0.33$ m

$\vec{N} = 2.7\,\hat{j}$ (N) during $\Delta t = 3.1$ s

$\mu_k = 0.46$ (b/w wrench & bike wheel)

$\vec{F}_f = \mu_k N(-\hat{k})$

$\vec{r} = R(-\hat{j})$ position vector of force application point for $\left\{ \begin{matrix} \vec{N} \\ \vec{F}_f \end{matrix} \right.$

Torque: $\vec{\tau}_{net} = \underbrace{\vec{r} \times \vec{N}}_{0} + \vec{r} \times \vec{F}_f$

$= RN \underbrace{\sin 180}_{0} \underbrace{((-\hat{j}) \times \hat{j})}_{0} + R\mu_k N \underbrace{\sin 90}_{1} \underbrace{(-\hat{j}) \times (-\hat{k})}_{\hat{i}}$ (by RHR)

$\boxed{\vec{\tau}_{net} = R\mu_k N\,\hat{i}}$ → produces an angular deceleration

Step 2: $\tau_{net} = I\alpha$ Analog of 2nd Newton's Law for rotational motion

$R\mu_k N\,\hat{i} = MR^2 \dfrac{w_f - w_0}{\Delta t}(-\hat{i})$ → scalar equation : $R\mu_k N = \dfrac{MR^2}{\Delta t}(w_0 - w_f)$

RHR : CW rotation means angular velocities point into the page or $(-\hat{i})$

Step 3: solve for w_f : $w_f = w_0 - \underbrace{\dfrac{\cancel{R}\mu_k N \Delta t}{MR^{\cancel{2}}}}_{\frac{rad}{s}}$ → rpm $= 230 \text{ rpm} - \underbrace{\dfrac{0.46 \cdot 2.7 \cdot 3.1}{1.9 \cdot 0.33} \dfrac{\cancel{rad}}{\cancel{s}} \cdot \dfrac{1 rev}{2\pi \cancel{rad}} \cdot \dfrac{60 \cancel{s}}{min}}_{58.6 \text{ rpm}}$

$\boxed{w_f = 171 \text{ rpm}}$