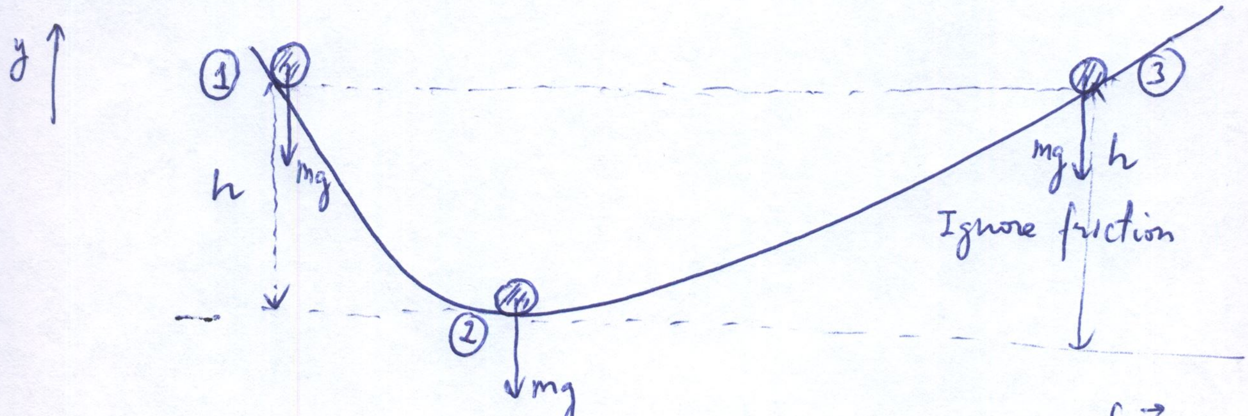
Ch 7: Conservation of Energy

Motion → Forces → Work & Energy

Worked with two types of forces
- Conservative: gravitational (work is conserved)
- Non-conservative: force of friction (work done is non-conservative)

. Work by the gravitational force is conserved.



→ Work done by gravitational force: $W = \int \vec{F} \cdot d\vec{r}$

$$= \vec{F} \cdot \underbrace{\int d\vec{r}}_{\Delta\vec{r}} = \vec{F} \cdot \Delta\vec{r}$$

$$= -mg\hat{j} \cdot \Delta\vec{r}$$

1 → 2 : $\Delta\vec{r} = (0 - h)\hat{j} = -h\hat{j}$

$W_{12} = -mg\hat{j} \cdot (-h\hat{j}) = mgh$ (Work done by gravity on ball from 1 to 2 is mgh)

$\vec{A} \cdot \vec{B} = AB\cos\theta \rightarrow \hat{j} \cdot \hat{j} = 1 \cdot 1 \cdot \cos 0 = 1$

2 → 3 $\Delta\vec{r} = (h - 0)\hat{j} = h\hat{j}$

$W_{23} = -mg\hat{j} \cdot h\hat{j} = -mgh$ (Work done by gravity on ball from 2 to 3 is $-mgh$)

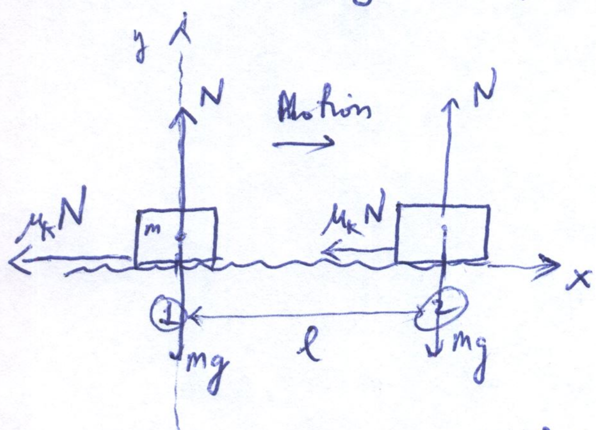
Total work done by gravity on ball 1 → 3 is $mgh - mgh = 0$

→ Work by gravity is <u>conserved</u>

. Work by a frictional force is not conserved:

1) Pushing a box of mass m ① → ② → ① on a rough surface

2) Friction is always against motion

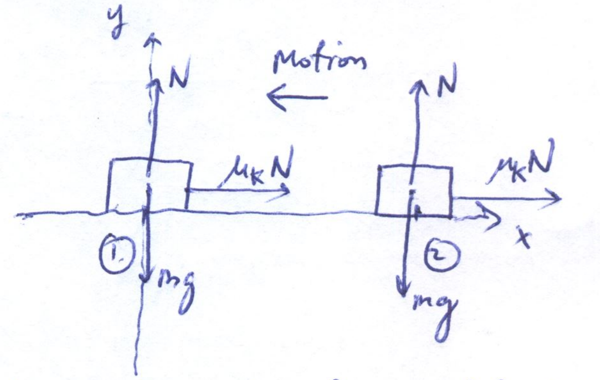


Work done by force of friction ① → ②

$$W_{12} = \vec{F} \cdot \int d\vec{r} = \vec{F} \cdot \Delta \vec{r}$$

$$= -\mu_k N \hat{i} \cdot (l - 0)\hat{i}$$

$$= -\mu_k N l$$



Work done by force of friction ② → ①

$$W_{21} = \vec{F} \cdot \int d\vec{r} = \vec{F} \cdot \Delta \vec{r}$$

$$= \mu_k N \hat{i} \cdot (0 - l)\hat{i}$$

$$= -\mu_k N l$$

$$W_{12} + W_{21} = -2\mu_k N l \neq 0$$

→ Total $\boxed{\text{work done by friction}}$ on box is $-2\mu_k N l = -2\mu_k m g l$ is not conserved. Friction force is non-conservative.

→ <u>Negative sign</u>: because friction never perform work, it consumes work (work was performed in this example by whatever force was applying on the box to move it from ① → ② → ①). That's why the work done by friction is always negative.

If all forces involved are conservative → Total mechanical energy is conserved.

↳ Gravitational potential energy $GPE = m \cdot g \cdot h$

$\Delta GPE = mg \cdot \Delta h$

Kinetic energy : $KE = \frac{1}{2} m v^2$

↳ $W = \int \vec{F} \cdot d\vec{r} = m \int \frac{dv}{dt} \cdot dr = m \int v\,dv$

Elastic potential energy $EPE = \frac{1}{2} k x^2$

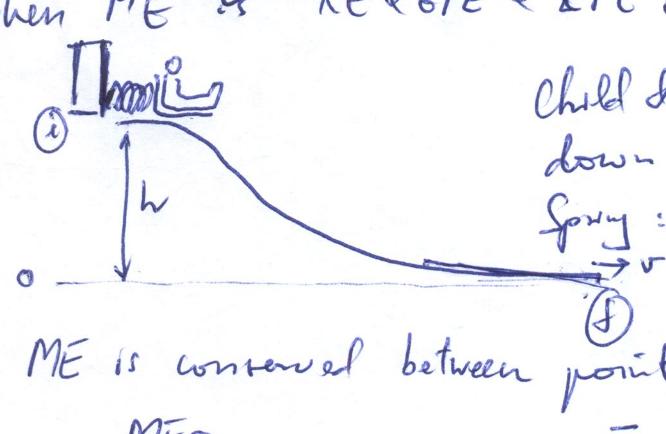
↳ $W = \int \vec{F} \cdot d\vec{r} = -k \int_0^x x \cdot dx = -\frac{1}{2} k x^2$ → Work done by spring is $-\frac{1}{2} k x^2$: spring receives $\frac{1}{2} k x^2$ → $EPE = \frac{1}{2} k x^2$

$d\vec{r}$ ↓ displacement in x

→ When mechanical energy involves only kinetic : $\Delta h = 0$ → Horizontal motion

→ When ME involves both KE & GPE : $\Delta h \neq 0$ → vertical motion

→ When ME involves KE & GPE & EPE :



Child & sled lanched by a spring down a hill.
Spring : k, compressed by Δx

ME is conserved between point (i) & point (f)

