Math descriptions

1D	2D	3D
position x	$\vec{r} = (x, y) = (r, \theta)$	$\vec{r} = (x, y, z) = (r, \theta, \varphi)$
velocity v	$\vec{v} = (v_x, v_y) = (v, \theta_v)$	$\vec{v} = (v_x, v_y, v_z) = (v, \theta_v, \varphi_v)$
acceleration a	$\vec{a} = (a_x, a_y) = (a, \theta_a)$	$\vec{a} = (a_x, a_y, a_z) = (a, \theta_a, \varphi_a)$

$\vec{r}$ = position vector

r : radius ; θ : angle theta ; φ : angle phi

(x, y) or (x, y, z) Cartesian coordinates

(r, θ) : polar coordinates ; (r, θ, φ) : spherical coordinates

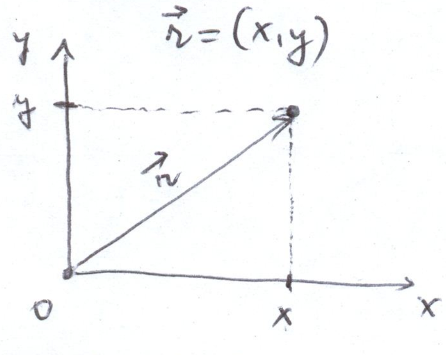
$\vec{v}$: velocity vector

$\vec{a}$: acceleration vector

More details : 2D

Cartesian

$\vec{r} = (x, y)$

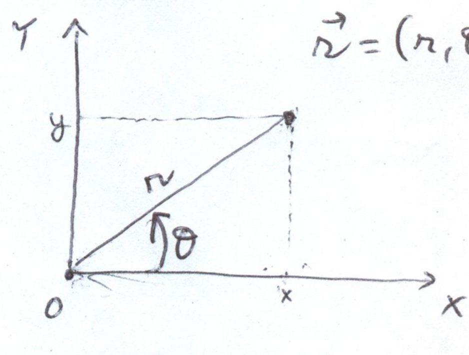


Origin of coordinates

Position defined by $\vec{r}$, its projections onto the axes determine the x & y Cartesian coordinates

Polar

$\vec{r} = (r, \theta)$



r : length or magnitude of position vector $\vec{r}$

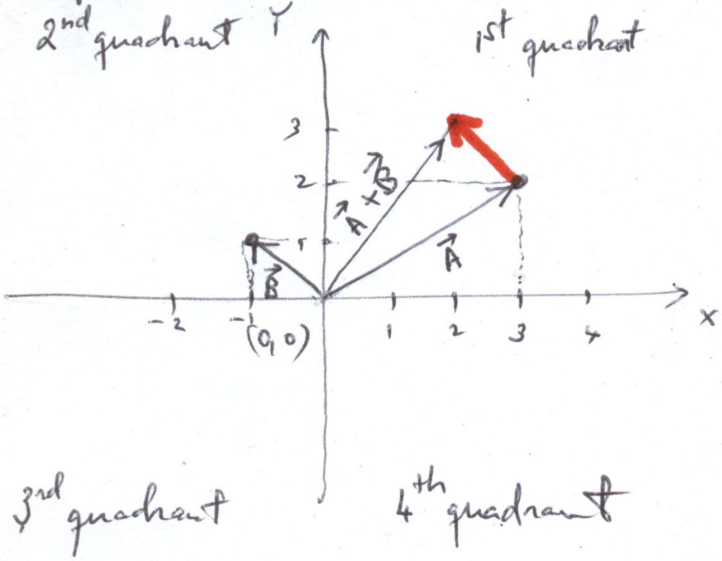
θ : angle from x-axis CCW (convention)

Cartesian $\longrightarrow$ Polar: $\begin{cases} r = \sqrt{x^2+y^2} & \text{Pythagoras Theorem} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) & \text{Trigonometry} \end{cases}$

Polar $\longrightarrow$ Cartesian $\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ Trigonometry

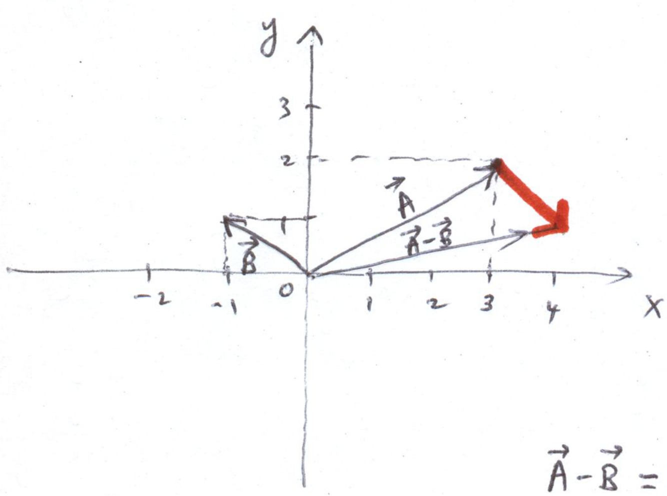
<u>Basic vector operations:</u> addition & subtraction $\begin{cases} \text{Graphically} \\ \text{Mathematically (using unit vectors)} \end{cases}$

<u>Graphical addition & subtraction</u>: $\vec{A}+\vec{B}$ & $\vec{A}-\vec{B}$



$\vec{A} = (3,2)$ (1st quadrant)
$\vec{B} = (-1,1)$ (2nd quadrant)

$\vec{A}+\vec{B}$: 1) Draw a copy of $\vec{B}$ from tip of $\vec{A}$
2) $\vec{A}+\vec{B}$ is from origin of $\vec{A}$ to the tip of the copy of $\vec{B}$
$\vec{A}+\vec{B} = (2,3)$



$\vec{A} = (3,2)$
$\vec{B} = (-1,1)$

$\vec{A}-\vec{B}$: 1) Draw a copy of $\vec{B}$ pointing in opposite direction $(-\vec{B})$
2) $\vec{A}-\vec{B}$ is from origin of $\vec{A}$ to tip of reversed copy of $\vec{B}$

$\vec{A}-\vec{B} = (4,1)$

Math addition & subtraction (vector):

Unit vectors:

$$\begin{cases} \text{vectors with length} = 1 \quad (\text{magnitude} = 1) \\ x\text{-direction}: \quad \hat{i} \quad (i\text{-hat}) \\ y\text{-direction}: \quad \hat{j} \quad (j\text{-hat}) \\ z\text{-direction}: \quad \hat{k} \quad (k\text{-hat}) \end{cases}$$

$$\vec{A} = 3\hat{i} + 2\hat{j} \qquad \left(\vec{A} = (A_x, A_y) = (3, 2) \begin{cases} A_x = 3 \\ A_y = 2 \end{cases}\right)$$

$$\vec{B} = -\hat{i} + \hat{j} \qquad \left(\vec{B} = (B_x, B_y) = (-1, 1) \begin{cases} B_x = -1 \\ B_y = 1 \end{cases}\right)$$

