6.38 | long jumper $m = 75\,kg$ $\left\{ \text{rest } v_0 = 0 \xrightarrow{t = 3.1s} \text{prejump speed } v = 10\,\frac{m}{s} \right\}$ Power output?

$$\bar{P} = \frac{\Delta \text{work}}{\Delta t} = \frac{KE_f - KE_i}{\Delta t} \overset{①}{=} \frac{\frac{1}{2}mv^2 - 0}{\Delta t} = \frac{\frac{1}{2}75 \cdot 10^2}{3.1} = 1210\ W$$

① $KE = \frac{1}{2}mv^2$ Why?

Ch.6 : Work $= \begin{cases} \vec{F}\cdot\Delta\vec{r} & \text{when } \vec{F} \text{ constant}, \ \Delta\vec{r}: \text{whole displacement vector} \\ \boxed{\int \vec{F}\cdot d\vec{r}} & \text{when } \vec{F} \text{ is not constant}, \ d\vec{r}: \text{"infinitesimal displacement" vector} \end{cases}$

↓ 2nd Newton's Law : $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$ (mass is constant)

$$\boxed{\text{Work}} = m\int \frac{d\vec{v}}{dt}\cdot \underbrace{d\vec{r}}_{\vec{v}} = m\int \vec{v}\cdot d\vec{v} \underset{1D}{=} m\int v\,dv = \boxed{\frac{1}{2}mv^2}$$

For motion, work is Kinetic energy!

Ch 7. Conservation of Energy :

Motion		Force		Work & Energy
Ch 2&3	→	"Agent that changes motion" Ch 4&5	→	· Definitions (Ch.6) & Conservation of energy (Ch.7)
Kinematic equations constant acceleration		Newton's Law		
1), 2), 3) (no time)		1) Law of inertia 2) $\vec{F} = \frac{d\vec{p}}{dt}$ 3) Law of action & reaction		$ME_i = ME_f$

Force & conservation of energy

↳ { Conservative e.g. gravitational force → Work done by a conservative force is conserved

Non-conservative : eg. friction → Work done by a non-conservative force is not conserved (lost)

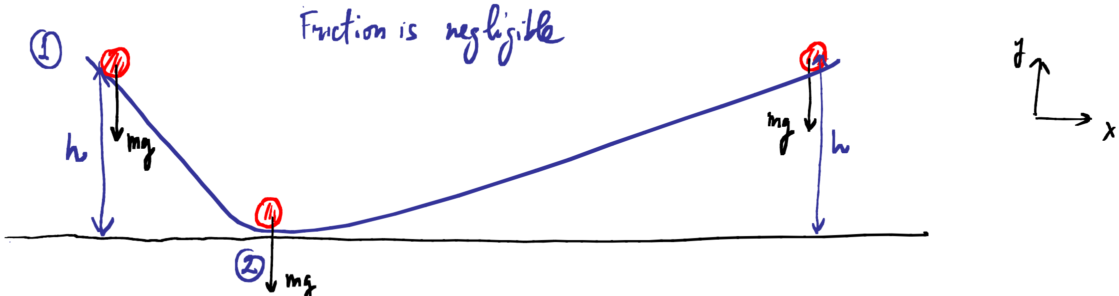
Gravitational force → Conservation of gravitational energy:

Friction is negligible

$h_1 = h$

$h_2 = 0$

$h_3 = h$



Ball after released from 1 will reach 2 and continue to rise up to 3 before turning back, this can be explained with conservation of gravitational potential energy:

(i) Work done by force of gravity:

$$W = \int \vec{F}\cdot d\vec{r} = -mg\hat{j}\cdot \underbrace{\int d\vec{r}}_{\Delta\vec{r}} = -mg\hat{j}\cdot\Delta\vec{r}$$

$$\begin{cases} 1\to 2 \begin{cases} \Delta\vec{r} = (0-h)\hat{j} = -h\hat{j} \\ \boxed{W_{12}} = -mg\hat{j}\cdot(-h\hat{j}) = mgh\,\underbrace{\hat{j}\cdot\hat{j}}_{1} = \boxed{mgh} \end{cases} \\ 2\to 3 \begin{cases} \Delta\vec{r} = (h-0)\hat{j} = h\hat{j} \\ \boxed{W_{23}} = -mg\hat{j}\cdot(h\hat{j}) = \boxed{-mgh} \end{cases} \end{cases}$$

Notes: 1) $\hat{j}\cdot\hat{j} = \underbrace{|\hat{j}|}_{1}\underbrace{|\hat{j}|}_{1}\underbrace{\cos 0}_{1} = 1$

2) We ignored horizontal displacement, since force of gravity only does work in the vertical direction ($W = \vec{F}\cdot\Delta\vec{r}$, $W=0$ if displacement and force vectors are perpendicular $\cos 90° = 0$)

3) From $1\to 2$ force of gravity did work on ball → "Work done $= W_{12} = +mgh$" was positive

From $2\to 3$ force of gravity received work back from ball → "Work done $= W_{23} = -mgh$" was negative

⇒ When ball goes $1\to 2\to 3$ total work done by force of gravity (conservative force) is $W_{12}+W_{23} = mgh - mgh = 0$ or work done by the grav. force is conserved.

4) This means $\begin{cases} 1\to 2 \text{ Ball acquires energy: gaining kinetic energy } \frac{1}{2}mv^2 \\ 2\to 3 \text{ ball returns energy: losing kinetic energy} \end{cases}$ $v_1 = 0$, $v_2 = max$, $v_3 = 0$

v_2: { 1 max. height $h_1 = h$, $v_1 = 0$ — 2 min height $h_2 = 0$, $v_2 = v_{max}$ — 3 max height $h_3 = h$, $v_3 = 0$

grav. potential energy goes to kinetic energy

GPE → KE

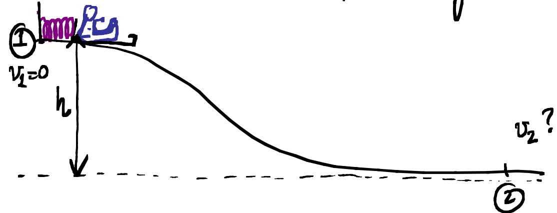
$$\cancel{m}gh = \frac{1}{2}\cancel{m}v_2^2 \to v_2 = \sqrt{2gh} \to v_{max} = \sqrt{2gh}$$

5)

	1	2	3
GPE	mgh	0	mgh
+ KE	0	$\frac{1}{2}mv_2^2$	0
M.E (Mechanical energy)	mgh	$\frac{1}{2}mv_2^2 = mgh$	mgh

→ ME is constant $1\to 2\to 3$
↓
ME is conserved!

Most general M.E. also involves elastic potential energy (EPE)



→ Hill doesn't have a defined angle!
↳ very hard to deal with kinematic & Newton's equations → convenient coord. system varies from point to point!

→ Ignore friction

In this case $ME = GPE + KE + EPE$

GPE : work done by force of gravity $\begin{cases} \vec{F} = -mg\hat{j} \; ; \; d\vec{r} = dy\,\hat{j} \\ W = \int \vec{F}\cdot d\vec{r} = -mg\underbrace{\hat{j}\cdot\hat{j}}_{1} \underbrace{\int_h^0 dy}_{y]_h^0 = -h} = mgh \end{cases}$

KE : work done by motion (2nd Newton's law) $\begin{cases} F = m\frac{dv}{dt} \\ dr = dx \\ W = \int \vec{F}\cdot d\vec{r} = m\int \underbrace{\frac{dv}{dt}dx}_{v} = \frac{1}{2}mv^2 \end{cases}$

EPE : work done by force of spring $\begin{cases} F = -kx \\ dr = dx \\ W = -k\int x\,dx = -\frac{1}{2}kx^2 \end{cases}$

EPE : energy stored by pulling or compressing spring a distance of x (natural length → distance 0) is $\frac{1}{2}kx^2$

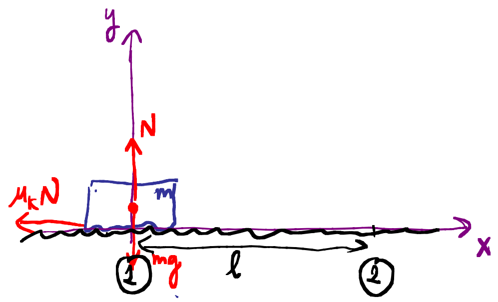
Irregular slope → conservation of ME :

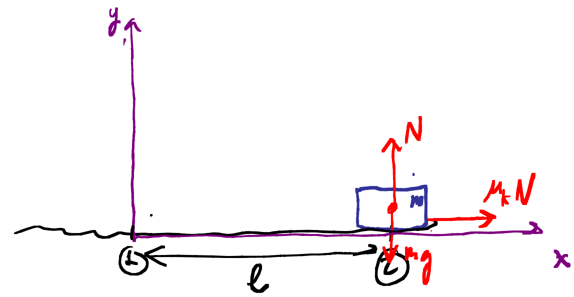
$$ME_{①} = ME_{②}$$

$$mgh + 0 + \frac{1}{2}kx^2 = 0 + \frac{1}{2}mv_2^2 + 0$$

E.g. solve for speed at bottom of hill → $v_2 = \sqrt{\frac{2}{m}\left(mgh + \frac{1}{2}kx^2\right)}$

Work done by force of friction (non-conservative) is not conserved: why?