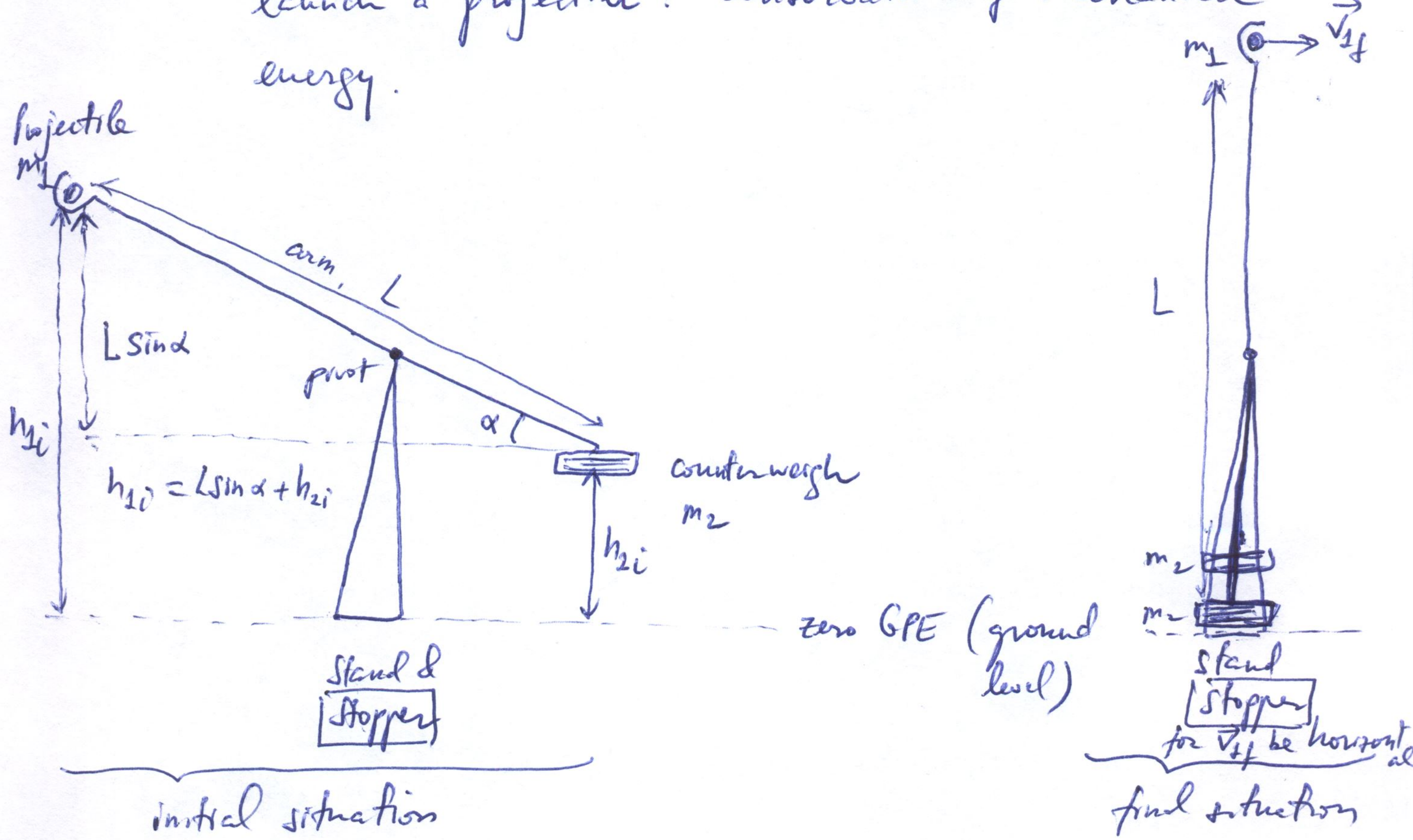
$$ME_{(i)} = ME_{(f)}$$

$$mgh + \underbrace{\frac{1}{2} m \cdot 0^2}_{0} + \frac{1}{2} k \Delta x^2 = \underbrace{mg \cdot 0}_{0} + \frac{1}{2} m v^2 + 0$$

(ground is zero GPE)

PP Set 3 :

(7.1) Catapult or trebuchet: uses a counterweigh to launch a projectile: conservation of mechanical energy.



→ Goal: to predict where projectile will land

(B) projectile motion (once we have v_{1f}; final velocity for projectile as it leaves the catapult)

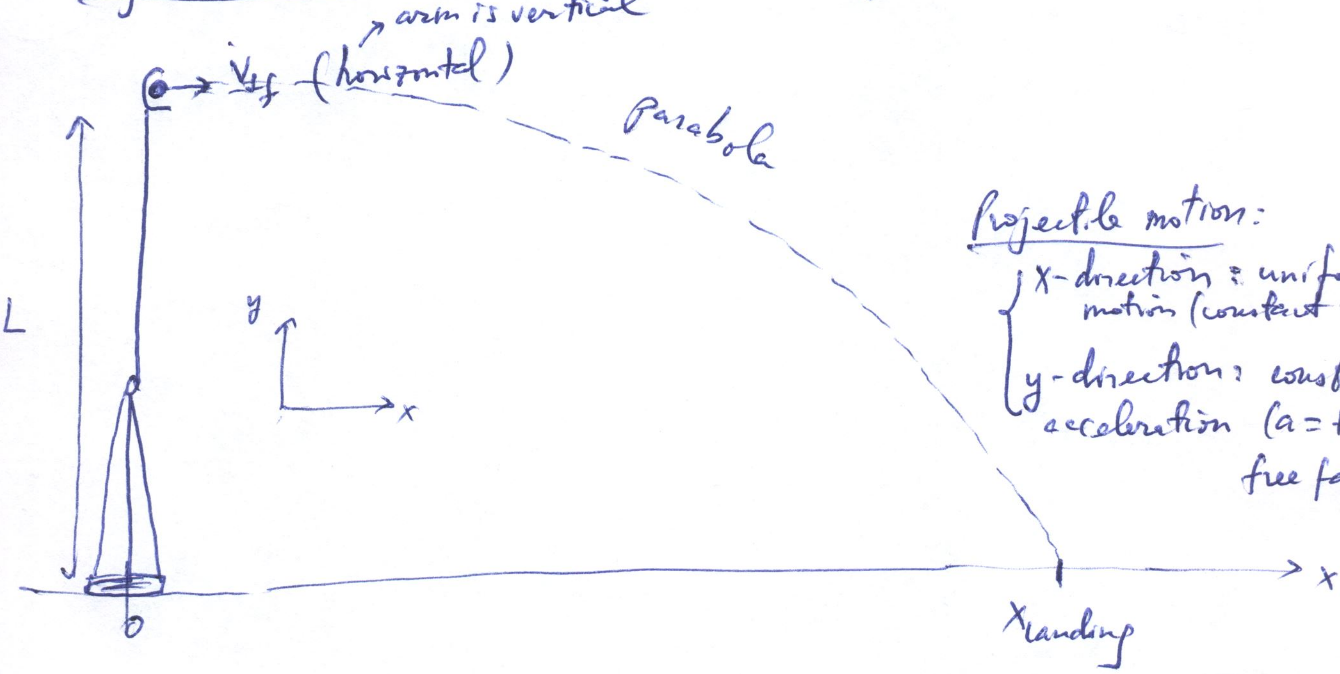
(A) Conservation of ME b/w (i) & (f)

↳ { Projectile initially @ rest; counterweigh initially @ height h_{2i}; Projectile finally with KE $\frac{1}{2}m_1 v_{1f}^2$; counterweigh finally has height 0

$$\underbrace{m_1 g h_{1i} + m_2 g h_{2i}}_{ME_i} = \underbrace{\tfrac{1}{2} m_1 v_{1f}^2 + m_1 g \cdot L}_{ME_f}$$

Solve for v_{1f} → $v_{1f} = \sqrt{\frac{2}{m_1}\left(m_1 g h_{1i} + m_2 g h_{2i} - m_1 g L\right)}$

Projectile Motion part (B) of the analysis



Projectile motion:

- x-direction: uniform motion (constant velocity)
- y-direction: constant acceleration ($a = +g$) free fall

$$\underbrace{\text{Time from } x=0 \text{ to } \cancel{x=\text{landi}}\ x_{landing}}_{t} = \underbrace{\text{time from } y=L \text{ to } 0}_{\sqrt{\frac{2L}{g}}}$$

$$L = \cancel{v_{0y}}\cdot t + \frac{1}{2} a t^2$$

$$L = \frac{1}{2} g t^2$$

$$x_{landing} = v_{1f} \cdot \sqrt{\frac{2L}{g}}$$

$$= \sqrt{\frac{2}{m_1}\left(m_1 g h_{1i} + m_2 g h_{2f} - m_1 g L\right) \cdot \frac{2}{g} \cdot L}$$

$$\left(m_1 g (L \sin\alpha + h_{2i}) + m_2 g h_{2i} - m_1 g L\right)$$

$$(m_1 + m_2) g\, h_{2i} + m_1 g L (\sin\alpha - 1)$$

$$x_{landing} = \sqrt{\frac{4L}{m_1 g}\left[(m_1 + m_2) g \underbrace{h_{2i}}_{\text{initial height of counterweight}} - m_1 g L(1 - \underbrace{\sin\alpha}_{\text{arm angle}})\right]}$$

Ch: 8 Gravitation

Force of gravity = mg;
Mass m @ height h has a GPE = m.g.h (work)

Universal law of Gravitation: (applies to Earth, Moon, planets, galaxies, universe)
(ULG)

$$F = G\frac{m_1 \cdot m_2}{r^2}$$

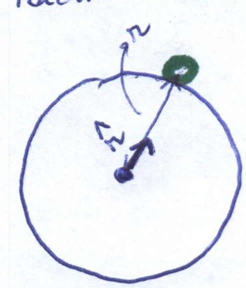
F: Force of gravitational attraction b/w m_1 & m_2 separated @ a distance r

G: Universal Gravitational Constant
$6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

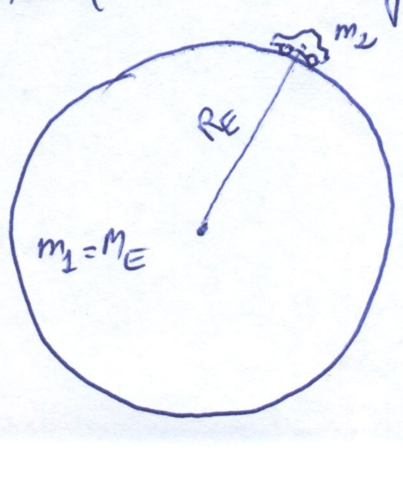
r: center-to-center separation between m_1 & m_2.
$\left(\frac{1}{r^2}\right)$ or "inverse square law"
(if r is doubled, F is one fourth)