Add:

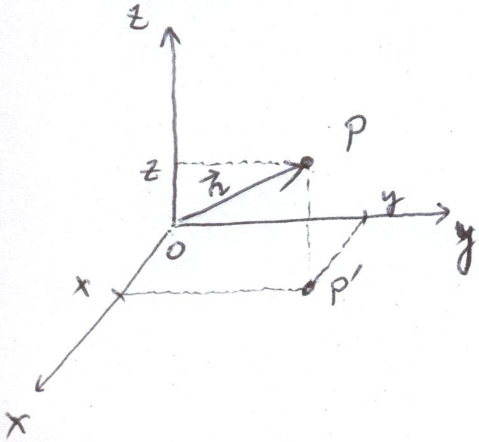
$$\begin{aligned} \vec{A} + \vec{B} &= (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j}) \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \qquad = (3-1)\hat{i} + (2+1)\hat{j} \\ &= 2\hat{i} + 3\hat{j} \end{aligned}$$

Subtract:

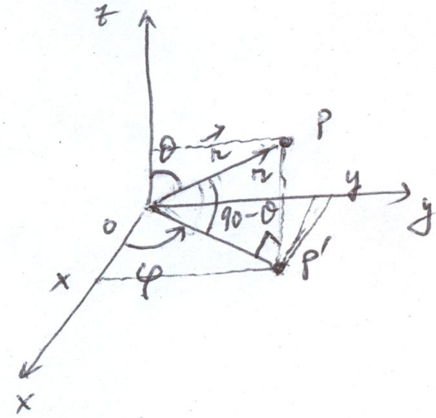
$$\begin{aligned} \vec{A} - \vec{B} &= (A_x\hat{i} + A_y\hat{j}) - (B_x\hat{i} + B_y\hat{j}) \\ &= (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} = (3+1)\hat{i} + (2-1)\hat{j} \\ &= 4\hat{i} + \hat{j} \end{aligned}$$

More details : 3D :

Cartesian : $\vec{r} = (x, y, z)$



Spherical : $\vec{r} = (r, \theta, \varphi)$



Cartesian
- x : coming out of page
- y : to the right
- z : vertical
- P : point in 3D space
  - Projection onto xy plane : P'
    - Projection of P' onto x-axis x
    - Projection onto y-axis y
  - Projection onto z-axis : z

Spherical
- Magnitude r : length of position vector $\vec{r}$
- Angle φ : from x-axis of direction OP' (P' was projection of P onto xy plane)
- Angle θ : of $\vec{r}$ from z-axis

Spherical $\longrightarrow$ Cartesian

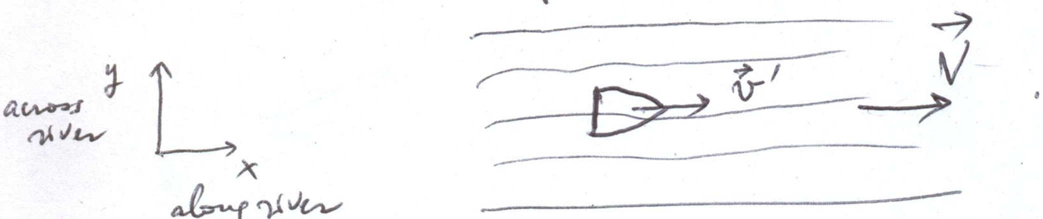
$$
\begin{cases}
x = OP' \cos\varphi = r\sin\theta\cos\varphi \\
y = r\sin\theta \cdot \sin\varphi \\
z = r\cos\theta
\end{cases}
$$

$$OP' = r\cos(90-\theta) = r\sin\theta$$

Relative Motion:

1) Boat going down stream: (faster than in a lake)

Top view (2D)



Velocity of water $\vec{V} = V\hat{i}$ (upper case V)

Velocity of boat wrt water $\vec{v}' = v'\hat{i}$ (lower case v prime)

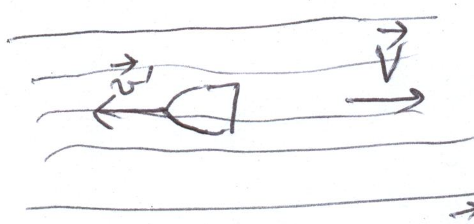
Velocity of boat wrt banks/ground: $\vec{v} = v\hat{i}$

$$\vec{v} = \vec{v}' + \vec{V}$$

$$= v'\hat{i} + V\hat{i} = (v'+V)\hat{i}$$

$$\Rightarrow \quad v = v' + V$$

2) Boat going upstream (slower than in a lake)



Velocity of water = $\vec{V} = V\hat{i}$

Velocity of boat wrt water: $\vec{v}' = -v'\hat{i}$

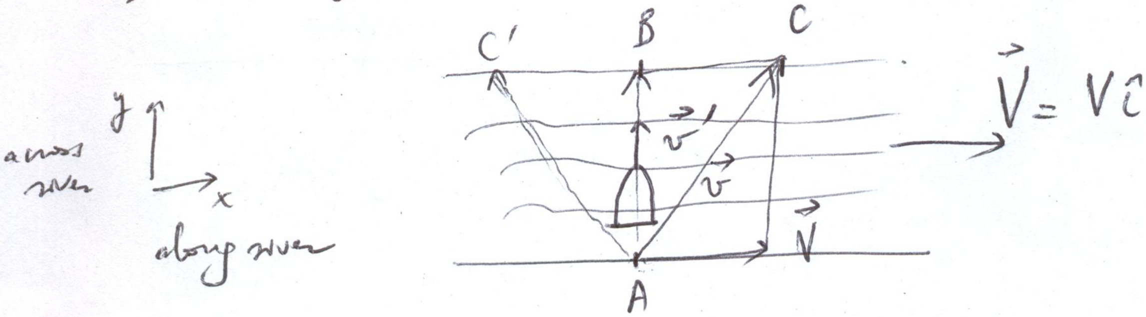
Velocity of boat wrt banks/ground: $\vec{v} = \vec{v}' + \vec{V}$

$$= -v'\hat{i} + V\hat{i}$$

$$= (-v' + V)\hat{i}$$

$$\rightarrow \quad v = -v' + V$$

3) Boat going across river:



→ Velocity of water: $\vec{V} = V\hat{i}$

→ Velocity of boat wrt water: $\vec{v}' = v'\hat{j}$

→ Velocity of boat wrt banks/ground: $\vec{v} = \vec{v}' + \vec{V}$

$$= v'\hat{j} + V\hat{i}$$

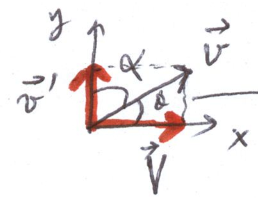
$$= V\hat{i} + v'\hat{j}$$

Conclusions:

→ Total velocity of boat wrt ground has 2 components (when it crosses river)

magnitude: $v = \sqrt{V^2 + v'^2}$

→ Direction of total velocity:



$\theta = \tan^{-1} \dfrac{v'}{V}$ wrt x-axis

$$\alpha = 90 - \theta$$

(direction of boat is θ wrt river banks or $\alpha = 90 - \theta$ wrt cross river direction AB)

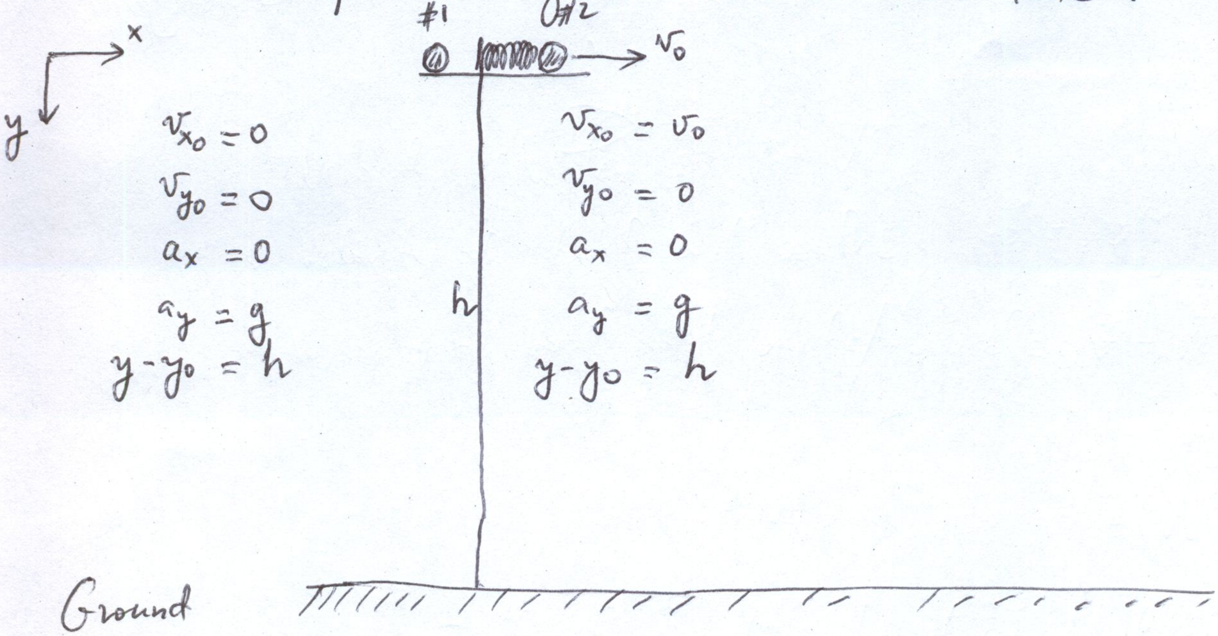
→ If you row the boat aiming straight across river (AB) you would end up @ C (to the right of B)

→ To end up @ B you should row your boat aiming @ C' (symmetrically opposite to C wrt AB) or a little bit upstream to compensate for velocity of water $\vec{V}$

Visual Experiment #2:

Two balls at same height h $\begin{cases} \#1 \text{ will be dropped} \\ \#2 \text{ will have an } \underline{\text{initial}} \\ \quad \underline{\text{horizontal}}\ \underline{\text{velocity}}\ v_0 \end{cases}$

Will they hit the ground at the same time?



__Statement:__ motions along y-direction for both are equal
→ time they take to travel the same vertical distance h will be the same.

Math: Kinematic eq. for constant acceleration (#2)

Ball#1 $h = \cancel{v_{y_0}}^{\,0} \cdot t_1 + \frac{1}{2} g t_1^2$ Ball #2 $h = \cancel{v_{y_0}}^{\,0} \cdot t_2 + \frac{1}{2} g t_2^2$ $\Rightarrow t_1 = t_2$

__Conclusion:__ they both hit the ground @ the same time.

Kinematic Equations for Constant Acceleration in 1D & 2D

$\boxed{1D}$ x, v, a

$v = v_0 + a \cdot t$ (1)

$x = x_0 + v_0 \cdot t + \frac{1}{2} a t^2$ (2)

$\boxed{2D}$ $\vec{r}, \vec{v}, \vec{a}$

$$\left.\begin{aligned} v_x &= v_{0x} + a_x \cdot t \\ v_y &= v_{0y} + a_y \cdot t \end{aligned}\right\} \vec{v} = \vec{v}_0 + \vec{a} \cdot t \quad (1)$$

$$\left.\begin{aligned} x &= x_0 + v_{0x} \cdot t + \frac{1}{2} a_x t^2 \\ y &= y_0 + v_{0y} \cdot t + \frac{1}{2} a_y t^2 \end{aligned}\right\} \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \quad (2)$$

$$\boxed{\begin{aligned} \vec{v} &= \vec{v}_0 + \vec{a} \cdot t \quad (1) \\ \vec{r} &= \vec{r}_0 + \vec{v}_0 \cdot t + \frac{1}{2} \vec{a} t^2 \quad (2) \end{aligned}}$$

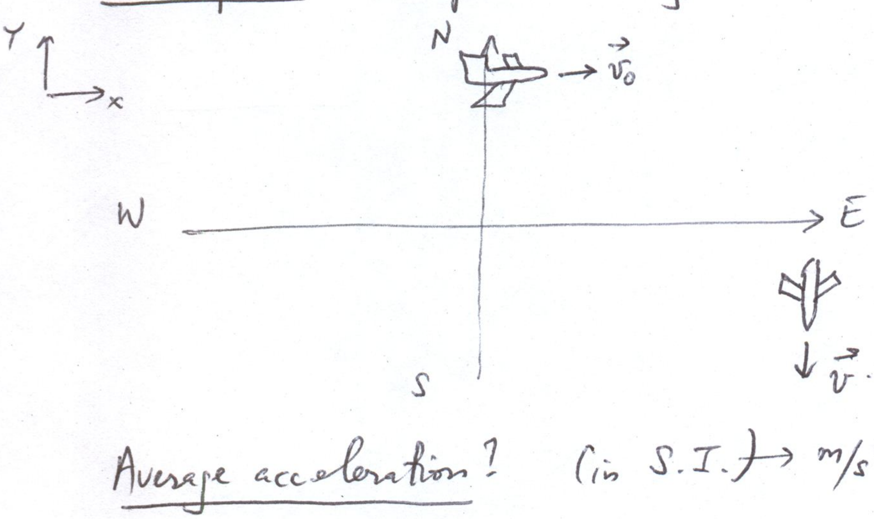
Position vector: $\vec{r} = x\,\hat{i} + y\,\hat{j}$