Pushing a box of mass m from ① to ②

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = -\mu_k N \hat{i}\cdot \underbrace{\int_1^2 d\vec{r}}_{\ell\hat{i}}$$

$$= -\mu_k N \ell$$

Pushing it back from ② to ①

$$W_{21} = \int_2^1 \vec{F}\cdot d\vec{r} = \mu_k N \hat{i}\cdot \underbrace{\int_2^1 d\vec{r}}_{(0-\ell)\hat{i}} = -\mu_k N\ell$$

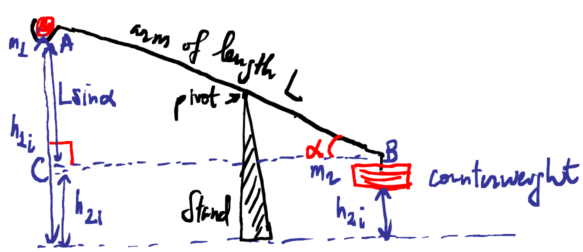
Conclusions:
1) "Work done" by friction is always negative! Friction never does work, it only receives work (negative sign)
2) Total "work done" by friction when box went back to same point is not zero: $W_{12} + W_{21} = -2\mu_k N\ell$ → Work done by force of friction is not conserved, force of friction is not conservative

PP 7.1 (set #3)

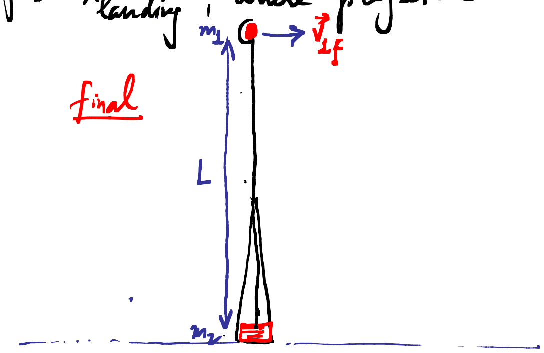
Physics of catapult: {(i) Conservation of energy; (ii) Projectile motion} it uses a counterweight to launch a projectile

Goal: predict w/ a derived formula for $x_{landing}$, where projectile will land.

initial



final



(i) Initial inclination of the arm: α

↳ $h_{1i} = h_{2i} + L\sin\alpha$ (Δ ABC)

(ii) Stand: {(a) Arm pivots about its tip; (b) Also stops the counterweight from swinging to left → to maximize range of projectile. →

why do we need initial & final situations defined? → To apply conservation of energy

Analysis
(i) Use conservation of energy to calculate $\vec{v}_{1f}$ (velocity of projectile when it leaves catapult)
(ii) Using $\vec{v}_{1f}$ we'll calculate range of projectile motion
Simultaneous a) & b)
a) uniform horizontal motion
b) constant acceleration vertical motion

(i) System of projectile (m_1) & counterweight (m_2), ignoring mass of the arm.

$$ME_{initial} \text{ (arm @ angle } \alpha \text{ above horizontal)} = ME_{final} \text{ (arm is vertical)}$$

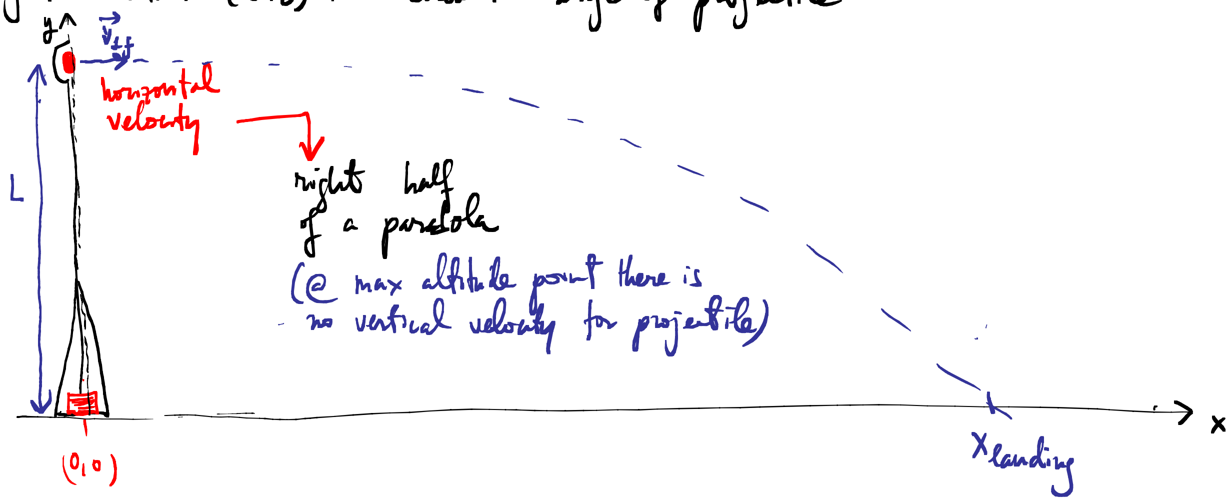
$$ME_{1,i} + ME_{2,i} = ME_{1,f} + ME_{2,f}$$

$$KE_{1,i} + PE_{1,i} + KE_{2,i} + PE_{2,i} = KE_{1,f} + PE_{1,f} + KE_{2,f} + PE_{2,f}$$

$$0 + m_1 g h_{1i} + 0 + m_2 g h_{2i} = \frac{1}{2} m_1 v_{1f}^2 + m_1 g L + 0 + 0$$

$\rightarrow$ solve for $v_{1f} = \sqrt{\frac{2}{m_1}\left(m_1 g h_{1i} + m_2 g h_{2i} - m_1 g L\right)}$

(ii) Projectile motion (Ch 3): calculate range of projectile



projectile motion
Simultaneous a) & b)
a) uniform horizontal motion
b) constant acceleration vertical motion

$\rightarrow$ Motion along perpendicular directions are independent: in time t when projectile drops vertical distance L due to constant acceleration of gravity, it will travel $x_{landing}$ in horizontal uniform motion.

Kin. eq. #2: $y - y_0 = v_{0y} \cdot t + \frac{1}{2} g t^2$

$$L = 0 \cdot t + \frac{1}{2} g t^2 \rightarrow t = \sqrt{\frac{2L}{g}}$$

no vertical velocity @ max. altitude point

Horizontal uniform motion: $x_{Landing} = v_{0x} \cdot t$

$$= v_{1f} \cdot \sqrt{\frac{2L}{g}}$$

$$x_{Landing} = \sqrt{\frac{4L}{m_1 g}\left(m_1 g h_{1_i} + m_2 g h_{2_i} - m_1 g L\right)}$$

$$x_{Landing} = \sqrt{\frac{4L}{m_1 g}\left[m_1 g L \underbrace{(\sin\alpha - 1)}_{\text{negative}} + \underbrace{(m_1 + m_2) g h_{2_i}}_{\text{positive}} \right]}$$

Predict where projectile will land given h_{2_i}

or solve for $h_{2_i} = \dfrac{\frac{x^2_{Landing}\, m_1 g}{4L} + m_1 g L (1 - \sin\alpha)}{(m_1 + m_2) g}$ → predict which h_{2_i} for projectile to hit target @ $x_{Landing}$

Ch 8: Gravity

- Universal Law of Gravitation
- Revised projectile motion
- Escape velocity
- Gravitational potential energy
- Circular orbital motion
- Planetary orbital motion & Kepler's Law

i) Universal Law of Gravitation

$F = mg$ only applies around the surface of the Earth!
At top of World Trade Center in NYC $g < 9.81 \frac{m}{s^2}$

Surface	Universal
$F = mg$	$F = G\frac{M_E m}{r^2}$
GPE $U = mgh$	$U = -G\frac{M_E m}{r}$

Universal: applies Earth, moon, planets, galaxies, universe
↳ essential for space exploration, cellular & GPS satellites, etc...

$$F = G\frac{m_1 m_2}{r^2}$$

- F: force of gravitational attraction b/w two masses m_1 & m_2
- G: universal gravitational constant $G = 6.67 \cdot 10^{-11} \frac{Nm^2}{kg^2}$
- m_1 & m_2: masses (kg)
- r: center-to-center separation b/w m_1 & m_2

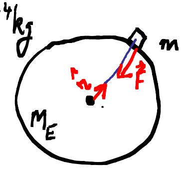
$\frac{1}{r^2} \leftrightarrow$ "inverse-square Law"

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

$\hat{r}$: radial unit vector

Example: use Universal Law of Grav. to calculate force of grav. attraction by Earth on an object of mass m at surface:

$M_E = 5.97 \times 10^{24}$ kg
$R_E = 6.37 \times 10^{6}$ m



(i) Center-to-center separation: $r \simeq R_E$ (height of object is negligible compared to R_E)
(ii) Direction of interaction given by $\hat{r}$
Direction of force of gravitational attraction $\vec{F}$ given by $-\hat{r}$
(iii) Magnitude of grav. attraction is $F = G\frac{M_E m}{R_E^2} = \frac{6.67 \cdot 10^{-11} \cdot 5.97 \cdot 10^{24}}{(6.37 \cdot 10^{6})^2} m$

$9.81344065 \frac{m}{s^2}$

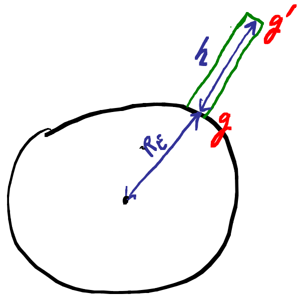
Conclusions: i) Universal Law of Gravitation confirms we can use $F = mg$ at surface!

ii) For objects @ height h above surface $\rightarrow r = R_E + h \rightarrow g' < g$!