F is a vector: $\vec{F} = -G\frac{m_1 \cdot m_2}{r^2}\hat{r}$

($\hat{r}$: radial unit vector)



→ Gravitational attraction of Earth on an object at surface



$M_E = 5.97 \times 10^{24}$ kg
$R_E = 6.37 \times 10^6$ m
$r \cong R_E$

$$\boxed{\frac{Nm^2}{kg^2} \cdot \frac{kg}{m^2} = \frac{N}{kg} \qquad \frac{kg\frac{m}{s^2}}{kg} = \frac{m}{s^2}}$$

ULG: $F = G\frac{M_E \cdot m_2}{R_E^2} = \underbrace{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(6.37 \times 10^6)^2}}_{9.81\frac{m}{s^2}} m_2$

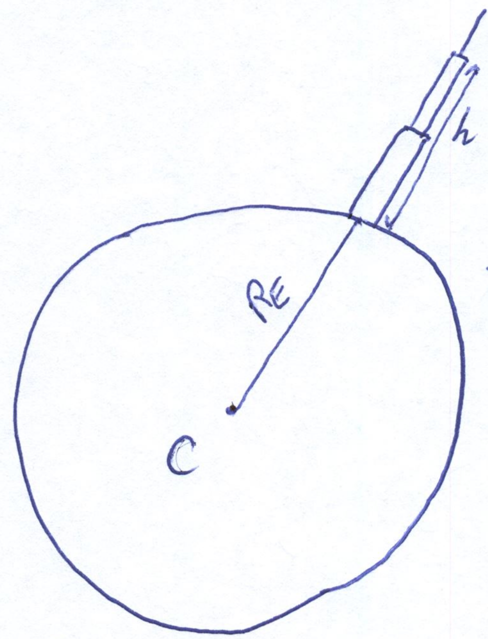
$F = \underbrace{9.81}_{g} \cdot m_2$

Conclusions

(i) Objects of mass m on surface of our planet
$h \ll R_E \rightarrow$ Force of gravitational attraction by Earth is $F = m \cdot g$ $\left(g = 9.81 \frac{m}{s^2}\right)$

(ii) Objects further away above surface:
$r = R_E + h \rightarrow g' < g$

(8.17) $g' < g$ @ top of Willis (formerly Sears) Tower in Chicago: $g' - g = -1.36 \frac{mm}{s^2}$ or $\Delta g \equiv \underset{\substack{\downarrow \\ \text{surface or street level}}}{g} - \underset{\substack{\searrow \\ \text{top of building}}}{g'} = 1.36 \frac{mm}{s^2}$



<u>Statements</u>: i) Sketch not to scale, h still much less than R_E: $\boxed{h \ll R_E}$

Guess h: 10m, 100m ✓, 1000m ✓, 10,000m

$$F = \underbrace{\boxed{\frac{G M_E}{r^2}}}_{g} m \; ; \quad \begin{cases} g' = \dfrac{G M_E}{(R_E + h)^2} & \text{(top of building of height } h\text{)} \\ g = \dfrac{G M_E}{R_E^2} & \text{(surface or street level)} \end{cases}$$

$$\Delta g = g - g' = G M_E \left[\frac{1}{R_E^2} - \frac{1}{(R_E + h)^2} \right] = G M_E \left[\frac{(R_E + h)^2 - R_E^2}{R_E^2 (R_E + h)^2} \right]$$

$$= G M_E \frac{2 R_E h + h^2}{R_E^2 (R_E + h)^2} = \underbrace{\boxed{\frac{G M_E}{R_E^2}}}_{g} \underbrace{\frac{2 R_E h + h^2}{(R_E + h)^2}}$$

$\boxed{\begin{array}{l} (R_E + h) \approx R_E \\ \underbrace{(2 R_E + h)}_{\approx 2R_E} h \approx 2 R_E h \end{array}}$

$$\Delta g \simeq g \cdot \frac{2 \cancel{R_E} h}{R_E^{\cancel{2}}} = g \frac{2h}{R_E} \rightarrow \boxed{h = \frac{\Delta g}{g} \frac{R_E}{2}}$$

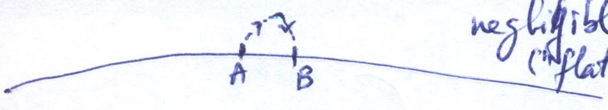
Projectile Motion: motion under force of gravitational attraction

So far

($-mg\hat{j}$: vertical)

↳ short-range projectile motion

↳ B/w A & B curvature is negligible ("flat" surface)



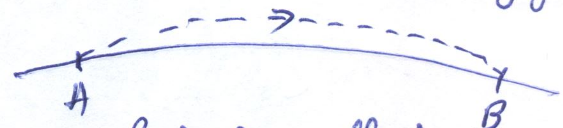
Parabolic trajectory

Correction

($-mg\hat{r}$: radial)

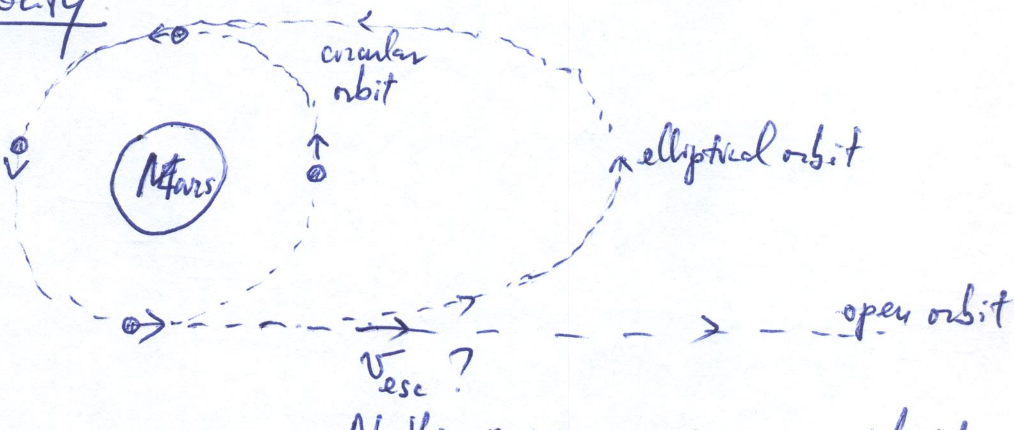
↳ long-range projectile motion

↳ curvature is not negligible



Part of an elliptical orbit!

Escape Velocity:



At this minimum escape velocity space probe can escape the gravitational attraction of Mars, & follows an open orbit

Total mechanical energy of an object trapped in a gravitational field (a satellite or probe in circular or elliptical orbit around a planet) is negative.

$$ME = KE + GPE = \frac{1}{2}mv^2 - \boxed{\frac{GM_E\, m}{r}} < 0$$

(extension of mgh)

$$\rightarrow h = \frac{1.36\times10^{-3}}{9.81} \times \frac{6.37\times10^{6}}{2} = 442\ m$$

$\downarrow$

{ Universal Law of Gravitation
{ & $h << R_E$

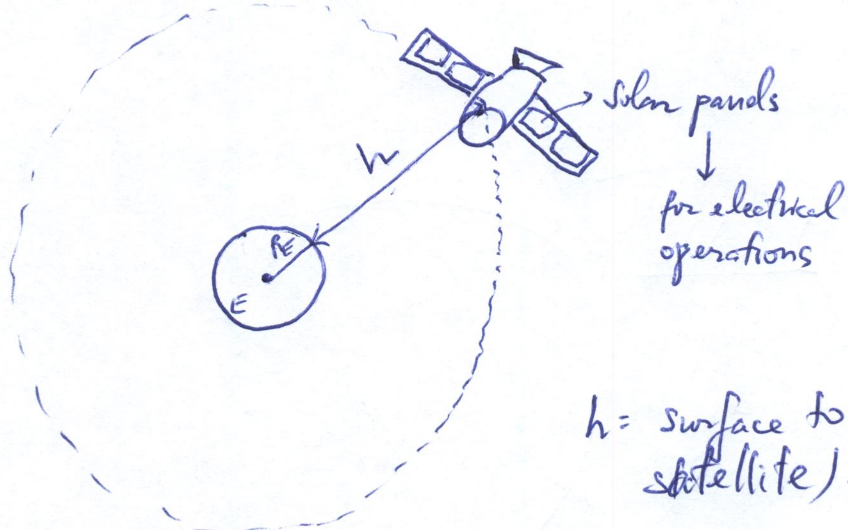
Circular Orbital Motion (Satellite's UCM)

$\hookrightarrow$ { constant speed v
$\boxed{a = \frac{v^2}{R}}$

Force of gravitational attraction

$\downarrow$

Orbital motion



h = surface to orbit (center of satellite) $\rightarrow$ Orbital radius $= R_E + h$

Orbital period T: time needed per orbit: $T = \frac{2\pi(R_E+h)}{v}$

Calculate v: $F = G\frac{M_E \cdot \cancel{m}}{(R_E+h)^{\cancel{2}}} = \cancel{m}\cdot\frac{v^2}{\cancel{R_E+h}}$