Velocity vector $\vec{v} = v_x\,\hat{i} + v_y\,\hat{j} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$

Acceleration vector $\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$

$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$

Example: Airplane taking a turn (2D motion)



Initially flying eastward with $\vec{v}_0 = 2100 \frac{km}{h}\hat{i}$

After 2.5 min it turned southward with final velocity $\vec{v} = -1800 \frac{km}{h}\hat{j}$

Average acceleration? (in S.I.) $\rightarrow m/s^2$ $\rightarrow$ conversions

$2100 \frac{km}{h} \cdot \frac{1000\,m}{km} \cdot \frac{h}{3600\,s} = \frac{2100}{3.6} \frac{m}{s} = 583.3 \frac{m}{s}$

Similarly $\frac{1800}{3.6} = 500 \frac{m}{s}$; $\Delta t = 2.5\ min = 150\,s$

→ Average acceleration vector (2D):

$$\vec{\bar{a}} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v} - \vec{v}_0}{\Delta t} = \frac{-500\hat{j} - 583.3\hat{i}}{150} = -3.9\hat{i} - 3.3\hat{j} \left(\frac{m}{s^2}\right)$$

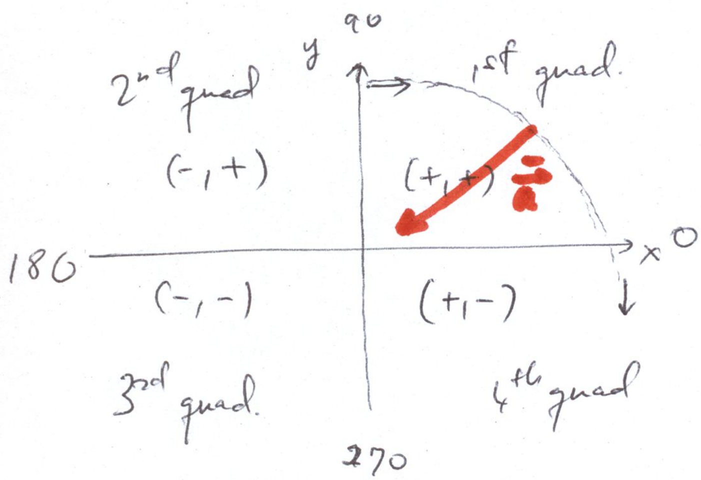
$$\vec{\bar{a}} = \begin{cases} \rightarrow \text{Cartesian components} \begin{cases} \bar{a}_x = -3.9 \ m/s^2 \\ \bar{a}_y = -3.3 \ m/s^2 \end{cases} \\ \rightarrow \text{Polar components.} \begin{cases} \bar{a} = \sqrt{3.9^2 + 3.3^2} = 5.1 \frac{m}{s^2} \\ \theta = \tan^{-1}\frac{\bar{a}_y}{\bar{a}_x} = \tan^{-1}\left(\frac{-3.3}{-3.9}\right) = 40.24^\circ \end{cases} \end{cases}$$

Magnitude of the average acceleration vector

→ 1st quad.

→ $\boxed{\text{Correction}}$: add 180°

→ Calculators can't distinguish quadrants!



⇒ Correct θ = 40.24° + 180°
= 220.24°
(3rd quad.).

Our average accel. vect. points in the third quadrant! ($\bar{a}_x < 0$, $\bar{a}_y < 0$)

Projectile Motion:

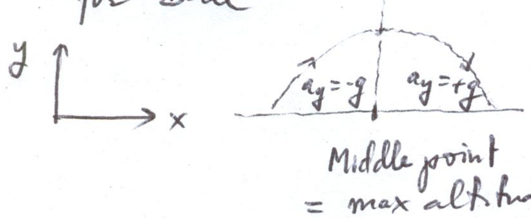
New physics? New equations? No, just a particular type of 2D motion of constant acceleration $\begin{cases} a_x = 0 \\ a_y = \pm g \end{cases}$

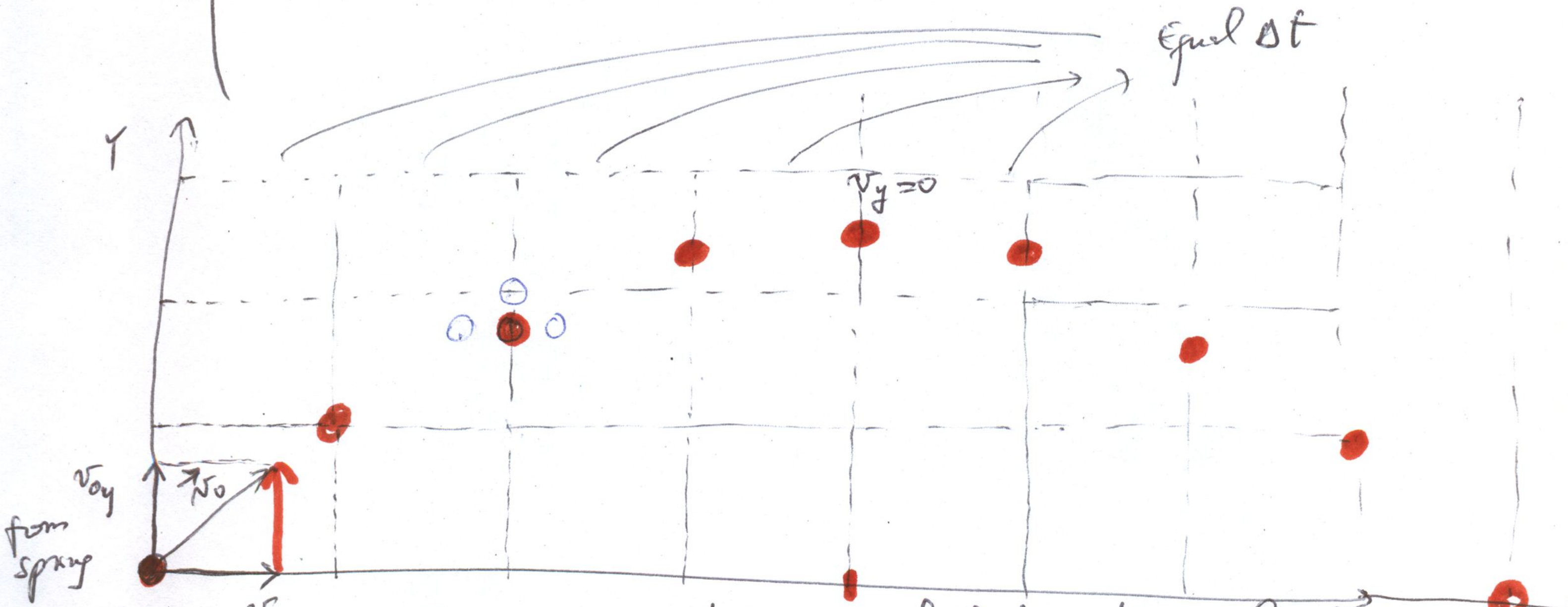
- Examples: baseball, basketball, soccer ball, bullets, short-range missiles (ground is flat); water from a sprinkler, etc...