8.17 One World Trade Center in NYC : at top : $g' = g - 1.67\cdot 10^{-3}\,\frac{m}{s^2}$ → calculate building height h (54)

$10\,m,\ 100\,m,\ 1000\,m,\ 10\,000\,m$

Step 1: Diagram:



Step 2: equation: ULG: $F = G\frac{M_E}{r^2}\,m$

$$\frac{GM_E}{r^2}\begin{cases} \text{street level: } r = R_E \rightarrow g = \frac{GM_E}{R_E^2} \\ \text{top of building } r = R_E + h \rightarrow g' = \frac{GM_E}{(R_E+h)^2} \end{cases}$$

Step 3: $$\Delta g = 1.67\cdot 10^{-3}\,\frac{m}{s^2} = g - g' \overset{ULG}{=} GM_E\left[\frac{1}{R_E^2} - \frac{1}{(R_E+h)^2}\right] = GM_E\,\frac{(R_E+h)^2 - R_E^2}{R_E^2(R_E+h)^2}$$

$$= \underbrace{\frac{GM_E}{R_E^2}}_{g}\,\frac{2R_Eh + h^2}{(R_E+h)^2} = g\cdot\frac{(2R_E+h)h}{(R_E+h)^2} \simeq g\,\frac{2\cancel{R_E}\cdot h}{R_E^{\cancel{2}\,1}}$$

Reasonable approximation

$R_E + h \simeq R_E$

$2R_E + h \simeq 2R_E$

$$\Delta g = g\,\frac{2h}{R_E} \rightarrow h = \frac{\Delta g\cdot R_E}{g\cdot 2} = \frac{1.67\cdot 10^{-3}\cdot 6.37\cdot 10^{6}}{9.81\cdot 2} = 542\,m$$

Alternative calculation of h:

$$\begin{cases} g = \dfrac{GM_E}{R_E^2} = 9.81344065\ \frac{m}{s^2} & \text{street level} \\ g' = \dfrac{GM_E}{(R_E+h)^2} \rightarrow R_E + h = \sqrt{\dfrac{GM_E}{g'}} & \text{top of building} \end{cases}$$

center-to-center separation: $r = R_E + h$

$$\boxed{h = \sqrt{\frac{GM_E}{g'}} - R_E} \quad (1)$$

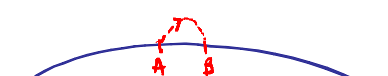
$$g' = g - \Delta g = 9.81344065\ \frac{m}{s^2} - \underbrace{0.00167\ \frac{m}{s^2}}_{1.67\ \frac{mm}{s^2}\ \text{given}} = 9.81177065\ \frac{m}{s^2}$$

$$(1)\quad h = \sqrt{\frac{6.67\cdot 10^{-11}\cdot 5.97\cdot 10^{24}}{9.81177065}} - 6.37\cdot 10^{6} = 6370542.0765 - 6370000 = 542\ m$$

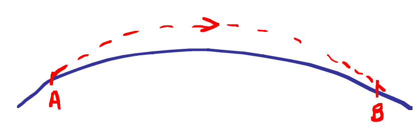
2) Revision of projectile motion: (due to ULG)

$\hookrightarrow$ Ch 3. { Simultaneous i) Uniform horizontal motion
ii) Constant acceleration vertical motion $a = \mp g$

ULG: { projectiles with long range which go higher up above surface
i) value of g changes with altitude
ii) radial gravitational attraction not vertical (specially long range or intercontinental missiles etc...)



i) AB is $\simeq$ flat
ii) Projectile's trajectory b/w A&B is a parabola



i) AB is curved (Earth's curvature)
ii) Projectile's trajectory is more elliptical than parabolic

3) Gravitational Potential Energy:

$$U = -\frac{GMm}{r} \quad \text{extension of } mgh$$

How we derived mgh

Work done by force of gravity: when an object of mass m is lifted from 0 to h is:

$$W = \int_0^h F\cdot dy = -mg\int_0^h dy = -mgh$$

minus sign: work done by gravity is negative since it receives work

$U = -W = mgh$ is the gravitational potential energy stored

Derive U using Univ. Law of Gravitational

Work done by $F = -\frac{GMm}{r^2}$ when an object of mass m is lifted from A to B is

$$W = \int_A^B \vec{F}\cdot d\vec{r} = -GMm\underbrace{\int_A^B \frac{dr}{r^2}}_{\left[-\frac{1}{r}\right]_A^B} = GMm\left[\frac{1}{r}\right]_A^B$$

$$U = -W = -GMm\left[\frac{1}{r}\right]_A^B = -GMm\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

ULG: zero potential is at $r = \infty \rightarrow$ reference for potential

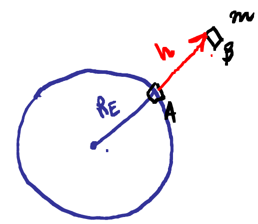
$r_A = \infty$, $r_B = r \rightarrow \boxed{U = -GMm\cdot\frac{1}{r}}$

Notes:

i) if 2 objects are ∞ apart or $r=\infty$, there is no gravitational attraction b/w them $\Rightarrow$ gravitational potential energy is zero $\rightarrow$ reference of potential or zero potential is @ $r=\infty$ $\left(U(r\to\infty) = \lim_{r\to\infty} -\frac{GMm}{r} = 0 \right)$

ii) when 2 objects are closer together, $r \to 0$, there is high gravitational attraction b/w them (ULG is an inverse square law on center-to-center separation) $\rightarrow$ grav. potential is non-zero.

iii) This seems counterintuitive to what we are used to but.



Let's calculate grav. potential energy stored when lifting an object of mass m from surface @ A to a height h @ B using $U = -\frac{GMm}{r}$:

$$\Delta U_{AB} = U_B - U_A = -GMm\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

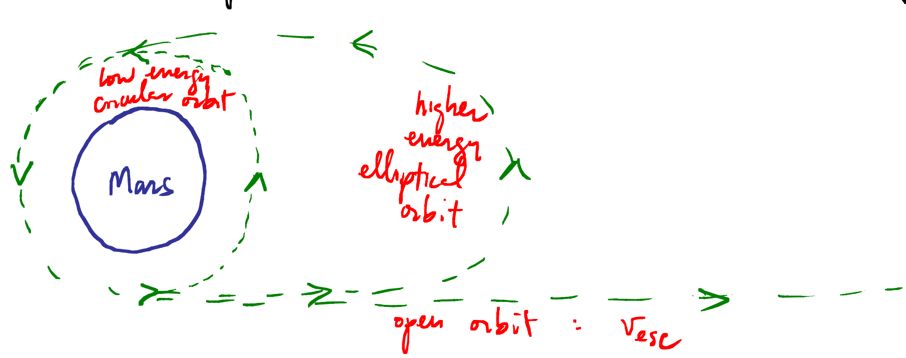
$$= -GMm\left(\frac{1}{R_E+h} - \frac{1}{R_E}\right) = -GMm\,\frac{R_E - (R_E+h)}{(R_E+h)R_E}$$

$$= \frac{GM_E\, m h}{(R_E+h)R_E}$$

For objects around surface ($h \sim 10\,m$): very good approximation is $R_E + h \simeq R_E$

$$\Delta U_{AB} \cong \underbrace{\frac{GM_E}{R_E^2}}_{g} mh = mgh$$

4) Escape velocity: v_{min} to get out of the grav. attraction of a massive object.
Space rocket needs to reach this v_{esc} to go to outer space



→ What is the formula for v_{esc}?