$\underbrace{\qquad}_{\text{Univ. Grav. attraction}}$ $\underbrace{\qquad}_{\text{Newton's 2nd Law}}$

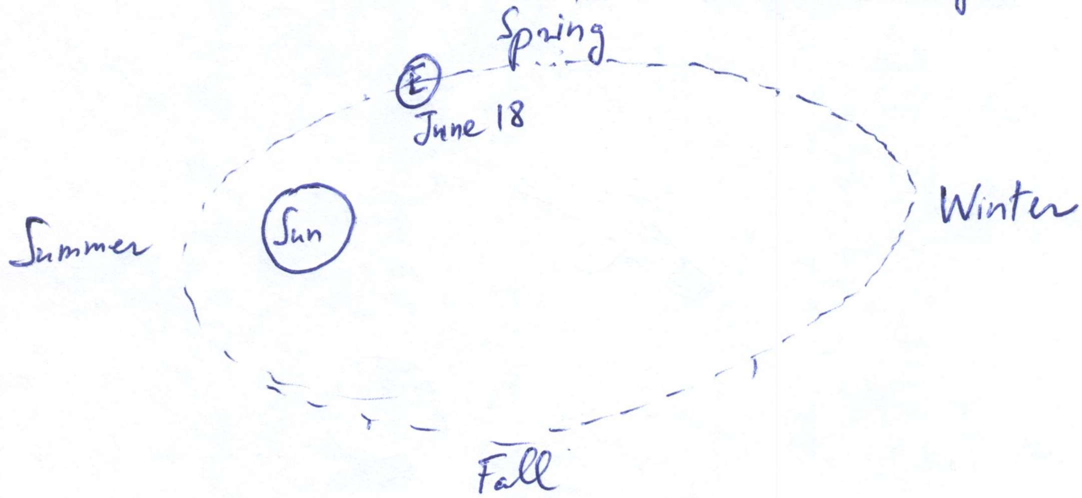
$$\rightarrow v = \sqrt{\frac{GM_E}{R_E+h}}$$

$$\boxed{T = \frac{2\pi(R_E+h)}{\frac{\sqrt{GM_E}}{(R_E+h)^{\frac{1}{2}}}} = \frac{2\pi}{\sqrt{GM_E}}(R_E+h)^{3/2}} \Rightarrow \boxed{T^2 = \frac{4\pi^2}{GM_E}(R_E+h)^3}$$

Planetary Orbital Motion: $R_E + h \equiv R$

$$T^2 = \frac{4\pi^2}{GM_E} R^3 \quad \text{or} \quad \boxed{T^2 \propto R^3}$$

Kepler's 3rd Law: period squared is directly proportional to the semimajor axis cubed (elliptical orbits)



Cell phone satellites: $h = 250\,km$

$$T = \frac{2\pi}{\sqrt{G \cdot M_E}} (R_E + h)^{3/2} = \frac{2\pi \left(6.37\times10^6 + 0.25\times10^6\right)^{3/2}}{\sqrt{6.67\times10^{-11} \times 5.97\times10^{24}}}$$

$$= 5400\,s \cdot \frac{1h}{3600\,s} = 1.5\,hrs.$$

When $ME = 0 \rightarrow v_{esc}$:

$$\frac{1}{2}\cancel{m} v_{esc}^2 - \frac{GM_E \cancel{m}}{r} = 0 \rightarrow v_{esc} = \sqrt{\frac{2GM_E}{r}}$$

Escape velocity for any object of mass m $\underbrace{@ \text{ surface}}_{r=R_E}$ of Earth:

$$\hookrightarrow v_{esc} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6}} = 11.2 \frac{km}{s} = 40320 \frac{km}{h}$$

Gravitational Potential Energy (extension for mgh)

How did we derive mgh? → Def. Work + $F_{app} = mg$

↳ Extension: Def. work + $F_{app} = -\frac{GM_E}{r^2} m$

$$\underbrace{\Delta U_{AB}}_{\text{Change of grav. potential energy from A to B}} = \int_A^B \vec{F} \cdot d\vec{r} = -GM_E m \underbrace{\int_A^B \frac{dr}{r^2}}_{\left[-\frac{1}{r}\right]_A^B} = GM_E m\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

Universal Law of Grav.

$$\Delta U_{AB} = U_A - U_B$$

Def: $\boxed{U \equiv -\frac{GM_E m}{r}}$

General expression for GPE

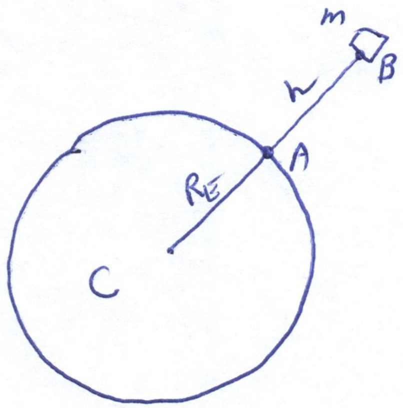
Reference for zero potential energy: $r \rightarrow \infty$; $U_\infty = 0$

$$\hookrightarrow \Delta U_{A\infty} = U_A - U_\infty = U_A = -\frac{GM_E m}{r_A}$$

If m is on surface: $r = R_E \rightarrow U = -\frac{GM_E m}{R_E} = -\underbrace{\frac{GM_E}{R_E^2}}_{g} m R_E$

$$U = -mgR_E$$

Proof of $U = -\frac{GM_E m}{r}$ is extension of mgh:



$$\Delta U_{AB} = -GM_E m\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$= -GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E+h}\right)$$

$$= -GM_E m\left(\frac{\cancel{R_E}+h - \cancel{R_E}}{R_E(R_E+h)}\right)$$

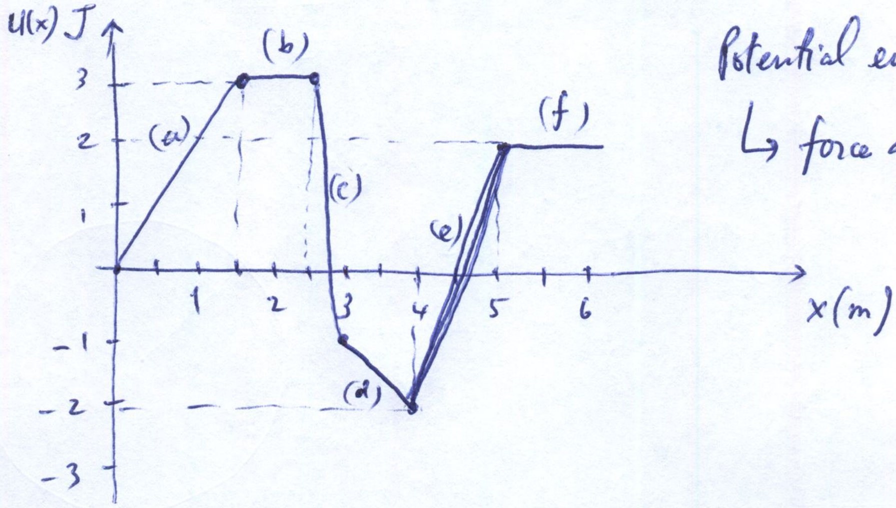
$$= -GM_E m\,\frac{h}{R_E(R_E+h)}$$

If $h \ll R_E$ (baseball, soccer ball, cars, water droplets ...)

$\hookrightarrow R_E + h \simeq R_E$

$\hookrightarrow \Delta U_{AB} = -\underbrace{\frac{GM_E}{R_E^2}}_{g} m h = -mgh$

7.26



Potential energy curve
↳ force applied F ?

$U \rightarrow W = \int F\,dx \rightarrow$

$\underbrace{\frac{dW}{dx}}_{slope} = F$

(a) $\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{1.5} = 2 \rightarrow F_{(a)} = 2N$

(b) $F_{(b)} = 0$

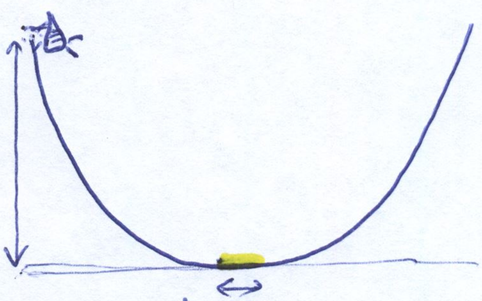
(c) $F_{(c)} = \frac{-1-3}{3-2.5} = -8N$

(d) $F_{(d)} = \frac{-2-(-1)}{4-3} = -1N$

(e) $F_{(e)} = \frac{2-(-2)}{5-4} = 4N$

(f) $F_{(f)} = 0N$

7.54



Bug starts sliding down from top of bowl.