  → basically any object with an initial velocity & let go under effect of gravity.

- Example we will focus on: ball ejected upward from rolling cart (visual experiment #1) →

Statements $\begin{cases} \text{parabolic trajectory for ball} : \vec{r} \begin{cases} x: \text{uniform motion (constant velocity)} \\ y: \text{constant acceleration } a_y = \begin{cases} -g & \text{upward} \\ +g & \text{downward} \end{cases} \end{cases} \end{cases}$





Statements $\begin{cases} \Delta x \text{ is constant due to horizontal uniform } x \text{ motion} \\ \Delta y \text{ is smaller on way up \& bigger on way down.} \end{cases}$

Equations for projectile motion: (no new equations! only 2D kinematic equations for constant acceleration $\vec{a}\begin{cases} a_x = 0 \\ a_y = \pm g \end{cases}$

$y \uparrow \rightarrow x$ $\left(a_y = \begin{cases} -g & \text{up} \\ +g & \text{down} \end{cases}\right)$

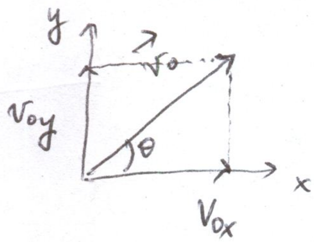
1) $$\vec{v} = \vec{v}_0 + \vec{a}\cdot t \quad \begin{cases} v_x = v_{0x} \\ v_y = v_{0y} \mp g\cdot t & \begin{cases} - : \text{upward part} \\ + : \text{downward part} \end{cases} \end{cases}$$

2) $$\vec{r} = \vec{r}_0 + \vec{v}_0\cdot t + \frac{1}{2}\vec{a}\cdot t^2 \quad \begin{cases} x = v_{0x}\cdot t \\ y = v_{0y}\cdot t \mp \frac{1}{2} g t^2 & \begin{cases} - : \text{upward part} \\ + : \text{downward part} \end{cases} \end{cases}$$

($\vec{r}_0 = (0,0)$ @ origin)

Easy & useful measurement in practical applications of projectile

Introduce: $\boxed{\text{aim angle } \theta}$ or the angle of the initial velocity $\vec{v}_0$



$$\vec{v}_0 = \begin{cases} v_{0x} = v_0 \cos\theta \\ v_{0y} = v_0 \cdot \sin\theta \end{cases}$$

2) $$\begin{cases} x = v_{0x}\cdot t = v_0 \cos\theta \cdot t \longrightarrow t = \dfrac{x}{v_0 \cos\theta} \\ y = v_0 \sin\theta \cdot t \mp \frac{1}{2} g t^2 \end{cases}$$

$$y = \cancel{v_0} \sin\theta \cdot \frac{x}{\cancel{v_0}\cos\theta} \mp \frac{1}{2} g \cdot \frac{x^2}{v_0^2 \cos^2\theta}$$

$$\boxed{y = x \tan\theta \mp \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\theta}}$$

Trajectory equation for a projectile motion

If θ & v_0 are known, this equation determines pairs of (x, y) which gives the trajectory.

Maximum altitude point: $(x_{max}, y_{max}) = \left(\frac{v_0^2 \sin 2\theta}{2g}, \frac{v_0^2 \sin^2\theta}{2g} \right)$

Proof: From kinematic eqs in 2D for constant acceleration:

y_{max}

Eq 1) $v_y = \underbrace{v_0 \sin\theta}_{v_{0y}} - gt$ (upward part)

@ max altitude: $v_y = 0 = v_0 \sin\theta - gt \rightarrow \boxed{t_{max} = \frac{v_0 \sin\theta}{g}}$

Eq 2) $y_{max} = \underbrace{v_0 \sin\theta}_{v_{0y}} \cdot \underbrace{\frac{v_0 \sin\theta}{g}}_{t_{max}} - \frac{1}{2} \cancel{g} \underbrace{\frac{v_0^2 \sin^2\theta}{g^{\cancel{2}}}}_{t_{max}^2} = \frac{1}{2} \frac{v_0^2 \sin^2\theta}{g}$

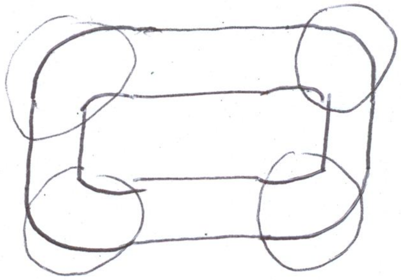
$x_{max} = v_{0x} \cdot t_{max} = v_0 \cos\theta \cdot \frac{v_0 \sin\theta}{g} = \frac{v_0^2 \cos\theta \sin\theta}{g} = \frac{v_0^2 \sin 2\theta}{2g}$

Trigonometry: $\cos\theta \sin\theta = \frac{1}{2} \sin 2\theta$

Uniform Circular Motion : UCM : circular motion with constant speed! (not constant velocity)