When m is under gravitational attraction of M its total mechanical energy is negative $\frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right) < 0$. When its ME = 0 → mass m gets onto the open orbit → $v = v_{esc}$

$$\frac{1}{2}\cancel{m}v_{esc}^2 - \frac{GM\cancel{m}}{r} = 0 \rightarrow \boxed{v_{esc} = \sqrt{\frac{2GM}{r}}}$$

i) Given M, v_{esc} doesn't depend on m

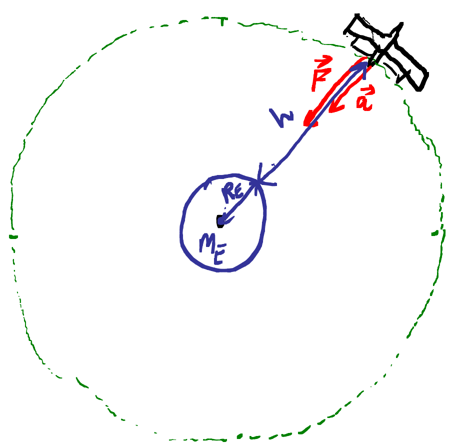
ii) smaller for larger r

v_{esc} smaller for higher orbit

5) Circular Orbital Motion: exploiting grav. potential energy (free)

↳ GPS, cellular satellites

UCM $\begin{cases} \text{i) Centripetal acceleration } a = \frac{v^2}{R} \\ \text{ii) Force of gravitational attraction provides } a: \quad F = ma \end{cases}$



i) Center-to-center separation is $r = R_E + h$ is the orbital radius → orbit's length is $2\pi(R_E+h)$

ii) Orbital period: time to complete one orbit's length: $T = \frac{2\pi(R_E+h)}{v}$

iii) Write v in terms of the gravitational parameters: use 2nd Newton's Law:

$$F_{net} = m\,a$$

$$\frac{GM_E\cancel{m}}{(R_E+h)^2} = \cancel{m}\cdot\frac{v^2}{(R_E+h)} \rightarrow v = \sqrt{\frac{GM_E\cancel{(R_E+h)}}{(R_E+h)^{\cancel{2}\,1}}}$$

$$v = \sqrt{\frac{GM_E}{R_E+h}}$$

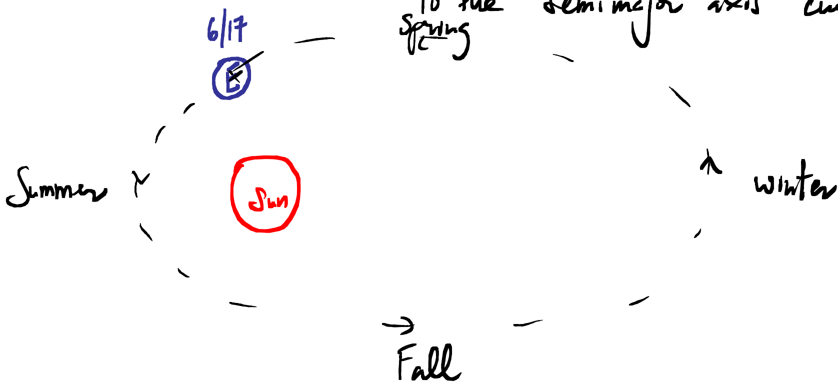
$$T = \frac{2\pi(R_E+h)^{\frac{3}{2}}}{(GM_E)^{\frac{1}{2}}}$$

$$\boxed{T^2 = \frac{4\pi^2(R_E+h)^3}{GM_E}}$$

6) Planetary Orbital Motion (e.g. Earth around the Sun):

$$R_E + h \longrightarrow R \quad \rightarrow \quad T^2 = \frac{4\pi^2}{GM_E}R^3 \quad \text{or} \quad T^2 \propto R^3$$

Kepler's Third Law: (elliptical orbits) → period squared is proportional to the semimajor axis cubed.



Ch9. Systems of Particles

When we describe two or more particles it is very important to locate the center of mass (CM) so to relate to previous equations that applied to a single particle. (CM of a boomerang follows a constant acceleration vertical motion; if added a uniform horizontal motion, its CM follows a projectile motion)

CM: average position of all components of a system, weighted by their masses:

→ described by position vector $\vec{R}$:

Discrete system

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M}$$

m_i = mass of component i
$\vec{r}_i$: position vector of i
$M = \sum_i m_i$
$\sum_i$: sum over all components
$\vec{R}$: "average position of all components weighted by their masses m_i"

Continuous system

$$\vec{R} = \frac{\int \vec{r}\, dm}{M}$$

dm = mass of an infinitessimal component
$\vec{r}$: position vector of that component
$M = \int dm$, total mass of system
$\vec{R}$ = "average position of all infinitessimal components weighted by masses dm"

Once CM is located, we can use $\vec{R}$ in previous equations applied to a single particle

2^{nd} Newton's Law can be applied to $\vec{R}$:

$$\boxed{\vec{F}_{net} = M \frac{d^2\vec{R}}{dt^2}}$$

$\vec{F}_{net}$ = net force on system of particles
→ (i) external
(ii) total internal forces b/w particles cancel by pairs due to 3^{rd} Newton's Law
M : total mass of system
$\vec{R}$: position vector of CM

Important conclusions: (i) Since $\vec{F}_{net} = \vec{F}_{ext}$ motion of CM of a system depends only on net external force on system. No internal interaction b/w components or particles will affect CM's motion.

(ii) Total linear momentum of a system of particles is

$$\vec{P} = M \cdot \vec{V}$$

M: total mass
$\vec{V}$: velocity vector of CM or $\frac{d\vec{R}}{dt}$

$$\vec{P} = M\vec{V} = M\frac{d\vec{R}}{dt} = \cancel{M}\frac{d}{dt}\frac{\sum_i m_i \vec{r}_i}{\cancel{M}} = \sum_i m_i \underbrace{\frac{d\vec{r}_i}{dt}}_{\vec{v}_i} = \sum_i \underbrace{m_i \vec{v}_i}_{\vec{p}_i} = \sum_i \vec{p}_i$$

$\vec{v}_i$: velocity of component i

$\vec{p}_i$: linear momentum of component i

$\Rightarrow \vec{P} = \sum_i \vec{p}_i$

(iii) 2nd Newton's Law of a system of particles: $\boxed{\vec{F}_{net} = \frac{d\vec{P}}{dt}}$ $\left\{ \begin{array}{l} \vec{F}_{net} = \vec{F}_{external} \\ \vec{P} \text{ total linear momentum of system} \end{array} \right.$

in general form

if $\vec{F}_{net} = 0 \rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \boxed{\vec{P} = \text{constant}}$: CONSERVATION OF LINEAR MOMENTUM (important in a system of particle)

↓ collisions

$$\left(\sum_i \vec{p}_i\right)_{initial} = \left(\sum_i \vec{p}_i\right)_{final}$$

Conservation of linear momentum & collisions b/w two particles:

↳ Ball #1 hits ball #2 on a 2D surface, @ collision $\vec{F}_{net,\,external} = 0$

(no net external force on system of two balls while they hit each other)

$\rightarrow \vec{P}_{initial} = \vec{P}_{final}$ or $\vec{P}_{1,initial} + \vec{P}_{2,initial} = \vec{P}_{1,final} + \vec{P}_{2,final}$

(before) (after)

Collisions:

1) Elastic: $\vec{P}$ is conserved, and total KE is conserved

↳ $\vec{P}_{initial} = \vec{P}_{final}$ or $m_1\vec{v}_{1initial} + m_2\vec{v}_{2initial} = m_1\vec{v}_{1final} + m_2\vec{v}_{2final}$

$KE_{initial} = KE_{final}$ or $\frac{1}{2}m_1 v_{1initial}^2 + \frac{1}{2}m_2 v_{2initial}^2 = \frac{1}{2}m_1 v_{1final}^2 + \frac{1}{2}m_2 v_{2final}^2$

Elastic collision in 2D → 2+1 equations

2) Inelastic: only $\vec{P}$ is conserved, total K.E is not conserved.

two colliding particles stick together after collision

↳ $\vec{P}_{initial} = \vec{P}_{final}$ or $m_1\vec{v}_{1initial} + m_2\vec{v}_{2initial} = m_1\vec{v}_{1final} + m_2\vec{v}_{2final}$

$\vec{v}_{1final} = \vec{v}_{2final} = \vec{v}_{final}$

$$m_1\vec{v}_{1initial} + m_2\vec{v}_{2initial} = (m_1+m_2)\vec{v}_{final}$$

Practice w/ problems involving 1D & 2D elastic collisions.

1D elastic collision (b/w 2 particles)

↳ 2 equations
$$\begin{cases} m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} & \text{Conservation of linear momentum} \\ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 & \text{Conservation of kinetic energy} \end{cases}$$

Given m_1, m_2, v_{1i}, v_{2i} → calculate v_{1f}, v_{2f} → solve for a system of 2 equations with 2 unknowns:

1) $$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

2) $$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

3) $$v_{1i} + v_{1f} = v_{2i} + v_{2f}$$

2D elastic collisions (b/w two particles):

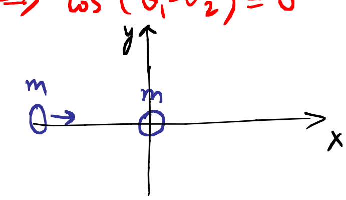
↳ 3 equations
$$\begin{cases} 1)\ m_1 v_{1i_x} + m_2 v_{2i_x} = m_1 v_{1fx} + m_2 v_{2fx} & \text{Conservation of linear momentum in } x \\ 2)\ m_1 v_{1i_y} + m_2 v_{2i_y} = m_1 v_{1fy} + m_2 v_{2fy} & \text{Conservation of linear momentum in } y \\ 3)\ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 & \text{Conservation of kinetic energy} \end{cases}$$

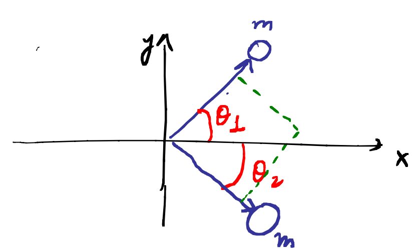
Given $m_1, m_2, v_{1i_x}, v_{1i_y}, v_{2i_x}, v_{2i_y}$ we don't have enough equations to calculate all final velocities: $v_{1fx}, v_{1fy}, v_{2fx}, v_{2fy}$ (4 unknowns) → One additional piece of information is needed, e.g. final angle for one of the two particles, etc...