$d = 1.5cm$

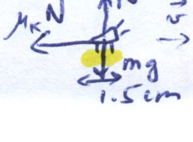
$\mu_k = 0.61$

otherwise bowl is frictionless

Statements: (i) Bug starts with $GPE = mgh$ ($v = 0 \rightarrow KE = 0$)

(ii) Loss due to work of friction: $\mu_k mg \cdot d$

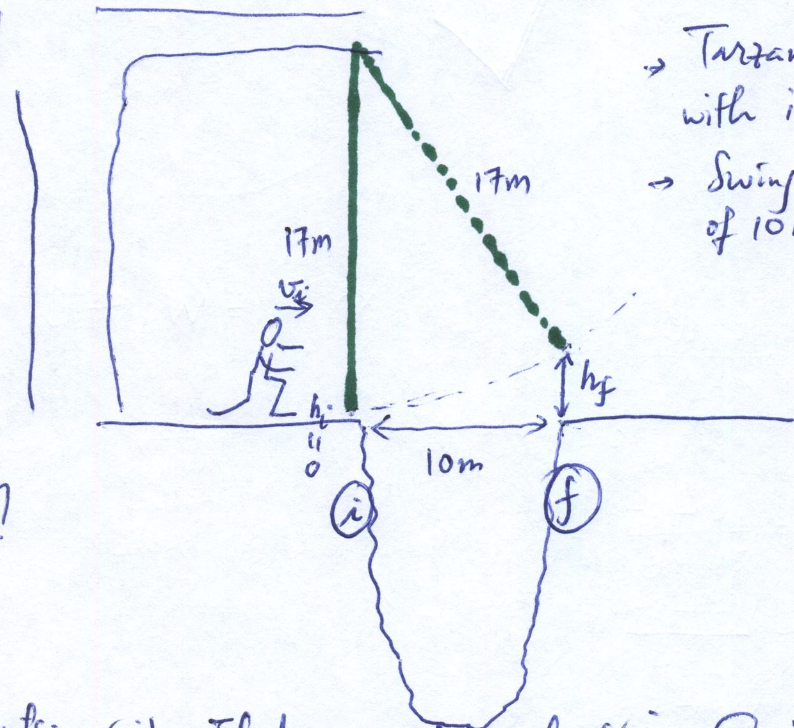
each time bug crosses sticky patch



How many times bug will cross sticky patch:

$$n = \frac{\cancel{m} \cdot g \cdot h}{\mu_K \cancel{m} g\, d} = \frac{\cancel{9.81} \cdot 0.11}{0.61 \cdot \cancel{9.81} \cdot 0.015} = 12.02 \rightarrow \boxed{n = 12}$$

7.62)



→ Tarzan runs toward vine with initial velocity v_i $(h_i = 0)$

→ Swings to the other side of 10m-wide gorge

$$h_f - h_i = \Delta h$$

$v_{i_{min}}$?

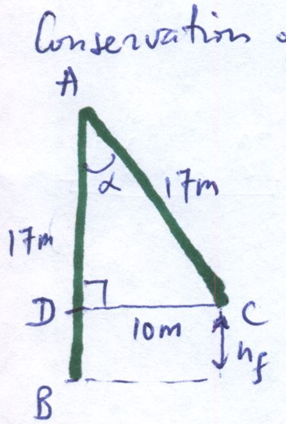
↓

Statements: (i) If he ran towards vine @ $V_{i_{min}}$, his v_f at the other side of the 10-m wide gorge would be 0 : $\begin{cases} ME_i = \frac{1}{2} m v_i^2 \\ ME_f = m g h_f \end{cases}$

(ii) If he ran at $v_i > V_{i,min}$, his $V_f > 0$ $\begin{cases} ME_i = \frac{1}{2} m v_i^2 \\ ME_f = \frac{1}{2} m v_f^2 + m g h_f \end{cases}$

Conservation of ME: tarzan gets to the other side by converting KE_i into GPE_f :

$$\frac{1}{2}\cancel{m} v_{i_{min}}^2 = \cancel{m} g h_f \rightarrow V_{i_{min}} = \sqrt{2 g h_f}$$



$AB = AC = 17m$

1) $h_f = AB - AD = AB - \sqrt{17^2 - 10^2} = 17 - \sqrt{17^2 - 10^2} = 3.25m$

$AD^2 + 10^2 = 17^2$

Using Trigonometry 2) $AD = AC\cos\alpha = 17\cdot\cos\alpha$

$$\sin\alpha = \frac{10}{17} \rightarrow \alpha = \sin^{-1}\frac{10}{17} = 36^\circ$$

$$h_f = AB - AD = 17 - 17\cos 36^\circ = 3.25\,m.$$

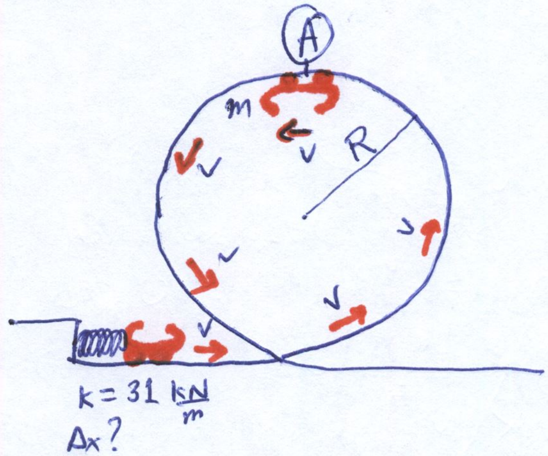
8.32 If $v_{esc} = 30\,\frac{km}{s}$ @ surface $(r = R_E)$ what is R_E?

$$v = v_{esc} \rightarrow ME = \frac{1}{2}\cancel{m}v_{esc}^2 - \frac{GM_E\cancel{m}}{r} = 0$$

at surface $r = R_E$ $\quad v_{esc}^2 = \frac{2GM_E}{R_E} \rightarrow R_E = \frac{2GM_E}{v_{esc}^2}$

$$R_E = \frac{2\cdot 6.67\times10^{-11}\cdot 5.97\cdot10^{24}}{(30\cdot10^3)^2}\,m =$$

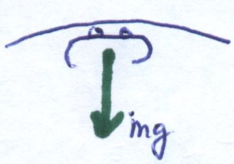
7.57



→ No friction
$m = 840\,kg$
$R = 6.2\,m$
Δx_{min}? so car will make top of loop

Statement: (i) $v \geq v_{min}$ for car to make it past the top point. $v_{min} = \sqrt{g\cdot R}$ why?

A → if car travels @ $v = v_{min}$ through the loop @A it would barely touches track: $N = 0$



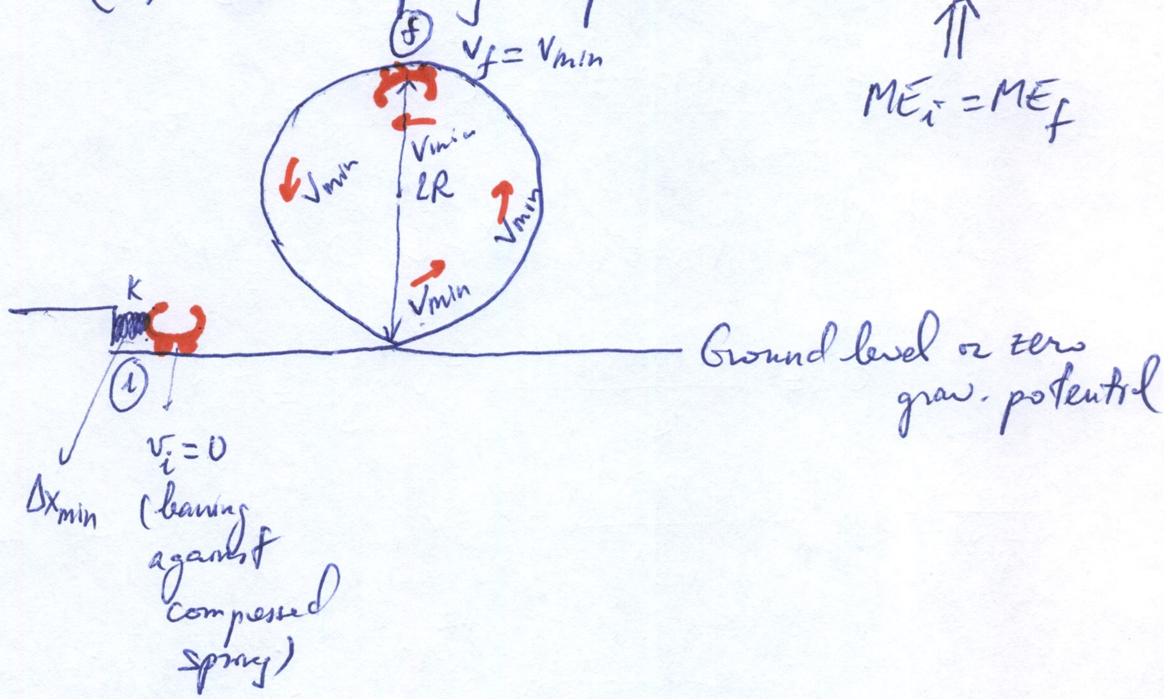
Only force on car @ A is mg which provides $a = \frac{v^2}{R} \Rightarrow \cancel{m}g = \cancel{m}\cdot\frac{v_{min}^2}{R}$

$\rightarrow v_{min} = \sqrt{gR}$

(ii) Certain spring compression $\Delta x_{min} \rightarrow v_{min}$

$\Uparrow$

$ME_i = ME_f$



$$ME_i = ME_f$$

$$\cancel{KE_i}^{0} + \cancel{GPE_i}^{0} + EPE_i = KE_f + GPE_f + \cancel{EPE_f}^{0}$$

$$\frac{1}{2}k\Delta x_{min}^2 = \frac{1}{2}mv_{min}^2 + mg2R + 0$$

$$\Delta x_{min} = \sqrt{\frac{mv_{min}^2 + 4mgR}{k}}$$

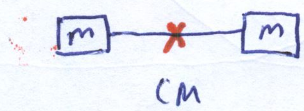
$$\underset{(v_{min}=\sqrt{gR})}{=} \sqrt{\frac{5mgR}{k}}$$

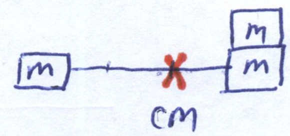
$$= \sqrt{\frac{5 \cdot 840 \cdot 9.81 \cdot 6.2}{31000}} = 2.87\,m$$

Spring compression min of 2.87m for roller coaster car to get v_{min} to make top of loop.