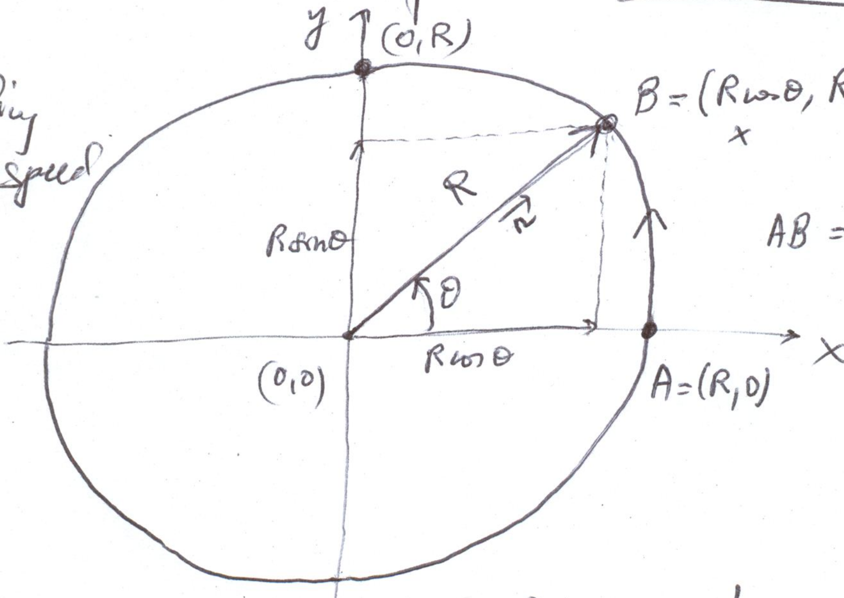
velocity: includes direction $\vec{v} = (\underset{\substack{\downarrow \\ \text{speed}}}{v}, \theta_v)$ → an object can have constant speed but changing angle → changing velocity. This is what happens in UCM : the speed is uniform along circular trajectory however as direction is changing → velocity is changing!

2D Motions
- → Projectile motion
- → UCM



Changing velocity: ⟶ in direction, not in magnitude, but still requires an <u>acceleration</u>

UCM
Object traveling @ uniform speed v



AB = arc → $\left[\theta = \frac{\text{arc}}{R} = \frac{v.t}{R}\right]$

@ time t, object is @ B defined by position vector $\vec{r} = x\hat{i} + y\hat{j}$

$$\vec{r} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j} = R\cos\left(\frac{v.t}{R}\right)\hat{i} + R\sin\left(\frac{v.t}{R}\right)\hat{j}$$

UCM =@t $\quad \vec{r} = R \cdot \cos\left(\frac{v \cdot t}{R}\right)\hat{i} + R\sin\left(\frac{v \cdot t}{R}\right)\hat{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \cancel{R}\left[-\frac{v}{\cancel{R}}\sin\left(\frac{v \cdot t}{R}\right)\hat{i} + \frac{v}{\cancel{R}}\cos\left(\frac{v \cdot t}{R}\right)\hat{j}\right]$$

$\boxed{\vec{v} \text{ is changing over time}}$

$$v = |\vec{v}| = \left| v\left(-\sin\left(\frac{v \cdot t}{R}\right)\hat{i} + \cos\left(\frac{v \cdot t}{R}\right)\hat{j}\right)\right|$$

$$= v \underbrace{\sqrt{\left(-\sin\frac{v \cdot t}{R}\right)^2 + \left(\cos\frac{v \cdot t}{R}\right)^2}}_{\sin^2\alpha + \cos^2\alpha = 1} = \text{constant}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -v\frac{d}{dt}\left[\sin\left(\frac{v \cdot t}{R}\right)\hat{i} - \cos\left(\frac{v \cdot t}{R}\right)\hat{j}\right]$$

$$= -v\left[\frac{v}{R}\cos\left(\frac{v \cdot t}{R}\right)\hat{i} + \frac{v}{R}\sin\left(\frac{v \cdot t}{R}\right)\hat{j}\right]$$

$$= -\frac{v^2}{R}\underbrace{\left[\cos\left(\frac{v \cdot t}{R}\right)\hat{i} + \sin\left(\frac{v \cdot t}{R}\right)\hat{j}\right]}_{\text{Magnitude} = 1}$$

$$\boxed{|\vec{a}| = \frac{v^2}{R}}_{UCM}$$

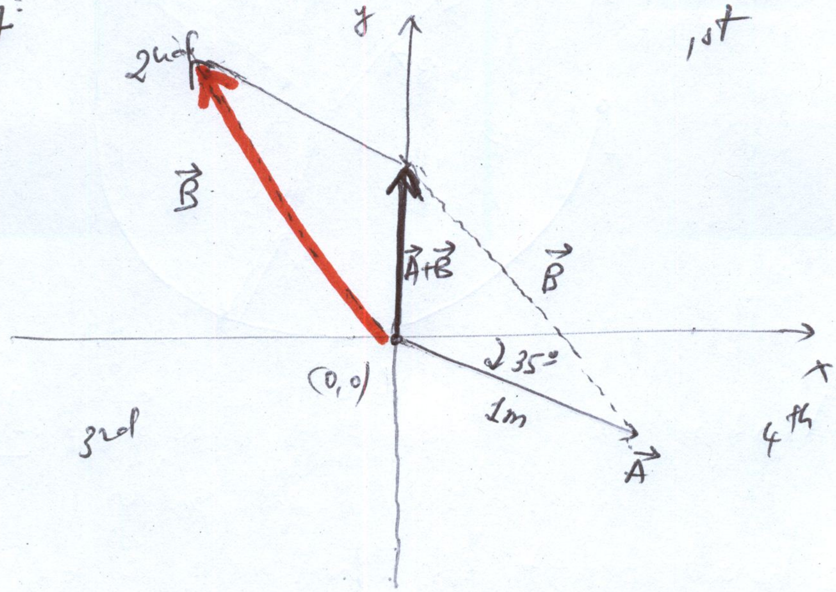
Acceleration connected with the change of direction of velocity.