→ Can derive
$$\begin{cases} 1)\ v_{1i}^2 = v_{1f}^2 + \frac{m_2^2}{m_1^2} v_{2f}^2 + \frac{2m_2}{m_1} v_{1f} v_{2f} \cos(\theta_2 - \theta_1) \\ 2)\ v_{1i}^2 = v_{1f}^2 + \frac{m_2}{m_1} v_{2f}^2 \\ 3)\ 0 = \left(\frac{m_2}{m_1} - 1\right) v_{2f} + 2 v_{1f} \cdot \cos(\theta_1 - \theta_2) \end{cases}$$

Particular case in 2D elastic collisions: $m_1 = m_2 \equiv m$ → 3) $0 = 2 v_{1f} \cos(\theta_1 - \theta_2)$

$\Rightarrow \cos(\theta_1 - \theta_2) = 0 \Rightarrow \theta_1 - \theta_2 = 90°$





Final velocities are perpendicular to each other when $m_1 = m_2$ in 2D elastic collision

An example of 2D elastic collision when $m_1 \neq m_2$

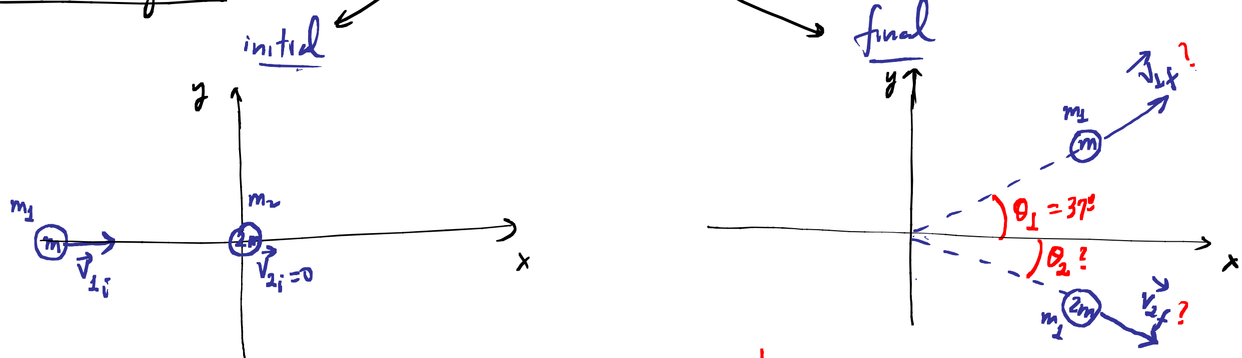
9.73) Proton ($m_1 = 1u$) collides elastically against a deuteron ($m_2 = 2u$) initially @ rest. Proton deflects 37° from its original direction. What fraction of its KE was transferred to the deuteron?

→ We need the KE_{1f} to compare with KE_{1i} → $KE_{2f} = KE_{1i} - KE_{1f}$

Fraction of KE_{1i} that was transferred to deuteron is $\dfrac{KE_{2f}}{KE_{1i}}$

→ We need v_{1f}! → use conservation of linear momentum & kinetic energy

Step 1: Diagram:



unknowns are v_{1f}, v_{2f}, θ_2

Step 2: Equations: 3 conservation equations original or derived ones

$$\begin{cases} 1)\ v_{1i}^2 = v_{1f}^2 + \dfrac{m_2^2}{m_1^2} v_{2f}^2 + \dfrac{2m_2}{m_1} v_{1f} v_{2f} \cos(\theta_2 - \theta_1) \\ 2)\ v_{1i}^2 = v_{1f}^2 + \dfrac{m_2}{m_1} v_{2f}^2 \\ 3)\ 0 = \left(\dfrac{m_2}{m_1} - 1\right) v_{2f} + 2v_{1f} \cdot \cos(\theta_1 - \theta_2) \end{cases}$$

For this problem:

$$\begin{cases} 1)\ v_{1i}^2 = v_{1f}^2 + 4 v_{2f}^2 + 4 v_{1f} v_{2f} \cos(\theta_2 - 37°) \\ 2)\ v_{1i}^2 = v_{1f}^2 + 2 v_{2f}^2 \\ 3)\ 0 = v_{2f} + 2v_{1f} \cdot \cos(\theta_2 - 37°) \end{cases}$$ (cos is an even function)

Also: $\boxed{4)\ v_{1i} \cos 37° = v_{1f} + 2 v_{2f} \cos(\theta_2 - 37°)}$

Proof: Conservation of linear momentum: $\begin{cases} i)\ P_{ix} = P_{fx} \\ ii)\ P_{iy} = P_{fy} \end{cases}$

(i) $\cancel{m} v_{1i} = \cancel{m} v_{1f} \cos 37° + 2\cancel{m} v_{2f} \cos\theta_2 \xrightarrow{\cdot \cos 37°} v_{1i} \cos 37° = v_{1f} \cos^2 37° + 2v_{2f} \cos\theta_2 \cos 37°$

(ii) $0 = v_{1f} \sin 37° - 2v_{2f} \sin\theta_2 \xrightarrow{\cdot \sin 37°} 0 = v_{1f} \sin^2 37° - 2v_{2f} \sin\theta_2 \sin 37°$

Adding (i)+(ii): $v_{1i} \cos 37° = v_{1f} + 2v_{2f} \underbrace{(\cos\theta_2 \cos 37° - \sin\theta_2 \sin 37°)}_{\cos(\theta_2 - 37°)}$ (4)

$v_{2f} \cdot$ Eq 3) & use Eq 4) : $2v_{2f}\cos(\theta_2 - 37^\circ) = v_{1i}\cos 37^\circ - v_{1f}$

$$3)\quad 0 = v_{2f}^2 + \underbrace{2v_{1f}\cdot\cos(\theta_2-37^\circ)\,v_{2f}}_{v_{1f}(v_{1i}\cos 37^\circ - v_{1f})}$$

$$0 = v_{2f}^2 + v_{1f}v_{1i}\cos 37^\circ - v_{1f}^2 \qquad (5)$$

$$\text{Eq 2)}: v_{1i}^2 = v_{1f}^2 + 2v_{2f}^2 \Rightarrow v_{2f}^2 = \frac{1}{2}(v_{1i}^2 - v_{1f}^2) \qquad (6)$$

Use (6) in (5) :

$$2\cdot\left[0 = \frac{1}{2}(v_{1i}^2 - v_{1f}^2) + v_{1f}v_{1i}\cos 37^\circ - v_{1f}^2\right]$$

$$0 = v_{1i}^2 - v_{1f}^2 + 2v_{1f}v_{1i}\cos 37^\circ - 2v_{1f}^2$$

$$0 = -3v_{1f}^2 + v_{1i}^2 + (2v_{1i}\cos 37^\circ)\cdot v_{1f}$$

$$\boxed{\underbrace{3}_{a}v_{1f}^2 \underbrace{-(2v_{1i}\cos 37^\circ)}_{b}v_{1f} \underbrace{-v_{1i}^2}_{c} = 0}$$ Quadratic polynomial in v_{1f}

$$\rightarrow v_{1f} = \frac{2v_{1i}\cos 37^\circ \pm\sqrt{4v_{1i}^2\cos^2 37 + 12v_{1i}^2}}{6}$$

$$v_{1f} = v_{1i}\left[\frac{\cos 37^\circ}{3} \pm \underbrace{\frac{\sqrt{\cos^2 37^\circ + 3}}{3}}_{0.6357}\right]$$

$-$ eliminated as [ ] needs to be + (v_{1f} & v_{2f} are magnitudes or lengths of final velocity vectors)

$$\boxed{v_{1f} = v_{1i}\, 0.902}$$

To get v_{2f} use eq 2) : $v_{2f}^2 = \frac{1}{2}(v_{1i}^2 - v_{1f}^2) = \frac{v_{1i}^2}{2}\underbrace{\left(1 - \frac{v_{1f}^2}{v_{1i}^2}\right)}_{1-0.902^2 = 0.1864}$

$$v_{2f}^2 = 0.0932\, v_{1i}^2$$

$$\boxed{v_{2f} = v_{1i}\, 0.305}$$

To get θ_2 use eq 3) : $0 = v_{2f} + 2v_{1f}\cdot\cos(\theta_2 - 37^\circ)$

$$\rightarrow \cos(\theta_2 - 37^\circ) = \frac{-v_{2f}}{2v_{1f}} = -\frac{\cancel{v_{1i}}\,0.305}{2\,\cancel{v_{1i}}\,0.902} = -\frac{0.305}{1.804}$$

$$\rightarrow \theta_2 - 37^\circ = \cos^{-1}\left(-\frac{0.305}{1.804}\right) = 99.73^\circ \rightarrow \boxed{\theta_2 = 62.73^\circ}$$ (cw below x-axis)

Fraction of KE of proton transferred to deuteron: was 18.6 %

$$\frac{KE_{2f}}{KE_{1i}} = \frac{KE_{1i} - KE_{1f}}{KE_{1i}} = 1 - \frac{KE_{1f}}{KE_{1i}} = 1 - \frac{\cancel{\frac{1}{2}m_1}\, v_{1f}^2}{\cancel{\frac{1}{2}m_1}\, v_{1i}^2} = 1 - \frac{\cancel{v_{1i}^2}\, 0.902^2}{\cancel{v_{1i}^2}} = 0.186$$