Ch 9 System of Particles:

Center of Mass (CM): average position of all components of a system, weighted by their masses





(The right component position weight is double that of the left one)

$\vec{R}$ = position vector of the CM

- Discrete systems: $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{M}$
  - m_i: mass of component i
  - $\vec{r}_i$: position vector of component i
  - $M = \sum_i m_i$ Total mass of system
- Continuous system: $\vec{R} = \dfrac{\int \vec{r}\, dm}{M}$
  - dm: mass of infinitesimal component
  - $\vec{r}$: position vector of infinitesimal component
  - $M = \int dm$ Total mass of system

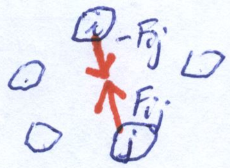
Newton's 2nd Law for a system of particles:

$$\boxed{\underbrace{\vec{F}_{net}}_{\text{Net force on system}} = \underbrace{M}_{\text{Total mass of system}} \cdot \frac{d^2 \vec{R}}{dt^2}}$$

$\vec{R}$ → position vector of CM of system.

i) <u>Subtlety</u>: force of interaction b/w components:

looking at pair interaction: $\begin{cases} F_{ij} \text{ (i on j)} \\ -F_{ij} \text{ (j on i)} \end{cases}$



Internal interactions occur in pairs of equal & opposite forces (Newton's 3rd Law) → sum of all internal forces is 0

→ $\vec{F}_{net}$: net <u>external</u> force on system.

(internal interactions are not relevant to motion of the system)

ii) <u>Linear momentum of a system:</u> $\vec{P}$

$$\underset{\substack{\downarrow \\ \text{Total mass of system}}}{\vec{P} \equiv M} \cdot \underset{\substack{\downarrow \\ \text{Velocity of CM}}}{\vec{V}} = M \cdot \frac{d\vec{R}}{dt} = M \cdot \frac{d\left(\frac{\sum_i m_i \vec{r}_i}{M}\right)}{dt} = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right)$$

$$= \sum_i m_i \cdot \underbrace{\frac{d\vec{r}_i}{dt}}_{\vec{v}_i} = \sum_i m_i \vec{v}_i$$

$$= \sum_i \vec{P}_i$$

$$\vec{P} = \sum_i \vec{P}_i$$

Newton's 2nd Law for System of Particles:

$$\boxed{\vec{F}_{net} = \frac{d\vec{P}}{dt}}$$

More general than the previous $\vec{F}_{net} = M \dfrac{d^2\vec{R}}{dt^2}$.

↳ If M = constant.

$$\vec{F}_{net} = \frac{d\left(M \cdot \frac{d\vec{R}}{dt}\right)}{dt} = M \cdot \frac{d^2\vec{R}}{dt^2}$$

↳ $\vec{F}_{net} = 0 \Rightarrow \dfrac{d\vec{P}}{dt} = 0 \Leftrightarrow \vec{P} = \text{constant} \rightarrow$ $\boxed{\text{Conservation of linear momentum}}$

Collisions

$\Downarrow$

$\vec{F}_{net} = 0$

$\Downarrow$

Conservation of linear momentum

Inelastic

$\vec{P}_i = \vec{P}_f$ Collision of 2 objects: $m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$

$KE_i \neq KE_f \Rightarrow KE_i - KE_f > 0 \rightarrow$

→ The two colliding components stick together after the collision: $\vec{v}_{1f} = \vec{v}_{2f} \equiv \vec{v}_f$

$\Rightarrow m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$

Elastic

$\vec{P}_i = \vec{P}_f \rightarrow m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$

$KE_i = KE_f \rightarrow \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$

1D Elastic collision:

$$\begin{cases} m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \\ \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \end{cases}$$

If m_1, m_2, v_{1i}, v_{2i} are given, we can solve for v_{1f} & v_{2f}

1) $v_{1f} = \dfrac{m_1 - m_2}{m_1 + m_2} v_{1i} + \dfrac{2m_2}{m_1 + m_2} v_{2i}$

2) $v_{2f} = \dfrac{2m_1}{m_1 + m_2} v_{1i} + \dfrac{m_2 - m_1}{m_1 + m_2} v_{2i}$

3) $v_{1i} + v_{1f} = v_{2i} + v_{2f}$

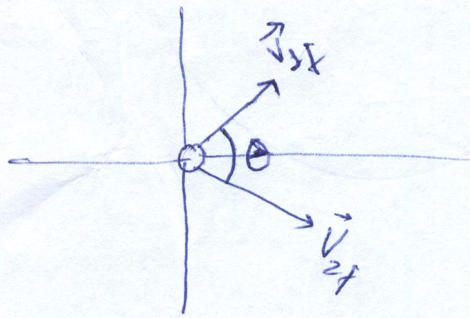
2D Elastic Collision :

$$\begin{cases} m_1 V_{1ix} + m_2 V_{2ix} = m_1 V_{1fx} + m_2 V_{2fx} \\ m_1 V_{1iy} + m_2 V_{2iy} = m_1 V_{1fy} + m_2 V_{2fy} \\ \frac{1}{2} m_1 V_{1i}^2 + \frac{1}{2} m_2 V_{2i}^2 = \frac{1}{2} m_1 V_{1f}^2 + \frac{1}{2} m_2 V_{2f}^2 \end{cases}$$

$\boxed{\text{3 equations}}$

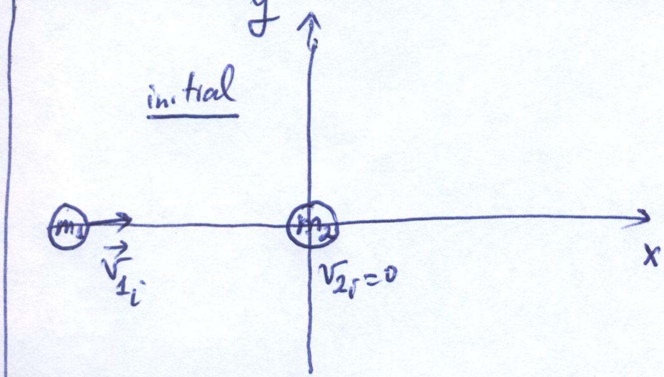
Given $m_1, m_2, V_{1ix}, V_{1iy}, V_{2ix}, V_{2iy} \Rightarrow$ V_{1fx} V_{1fy} V_{2fx} V_{2fy}

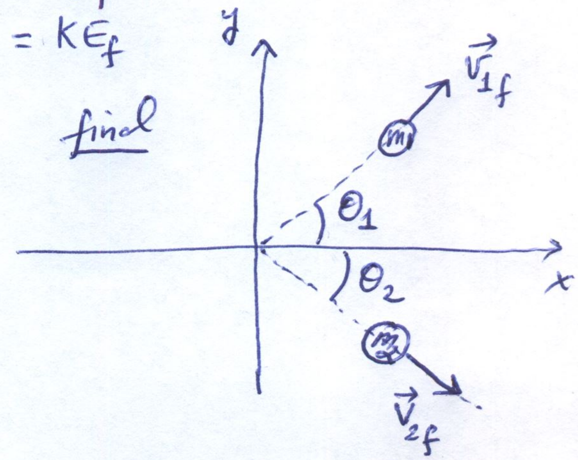
Need one more data : e.g. angle b/w final velocities : angle θ.