3.42 **Vector addition in 2D**

$$\begin{cases} \vec{A} = (1m,\ 35^\circ \text{ CW from } x\text{-axis}) & \text{Polar} \\ \vec{B} = (1.8m,\ \theta) \end{cases}$$

θ ↑ so that $\vec{A}+\vec{B}$ is in y-direction

Graphically:



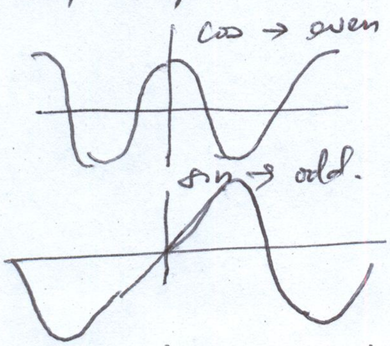
Statement: $\vec{B}$ will be in 2nd quad.
So diagonal of quadrilateral of sides $\vec{A}$ & $\vec{B}$ will point along + y-axis

Mathematically:
→ Cartesian coords are best suited for addition & subtraction
→ Polar coords " " " " multiplication & division.

$$\vec{A} = (1, -35^\circ) = (1\cos(-35^\circ),\ 1\sin(-35^\circ)) = (\cos 35^\circ, -\sin 35^\circ)$$

$$\vec{A} = \cos 35^\circ\,\hat{i} - \sin 35^\circ\,\hat{j}$$

Review:



$$\vec{B} = (1.8, \theta) = 1.8\cos\theta\,\hat{i} + 1.8\sin\theta\,\hat{j}$$

$$\vec{A}+\vec{B} = (\cos 35^\circ + 1.8\cos\theta)\hat{i} + (-\sin 35^\circ + 1.8\sin\theta)\hat{j}$$

For $\vec{A}+\vec{B}$ to be along y-direction: $\Rightarrow$ x-component should be zero: $\cos 35^\circ + 1.8\cos\theta = 0$

$$\underbrace{\cos\theta = -\frac{\cos 35^\circ}{1.8}} \rightarrow \theta = \cos^{-1}\left[-\frac{\cos 35^\circ}{1.8}\right]$$

$$= 117^\circ \quad (2^{nd}\text{ quad}).$$

Now if $\vec{A}+\vec{B}$ points in $(-)$y-direction $\theta = -117^\circ$ (3^{rd} quad.).

b/c: cos is an even function

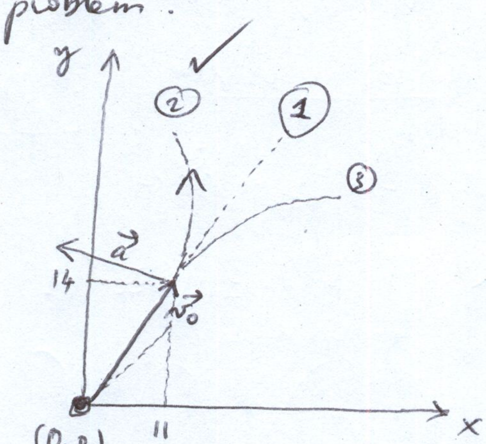
($\cos 117^\circ = \cos(-117^\circ)$)

(3.54) **Statement:** particle undergoing constant acceleration 2D.

Info: $\begin{cases} \vec{v}_0 = 11\hat{i} + 14\hat{j}\ \frac{m}{s} & @\ \vec{r} = (0,0) \text{ origin} \\ \vec{a} = -1.2\hat{i} + 0.26\hat{j}\ \frac{m}{s^2} \end{cases}$

a) When does particle cross y-axis?

Does this question make sense? Answer will help understand this problem.



→ $\vec{v}_0$ is more vertical than horizontal, will stay along direction (1) unless there is a change of velocity (including direction)

→ $\vec{a}$ points to 2^{nd} quad. → particle will bend to the left.

→ Yes it will cross y-axis @ some point

Statement: t can be calculated using kinematic eqs for constant acceleration in 2D eqs 1 and/or 2

Eq 1: $\vec{v} = \vec{v_0} + \vec{a} \cdot t$

Eq 2: $\vec{r} = \vec{r_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$ ← since we have some info on final position: $\boxed{x=0}$ when it crosses y-axis.

$(x_0, y_0) = (0,0)$

$$\begin{cases} x = 0 = 0 + v_{0x} \cdot t + \frac{1}{2} a_x t^2 \rightarrow 11 \cdot \cancel{t} + \frac{1}{2}(-1.2) t^{\cancel{2}} = 0 \rightarrow t = \frac{22}{1.2} s \\ y = 0 + v_{0y} \cdot t + \frac{1}{2} a_y t^2 \end{cases}$$

$\boxed{t = 18.3 s}$

@ this time particle crosses y-axis.

b) Particle y position @ $t = 18.3 s$

$$y = 14 \times 18.3 + \frac{1}{2} \times 0.26 \times 18.3^2 = 300 \text{ m}$$

c) Final $\vec{v}$ @ $t = 18.3 s$

$$\vec{v} = 11\hat{i} + 14\hat{j} + (-1.2\hat{i} + 0.26\hat{j}) \times 18.3$$

$$= (11 - 1.2 \times 18.3)\hat{i} + (14 + 0.26 \times 18.3)\hat{j} \text{ m/s}$$

$$= -10.96\,\hat{i} + 18.8\hat{j} \text{ m/s} \quad (2^{nd} \text{ quad.})$$

Magnitude & direction of $\vec{v} = (v, \theta)$

$$= \Big(\underbrace{\sqrt{(-10.96)^2 + 18.8^2}}_{21.7},\ \underbrace{\tan^{-1}\left(\frac{18.8}{-10.96}\right)}_{-60^\circ}\Big)$$

-60° **+180°**

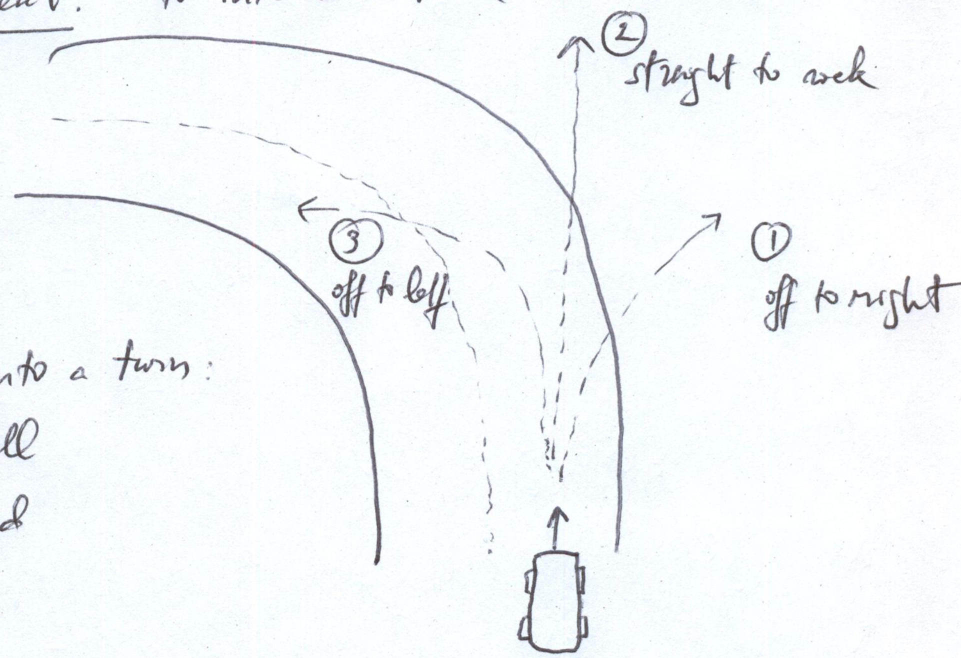
$$\vec{v} = \left(21.7 \tfrac{m}{s},\ 120^\circ\right)$$