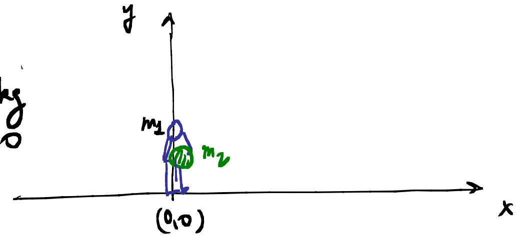
9.59

Step 1: Diagram:

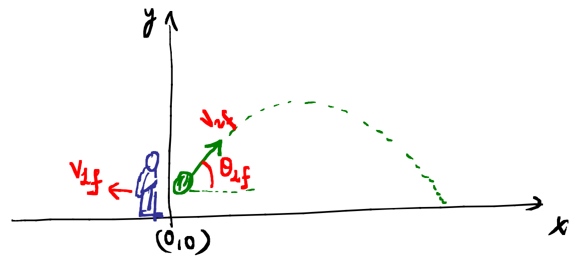
no friction → $\vec{F}_{net,\,ext} = 0$ $\Rightarrow \vec{P}_i = \vec{P}_f$

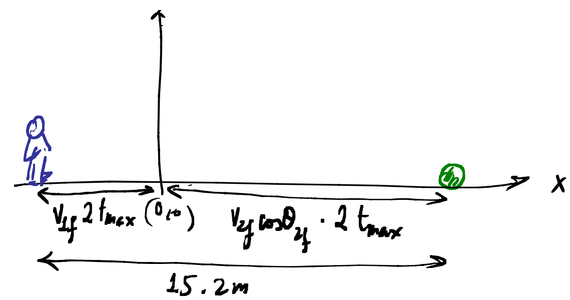
$m_1 = 65\,kg$
$m_2 = 4.5\,kg$
$\vec{v}_{1i} = \vec{v}_{2i} = 0$
$v_{2f} = 12\,\frac{m}{s}$

initial



final





Step 2: Equations:

Conservation of linear momentum in x:

$$0 = m_1 v_{1f} + m_2 v_{2f} \cos\theta_{2f}$$

$$\Rightarrow v_{1f} = -\frac{m_2}{m_1} v_{2f} \cos\theta_{2f} = -\frac{4.5}{65}\, 12 \cos\theta_{2f} = -0.83 \cos\theta_{2f}$$?

Projectile motion: t_{max}: time for rock to get to max. altitude point $\begin{cases} v_y = 0 \\ v_x = v_{2f}\cos\theta_{2f} \end{cases}$

$$v_y = v_{0y} - g \cdot t$$

$$0 = v_{2f}\sin\theta_{2f} - g \cdot t_{max} \rightarrow t_{max} = \frac{v_{2f}\sin\theta_{2f}}{g}$$

$$15.2\,m = 2 \cdot t_{max}\left(v_{2f}\cos\theta_{2f} - v_{1f}\right) = 2 \cdot t_{max}\left(v_{2f}\cos\theta_{2f} + 0.83\cos\theta_{2f}\right)$$

$$= 2 \cdot t_{max}\left(v_{2f} + 0.83\right)\cos\theta_{2f} = \frac{1}{g} v_{2f}\left(v_{2f} + 0.83\right) \underbrace{2\sin\theta_{2f}\cos\theta_{2f}}_{\sin(2\theta_{2f})}$$

$$15.2 = \frac{12\,(12 + 0.83)}{9.81} \sin(2\theta_{2f}) \rightarrow 2\theta_{2f} = \sin^{-1}\frac{15.2 \cdot 9.81}{12 \cdot 12.83} \rightarrow \boxed{\theta_{2f} = \frac{75.58}{2} = 37.8^\circ}$$

9.76 O_2 $m_1 = 32u$ $+x$ $v_{1i} = 580\,\frac{m}{s}$ collides with O $(m_2 = 16u)$ $\vec{v}_{2i} = \left(870\,\frac{m}{s},\, 27^\circ\right)$

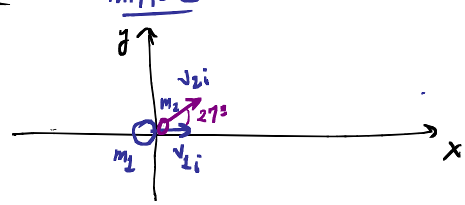
They stick together after collision (inelastic collision) to form O_3 or ozone. Find ozone's velocity:

$\vec{F}_{net,\,ext} = 0 \rightarrow \vec{P}_i = \vec{P}_f$

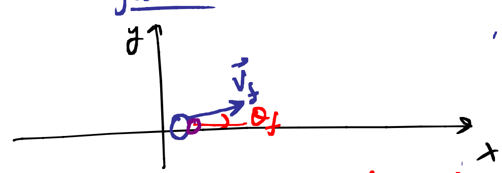
Step 1: Diagram:

$m_1 = 32\,u$
$m_2 = 16\,u$

initial



final



They stick together after collision

Step 2 Equations : conservation of linear momentum

$$\vec{P}_i = \vec{P}_f$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$2\vec{v}_{1i} + \vec{v}_{2i} = 3\vec{v}_f$$

in x : $2 v_{1i} + v_{2i} \cos 27° = 3 v_f \cos \theta_f$

in y : $0 + v_{2i} \sin 27° = 3 v_f \sin \theta_f$

system of 2 equations w/ 2 unknowns

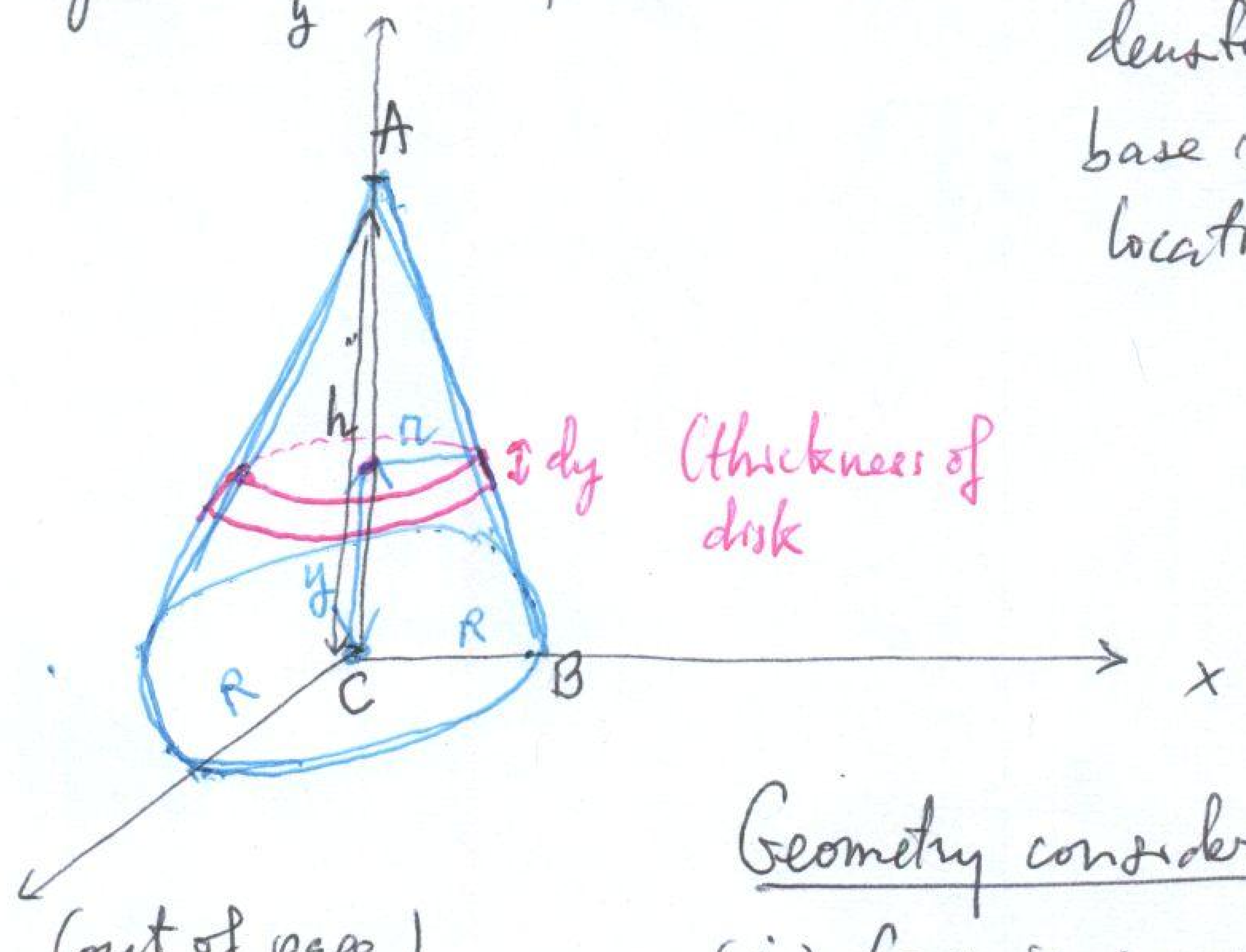
$$\frac{2 \cdot 580 + 870 \cdot \cos 27°}{870 \sin 27°} = \frac{1}{\tan \theta_f} \rightarrow \boxed{\theta_f} = \tan^{-1} \frac{870 \sin 27°}{2 \cdot 580 + 870 \cos 27°} = \boxed{11.53°}$$

$$\rightarrow \boxed{v_f} = \frac{v_{2i} \sin 27°}{3 \sin \theta_f} = \frac{870 \cdot \sin 27°}{3 \cdot \sin 11.53°} = \boxed{658.29 \ \frac{m}{s}}$$

9.43)

Step 1: Diagram with information:

solid cone, uniform mass density ρ; height h, base radius R → find location of CM.