2D Elastic Collisions : $\Rightarrow$ 3 eqs.

$$\begin{cases} P_{i_x} = P_{f_x} \\ P_{i_y} = P_{f_y} \\ KE_i = KE_f \end{cases}$$





Note: with 3 eqs we can't calculate all of $\vec{v}_{1f}$ & $\vec{v}_{2f}$ $\begin{cases} (v_{1fx}, v_{1fy}) \\ (v_{2fx}, v_{2fy}) \end{cases}$

$\hookrightarrow$ Need one more data $\begin{cases} v_{1f} \text{ or } \theta_1 \\ \text{or } v_{2f} \text{ or } \theta_2 \end{cases}$

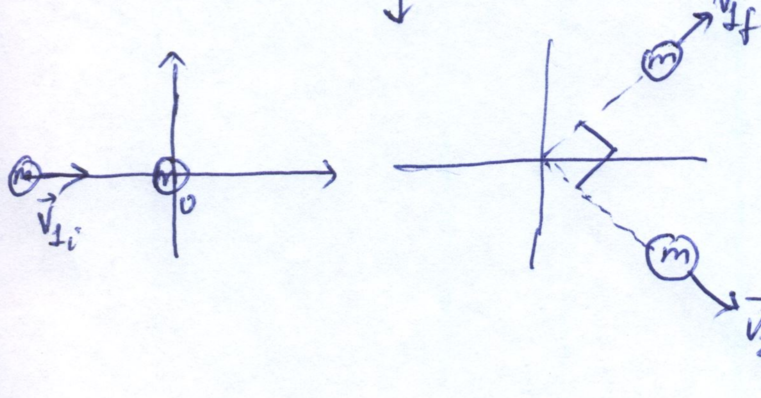
$\hookrightarrow$ Can derive :

$$\begin{cases} 1) \quad v_{1i}^2 = v_{1f}^2 + \dfrac{m_2^2}{m_1^2} v_{2f}^2 + \dfrac{2m_2}{m_1} v_{1f} v_{2f} \cos(\theta_2 - \theta_1) \\ 2) \quad v_{1i}^2 = v_{1f}^2 + \dfrac{m_2}{m_1} v_{2f}^2 \\ 3) \quad 0 = \left(\dfrac{m_2}{m_1} - 1\right) v_{2f} + 2 v_{1f} \cos(\theta_1 - \theta_2) \end{cases}$$

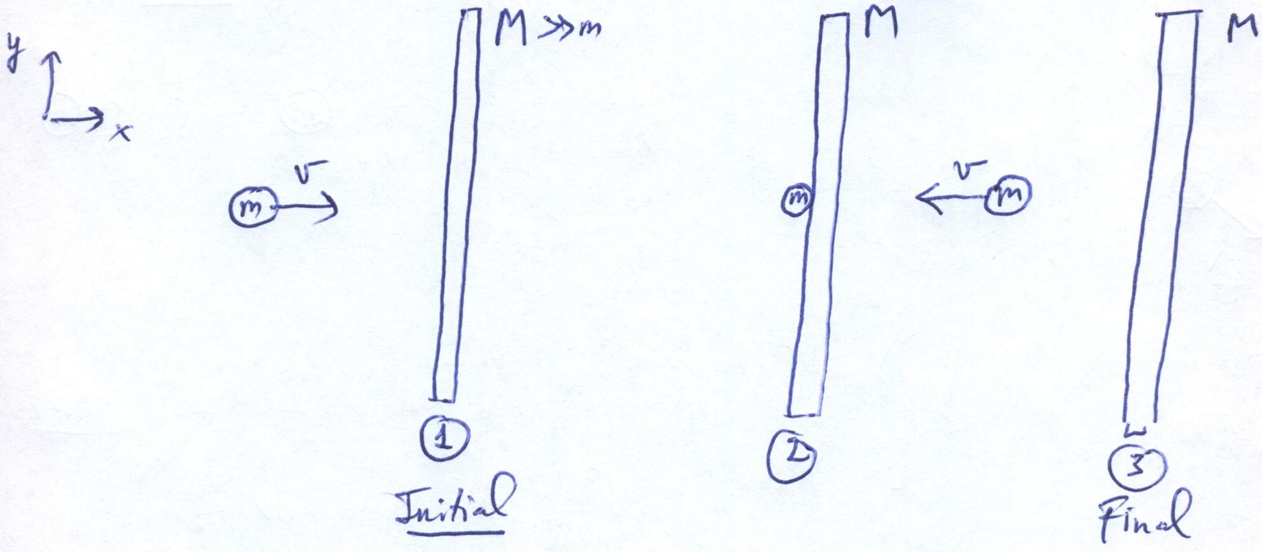
Note: when $\boxed{m_2 = m_1}$ $\quad$ 3) $\rightarrow$ $\quad 0 = 2 v_{1f} \cos(\theta_1 - \theta_2)$

$\hookrightarrow$ if $v_{1f} \neq 0 \Rightarrow \cos(\theta_1 - \theta_2) = 0$

or $\underbrace{\theta_1 - \theta_2}_{\text{angle b/w two final directions}} = 90°$



Air Pressure: Collision of gas molecules (hard balls) with containing walls:



Note: - No deformation for neither air molecule (hard ball) nor wall. → Molecule-wall collision is elastic. → KE is conserved → air molecule bounce back with same speed but in opposite direction.

- Collision: linear momentum is conserved:

$$P_{ix} = P_{fx} \qquad (\vec{F}_{net} = 0)$$

$$\downarrow$$

$$mv = m(-v) + \underbrace{2mv}$$

↓ Lin. momentum acquired by the wall

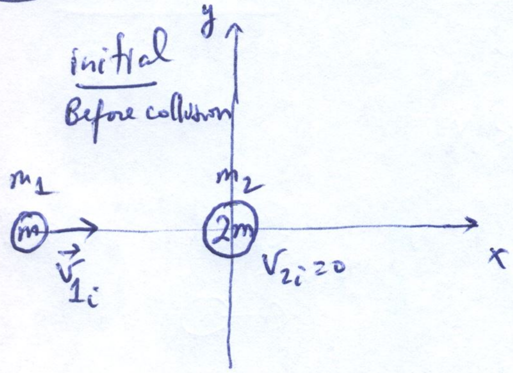
$$2mv = M \cdot V$$

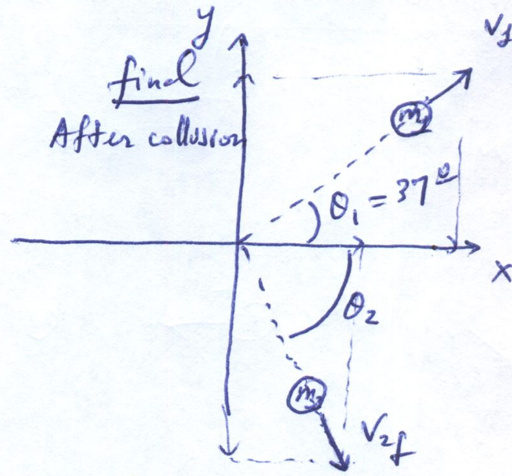
$$V = \frac{2m}{M} v$$

↓ speed acquired by wall is insignificant $M \gg m$

However: this transfer of momentum molecule to wall is the origin of air pressure

9.71 2D elastic collision proton against stationary deuteron
($m_1 = 1u$) ($m_2 = 2u$)





Statements

1) 2D elastic collision $\begin{cases} P_{ix} = P_{fx} \\ P_{iy} = P_{fy} \\ KE_i = KE_f \end{cases}$

2) Since $m_1 \neq m_2$ final directions are not perpendicular to each other

3) 3 eqs, can solve for 3 unknowns: v_{1f}, v_{2f}, θ_2

a) Fraction of KE that proton transfered to deuteron

Why p transfered some KE to d? – Since d started stationary and acquired a speed after collision

How do we know p did transfer only a fraction of its KE_i?
– Since it was travelling @ $\theta_1 = 37^\circ$ after collision, it retained part of its KE_i.

$$\frac{KE_{2f}}{KE_{1i}} = \frac{KE_{1i} - KE_{1f}}{KE_{1i}} = 1 - \frac{KE_{1f}}{KE_{1i}} = 1 - \frac{\cancel{\frac{1}{2}}\cancel{m_1} v_{1f}^2}{\cancel{\frac{1}{2}}\cancel{m_1} v_{1i}^2} = \boxed{1 - \frac{v_{1f}^2}{v_{1i}^2}}$$

Solve for v_{1f} in term of v_{1i} using the 3 eqs for 2D elastic collisions:

1) $v_{1i}^2 = v_{1f}^2 + 4v_{2f}^2 + 4v_{1f}v_{2f}\cos(\theta_2 - 37°)$

2) $v_{1i}^2 = v_{1f}^2 + 2v_{2f}^2$

3) $0 = v_{2f} + 2v_{1f}\cos(\theta_2 - 37°)$ $\Rightarrow \cos(\theta_2 - 37°) = -\frac{v_{2f}}{2v_{1f}}$

~~$\rightarrow$ 1) $v_{1i}^2 = v_{1f}^2 + 4v_{2f}^2 - 2v_{2f}^2 = v_{1f}^2 + 2v_{2f}^2$~~

$\cos(\theta_2 - 37°) = \cos\theta_2 \cos 37° + \sin\theta_2 \sin 37°$

$\rightarrow \quad P_{ix} = P_{fx}$

$m_1 v_{1i} = m_1 v_{1f}\cos 37° + m_2 v_{2f}\cos\theta_2$

$[v_{1i} = v_{1f}\cos 37° + 2v_{2f}\cos\theta_2] \times \cos 37° \rightarrow$ (i) $\boxed{v_{1i}\cos 37° = v_{1f}\cos^2 37 + 2v_{2f}\cos\theta_2\cos 37°}$

$\rightarrow \quad P_{iy} = P_{fy}$

$[0 = v_{1f}\sin 37° - 2v_{2f}\sin\theta_2] \xrightarrow{\times \sin 37°}$ (ii) $\boxed{0 = v_{1f}\sin^2 37° - 2v_{2f}\sin\theta_2\sin 37°}$