Geometry considerations:

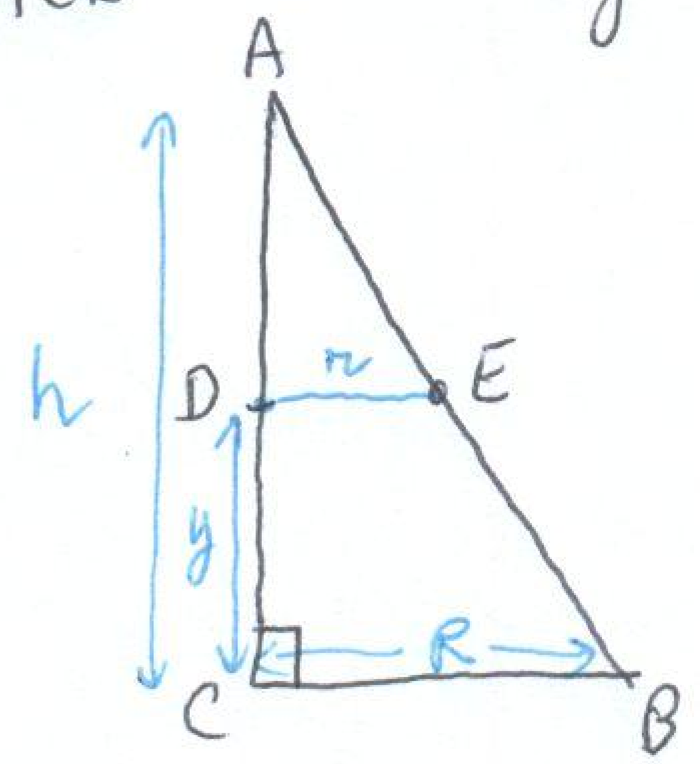
(i) Cone is symmetric wrt y-axis

$$\Rightarrow \vec{R} = y_{cm}\,\hat{j} \Rightarrow y_{cm} = \frac{\int y\,dm}{M}$$

$$\vec{R} \equiv \frac{\int \vec{r}\,dm}{M}$$

(3D problem but 1D calculation!)

(ii) $\overset{\triangle}{ACB}$ is a right triangle:



Similar triangles: $\overset{\triangle}{ACB}$ & $\overset{\triangle}{ADE}$

$$\hookrightarrow \frac{r}{h-y} = \frac{R}{h}$$

$$r = \frac{R}{h}(h-y)$$

$$\boxed{r = R\left(1 - \frac{y}{h}\right)}$$

Step 2: Relevant equations:

1) Def of CM: $y_{cm} = \dfrac{\int y\, dm}{M}$

2) Density $\rho = \dfrac{dm}{d\,vol} \Rightarrow dm = \rho \underbrace{d\,vol}$

infinitesimal volume of disk of radius r & thickness dy

3) Volume of a disk or cylinder: base area × height (thickness)

$\hookrightarrow d\,vol = \pi r^2 dy$

$$\Rightarrow dm = \rho \pi r^2 dy = \rho \pi R^2 \left(1 - \frac{y}{h}\right)^2 dy$$

$\boxed{\text{4) volume of a cone of height } h\text{, base radius } R \text{ is } \dfrac{\pi R^2 h}{3}}$

Step 3: solve for y_{cm}:

$$\boxed{y_{cm}} = \frac{1}{M}\int_0^h \rho \pi R^2 y \left(1 - \frac{y}{h}\right)^2 dy = \frac{\rho \pi R^2}{M}\int_0^h \left(y - \frac{2y^2}{h} + \frac{y^3}{h^2}\right) dy$$

$$= \frac{\rho \pi R^2}{M}\left[\frac{y^2}{2} - \frac{2}{3h}y^3 + \frac{1}{4h^2}y^4\right]_0^h = \frac{3}{h}\left[\frac{h^2}{2} - \frac{2}{3}h^2 + \frac{h^2}{4}\right]$$

$$= \frac{3}{2}h - 2h + \frac{3}{4}h$$

$$= \frac{6h - 8h + 3h}{4} = \boxed{\frac{h}{4}}$$

$$\rho = \frac{M}{V_{cone}} = \frac{3M}{\pi R^2 h}$$

$$\frac{\rho \pi R^2}{M} = \frac{3\cancel{M}}{\cancel{\pi R^2} h}\,\frac{\cancel{\pi R^2}}{\cancel{M}} = \frac{3}{h}$$

$$\Rightarrow \boxed{y_{cm} = \frac{h}{4}}$$

8.32

$v_{esc} = 30 \frac{km}{s}$

$\downarrow$

what is R_E' ?

For Earth $\left\{ \begin{array}{l} M_E = 5.97 \cdot 10^{24} kg \\ R_E = 6.37 \cdot 10^{6} m \end{array} \right\} v_{esc} = 11.2 \frac{km}{s}$

To escape grav. attraction $\rightarrow$ $ME = \frac{1}{2} m v_{esc}^2 - \frac{G M_E m}{R_E} = 0$

$$\Rightarrow R_E' = \frac{2 G M_E \cancel{m}}{\cancel{m} v_{esc}^2} = \frac{2 \cdot 6.67 \cdot 10^{-11} \cdot 5.97 \cdot 10^{24}}{(30 \cdot 10^3)^2} m$$

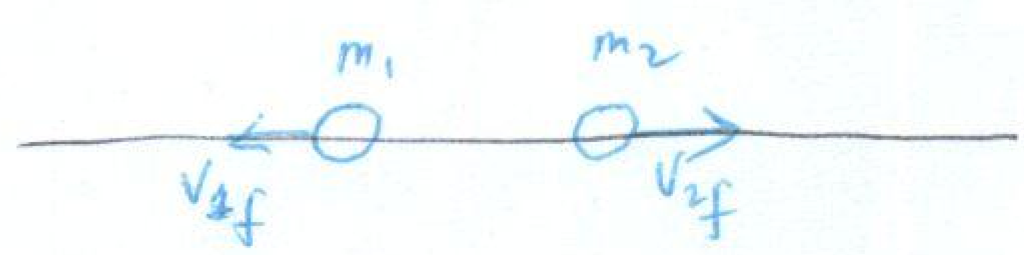
$$= 8.85 \cdot \frac{10^{13}}{10^8} = 0.885 \cdot 10^6 \, m$$

9.35

Step 1:

initial

final (after collision)



$m_1 = 1u$
$m_2 = 1u$ $\}$ $m_1 = m_2 = m$

$v_{1i} = 6.9 \cdot 10^6 \frac{m}{s}$

$v_{2i} = -11 \cdot 10^6 \frac{m}{s}$

v_{1f} ?
v_{2f} ?

1D elastic collision

$\left\{ \begin{array}{l} P_i = P_f \\ KE_i = KE_f \end{array} \right\} \rightarrow$ can solve for 2 unknowns.

Step 2: Relevant equations: 1D elastic collision:

1) $\boxed{v_{1f}} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2 m_2}{m_1 + m_2} v_{2i}$ $\;\underset{m_1 = m_2}{=}\; v_{2i}$

2) $\boxed{v_{2f}} = \frac{2 m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$ $\;\underset{m_1 = m_2}{=}\; v_{1i}$

When masses are equal the two colliding particles exchange their velocities

Step 3: $v_{1f} = v_{2i} = -11 \cdot 10^6 \frac{m}{s}$; $v_{2f} = v_{1i} = 6.9 \cdot 10^6 \frac{m}{s}$

9.47

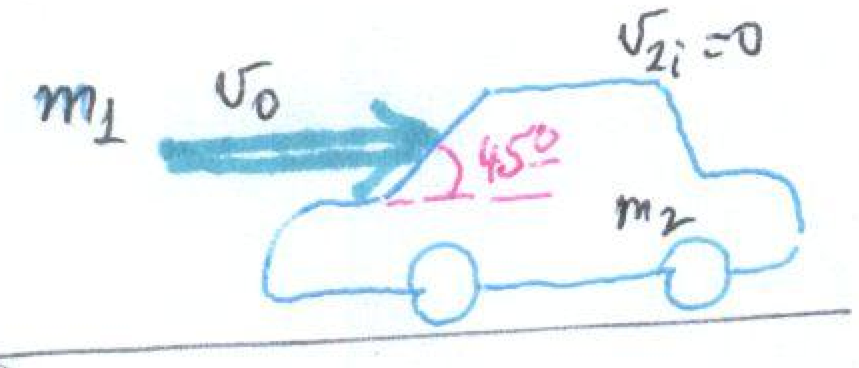
<u>Step 1:</u> Diagram with info :

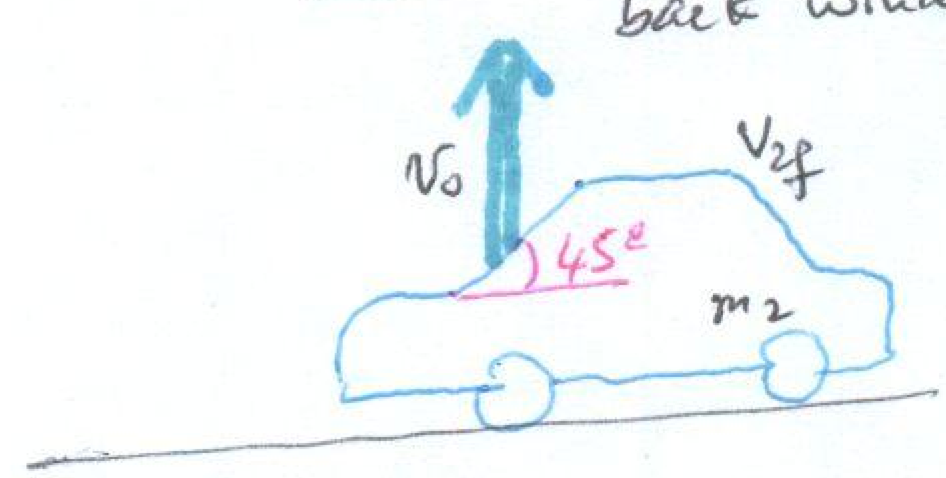
→ car initially at rest, received a push by a jet of water hitting its back window horizontally (leaving vertically), car acquired an acceleration a?

→ no friction $F_{net} = 0$ (no net external force on system of car & water)

<u>initial</u>

<u>final</u> (after water collided with back window)





m_1 : water
$\vec{v}_{1i} = v_0 \hat{i}$
m_2 = car
$v_{2i} = 0$ (at rest)

$\vec{F}_{net} = 0$
↓
$\vec{P}_i = \vec{P}_f$

$\vec{v}_{1f} = v_0 \hat{j}$
$\vec{v}_{2f}$ car after collision.

<u>Step 2:</u>

$$\vec{P}_i = \vec{P}_f$$

$$m_1 \vec{v}_{1i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

<u>Step 3:</u> Solve for $\vec{v}_{2f} = \dfrac{m_1 \vec{v}_{1i} - m_1 \vec{v}_{1f}}{m_2} = \left(\dfrac{1}{m_2} \vec{v}_{1i} - \dfrac{1}{m_2} \vec{v}_{1f} \right) m_1$

$$\vec{v}_{2f} = \frac{v_0}{m_2} (\hat{i} - \hat{j}) m_1$$

Acceleration $\vec{a}$ acquired by car: $\vec{a} = \dfrac{d\vec{v}_{2f}}{dt} = \dfrac{v_0}{m_2} (\hat{i} - \hat{j}) \dfrac{dm_1}{dt}$

⇓

$v_0, m_2, \hat{i}, \hat{j}$ are time independent!

→ a) Forward acceleration of car: $\boxed{a_x = \dfrac{v_0}{m_2} \dfrac{dm_1}{dt}}$

(downward acceleration is felt by car's suspension : $a_y = -\dfrac{v_0}{m_2} \dfrac{dm_1}{dt}$)

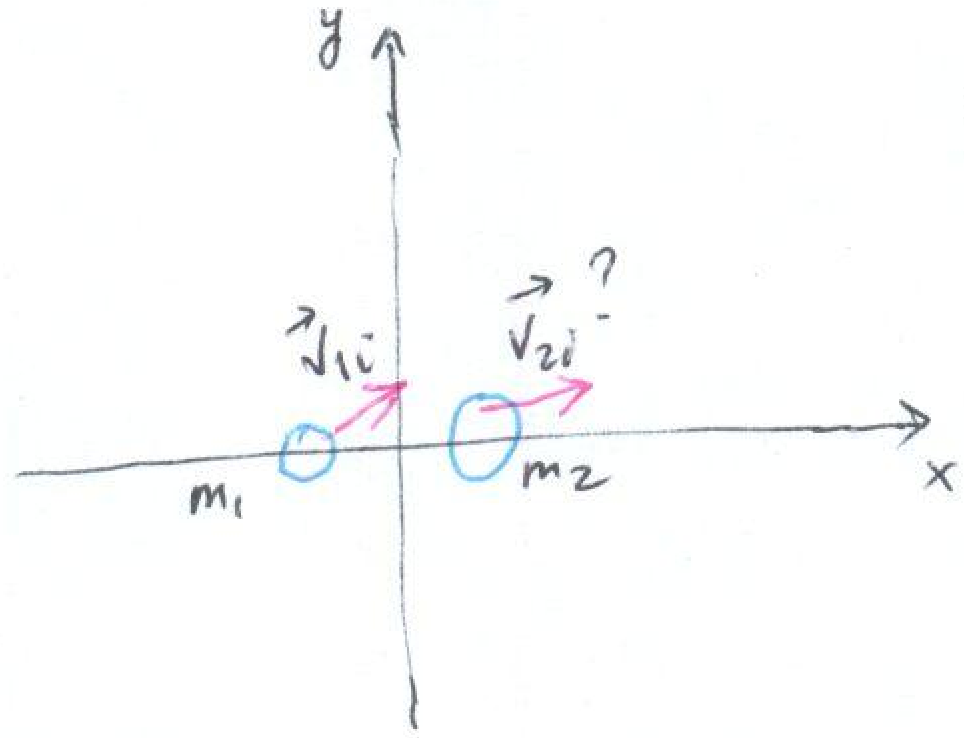
b) Max speed car can reach? ⇒ speed of water v_0

Once car reaches speed of v_0 in x-direction water can no longer provide a push for any further acceleration

9.28

Step 1:

Initial



15.1

22.6

$m_1 = 1u$

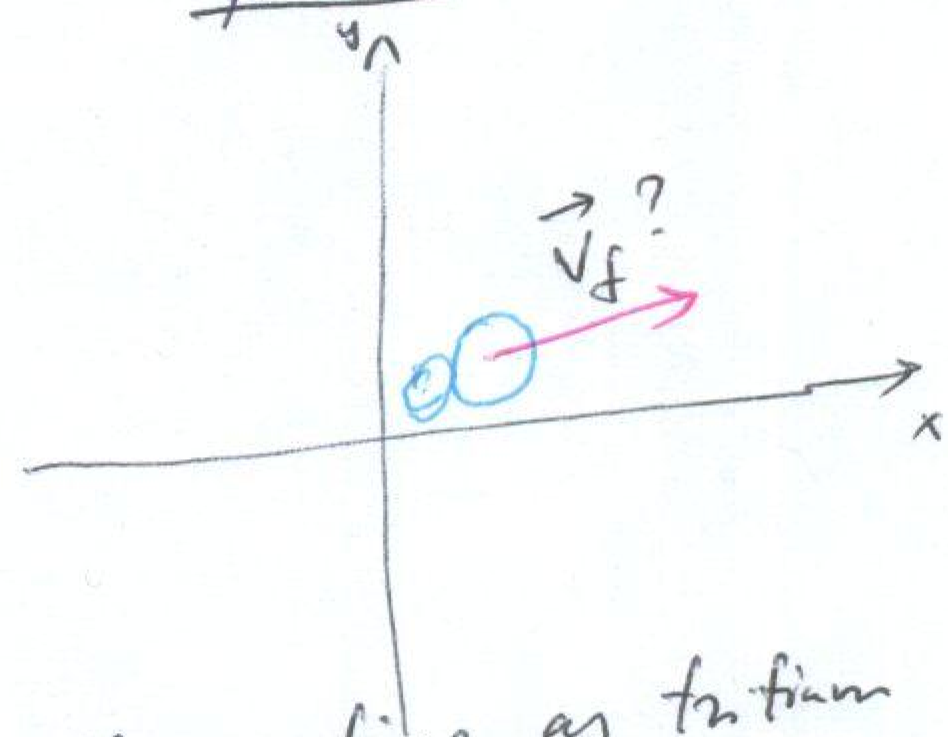
$m_2 = 2u$

$\vec{v}_{1i} = 23.5\hat{i} + 14.4\hat{j}\ (\cdot 10^6 \frac{m}{s})$

$\vec{v}_{2i} = ?$

$\vec{F}_{net} = 0$

final



they combine as tritium

$m_1 + m_2 = 3u$

$\vec{v}_f = 15.1\hat{i} + 22.6\hat{j}\ (10^6 \frac{m}{s})$

Step 2:

$$\vec{P}_i = \vec{P}_f$$

$$(23.5\hat{i} + 14.4\hat{j}) + 2(v_{2ix}\hat{i} + v_{2iy}\hat{j}) = 3(15.1\hat{i} + 22.6\hat{j})$$

Step 3:

$$\begin{cases} \text{in } x: & 23.5 + 2v_{2ix} = 45.3 \rightarrow v_{2ix} = \frac{45.3-23.5}{2} = 21.8 \cdot 10^6 \frac{m}{s} \\ \text{in } y & 14.4 + 2v_{2iy} = 67.8 \rightarrow v_{2iy} = \frac{67.8-14.4}{2} = 53.4 \cdot 10^6 \frac{m}{s} \end{cases}$$