$(i) + (ii) \rightarrow \boxed{v_{1i}\cos 37° = v_{1f}\underbrace{(\cos^2 37° + \sin^2 37°)}_{1} + 2v_{2f}\underbrace{(\cos\theta_2\cos 37° - \sin\theta_2\sin 37)}_{\cos(\theta_2 - 37°)}}$

$3) \times v_{2f} \rightarrow 0 = v_{2f}^2 + \underbrace{2v_{1f}\, v_{2f}\cos(\theta_2 - 37°)}_{v_{1i}\cos 37° - v_{1f}}$

$0 = v_{2f}^2 + v_{1f}v_{1i}\cos 37° - v_{1f}^2$ (Quadratic eq in v_{1f})

$v_{1f}^2 - v_{1f}v_{1i}\cos 37° - v_{2f}^2 = 0$

$ax^2 + bx + c = 0$

$\begin{cases} x = v_{1f};\ a = 1; \\ b = -v_{1i}\cos 37° \\ c = -v_{2f}^2 \end{cases}$

$v_{2f}^2 = \frac{v_{1i}^2 - v_{1f}^2}{2}$

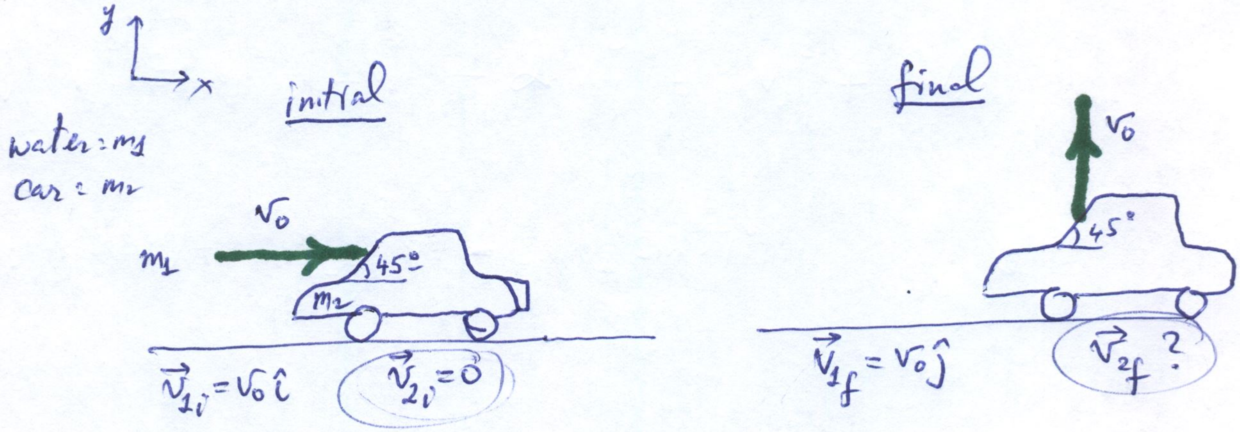
$3v_{1f}^2 - 2v_{1f}v_{1i}\cos 37° - v_{1i}^2 = 0$

$\rightarrow v_{1f} = \frac{2v_{1i}\cos 37° \pm \sqrt{4v_{1i}^2\cos^2 37 + 12v_{1i}^2}}{6}$

$\boxed{v_{1f} = \frac{1}{3}v_{1i}\left(\cos 37° + \sqrt{\cos^2 37° + 3}\right) = 0.902\, v_{1i}}$

$\rightarrow \frac{KE_{2f}}{KE_{1i}} = 1 - \frac{v_{1f}^2}{v_{1i}^2} = 1 - 0.902^2 = 0.186$ or 18.6% $\Big\|$ $v_{2f}^2 = \frac{v_{1i}^2}{2}\left(1 - \frac{v_{1f}^2}{v_{1i}^2}\right) = v_{1i}^2\, \frac{0.186}{2}$ $\rightarrow \boxed{v_{2f} = 0.305\, v_{1i}}$ $\Big\|$ $\theta_2 - 37° = \cos^{-1}\left(-\frac{0.305}{2 \cdot 0.902}\right) \rightarrow \theta_2 = 62.8°$

9.45] Car initially at rest, received a push by a jet of water hitting its back window. No friction.
Car acquired an acceleration a (?)



Statements: 1) Collision b/w jet of water & car
↳ $\boxed{\vec{F}_{net} \text{ on car \& water is } 0} \Rightarrow \boxed{\vec{P}_i = \vec{P}_f}$

2) Friction is ignore

3) m_1 is not given, however $\frac{dm_1}{dt}$ is given

$$\vec{P}_i = \vec{P}_f$$
$$m_1\vec{v}_{1i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$
$$\rightarrow \quad \vec{v}_{2f} = \frac{m_1\vec{v}_{1i} - m_1\vec{v}_{1f}}{m_2} = \frac{m_1}{m_2}v_0\hat{i} - \frac{m_1}{m_2}v_0\hat{j}$$

$$\rightarrow \quad \vec{a}_2 = \frac{d\vec{v}_{2f}}{dt} = \left(\frac{v_0}{m_2}\hat{i} - \frac{v_0}{m_2}\hat{j}\right)\frac{dm_1}{dt}$$

m_2 is constant (mass of car)
v_0 is constant (speed water)

a) Forward acceleration for car: $\boxed{a_{2x} = \frac{v_0}{m_2}\frac{dm_1}{dt}}$

Downward acceleration for car $a_{2y} = -\frac{v_0}{m_2}\frac{dm_1}{dt}$ (car's suspension feels push down),

b) Max speed car can reach?

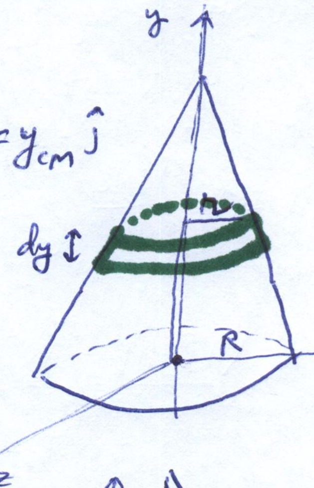
↓
is the speed of water v_0

Why it can't reach $v > v_0$? as soon as car reaches a speed a little bit more than v_0 it won't receive any more pushing from water → acceleration stops → can't go faster than v_0.

9.41 Find CM of continuous object $\vec{R} = \frac{1}{M}\int \vec{r}\,dm$

$\vec{R}$: position vector of CM

By symmetry of cone $\rightarrow \vec{R} = y_{cm}\,\hat{j}$



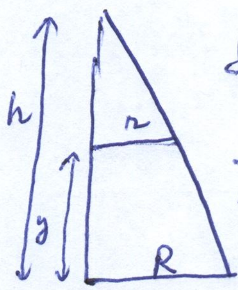
$$y_{cm} = \frac{1}{M}\int \underbrace{y}_{\text{vertical position of disk of mass } dm} dm$$

Density is $\rho = \dfrac{dm}{dV}$ → V: volume

$$\underbrace{dm}_{\text{mass of disk}} = \rho \underbrace{dV}_{\text{volume of disk of height } dy}$$

$\boxed{dV = \pi r^2 dy}$

r is a fraction of R



Similar triangles:

$$\frac{r}{h-y} = \frac{R}{h}$$

$$r = \frac{R}{h}(h-y)$$

$$\boxed{r = R\left(1 - \frac{y}{h}\right)}$$

$$dm = \rho\,\pi r^2 dy = \rho\pi R^2\left(1 - \frac{y}{h}\right)^2 dy$$

$$y_{cm} = \frac{1}{M}\int y\,dm = \frac{\rho\pi R^2}{M}\int_0^h y\left(1 - \frac{y}{h}\right)^2 dy = \frac{\rho\pi R^2}{M}\int_0^h y\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right)dy$$

$$= \frac{\rho\pi R^2}{M}\left[\frac{y^2}{2} - \frac{2}{3h}y^3 + \frac{1}{4h^2}y^4\right]_0^h = \frac{3}{h}\left[\frac{h^2}{2} - \frac{2}{3h}h^3 + \frac{1}{4h^2}h^4\right]$$

$$\rho_{cone} = \frac{M}{Vol} = \frac{M}{\pi R^2 \frac{h}{3}} \rightarrow \frac{\rho\pi R^2}{M} = \frac{3}{h}$$

$$= \frac{3h^2}{h}\underbrace{\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]}_{\frac{1}{12}}$$

$$\boxed{y_{cm} = \frac{h}{4}}$$

Collisions : b/w 2 objects

1D

Inelastic : 1 eq ($P_i = P_f$) 2 unknowns : v_{1f}, v_{2f}

↳ or $v_{1f} = v_{2f}$ (stick together)

↳ or one final vel. is given.

Elastic : 2 eqs ($P_i = P_f$ & $KE_i = KE_f$)

2 unknowns : v_{1f}, v_{2f}

2D

Inelastic : 2 eqs ($P_{ix} = P_{fx}$ & $P_{iy} = P_{fy}$)

Elastic : 3 eqs ($P_{ix} = P_{fx}$ & $P_{iy} = P_{fy}$ & $KE_i = KE_f$)

↳ $\boxed{m_1 = m_2}$ ⇒ $\theta_1 - \theta_2 = 90°$

↳ If θ_1 is given → $\theta_2 = \theta_1 - 90°$

Just calculate v_{1f}, v_{2f} using eqs for 2D elastic collisions