Ch 4 Motion & Forces

<u>description</u>: $\vec{r}, \vec{v}, \vec{a}, t$ $\qquad$ $\vec{F}$

<u>Statement</u>: Force is the agent that causes the accel. $\vec{a}$ or a change of motion

<u>Visual experiment</u>: to introduce $\vec{F}$ & its connection with $\vec{a}$

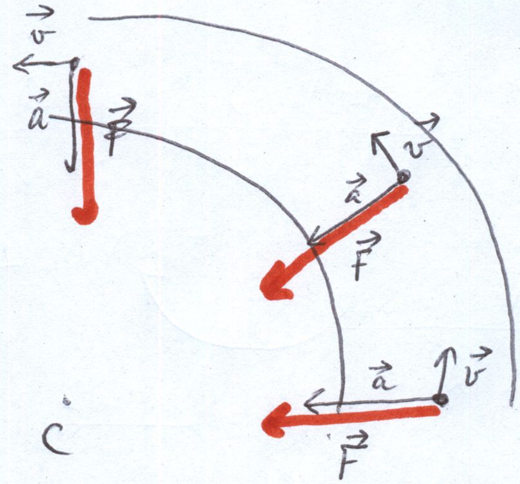


Driving a car into a turn:
- Downhill
- Icy road

Vehicle will follow path 2. Why? Lack of acceleration toward center of curvature b/c lack of agent or force to provide that acceleration which is the friction b/w tires & road.

<u>Conclusion</u>: vehicle entering a curve in forward direction will continue to do so if there is no force or agent that changes its direction.

<u>A force is needed to change a motion!</u>



Conclusion: 1) Force is a vector (it changes direction)

2) Force is agent to change direction of $\vec{v}$

Newton's Laws:

$\boxed{1^{st}}$: a body at rest will continue at rest, a body in uniform motion will continue in uniform motion unless there is a net force acting on the body.

Law of inertia

$\boxed{2^{nd}}$ $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$ $\left\{ \begin{array}{l} \vec{p}: \text{ linear momentum } \vec{p} = m\vec{v} \quad (kg\,\frac{m}{s}) \\ \vec{F}_{net} = \text{ superposition of all forces involved.} \end{array} \right.$

$$\vec{F}_{net} = \frac{d(m\vec{v})}{dt} = \underbrace{\frac{dm}{dt}}_{\text{important when mass is changing over time}}\vec{v} + m\underbrace{\boxed{\frac{d\vec{v}}{dt}}}_{\vec{a}}$$

If m is constant:

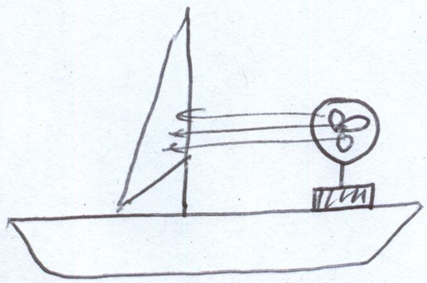
$$\boxed{\vec{F}_{net} = m.\vec{a}}$$

$$[F] = \frac{ML}{T^2} \xrightarrow{SI} kg\frac{m}{s^2} \equiv N \text{ (Newton)}$$

$\boxed{3^{rd}}$ Law of action & reaction:

If A exerts a force on B, B exerts an equal and opposite force on A.

Sailing without wind: Law of action & reaction



Info: → fan is fixed on boat

→ blows air on sail

Will boat move forward?

Yes? → Fan blows air molecules which in turn push sail (pushes)

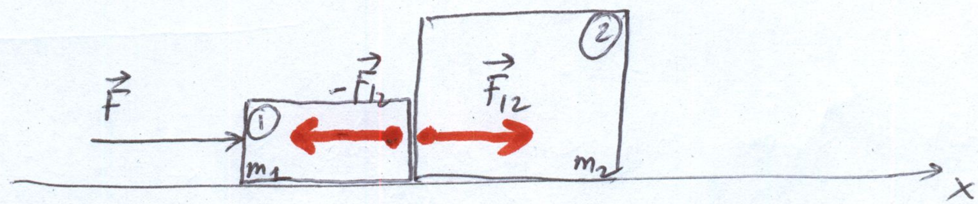
→ Law of action & reaction: air molecules push back on fan same force in opposite direction.

Since fan is attached to boat → air pushes back on boat → $\vec{F}_{net} = 0$

No.

1) Two boxes next to each other on a horizontal surface (no friction) force $\vec{F}$ is applied on box ① causing system of ① & ② to accelerate in x-direction : $\vec{F} = (m_1 + m_2)\vec{a}$

$\vec{a}$



a) What force is applied on box ② ? (Net force)

Can it be $\vec{F}$? $\boxed{\text{No}}$ $\vec{a}_1 = \vec{a}_2 = \vec{a}$

if $\vec{F} = (m_1 + m_2)\vec{a}$ it can't be also $\vec{F} = m_2\vec{a}$

→ What force makes m_2 move? $\vec{F}_{12}$: force applied by box ① on box ②. Without friction this is also the net force on box ② ⇒ $\boxed{F_{12} = m_2 a}$

b) What is the net force on box ① ?

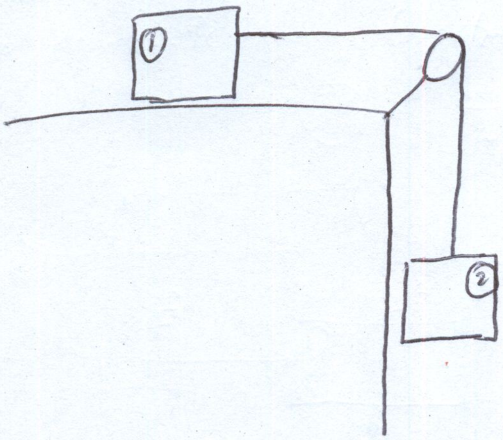
$$F_{net①} = \boxed{F - F_{12} = m_1 a}$$

c) Summary:

Net/Total force on system	Net force on ①	Net force on ②
F	$F - F_{12}$	F_{12}

Note: Net force on ① + Net force on ② = $F - F_{12} + F_{12} = F$ = Net force on system (F_{12} & $-F_{12}$ are internal forces for this system)

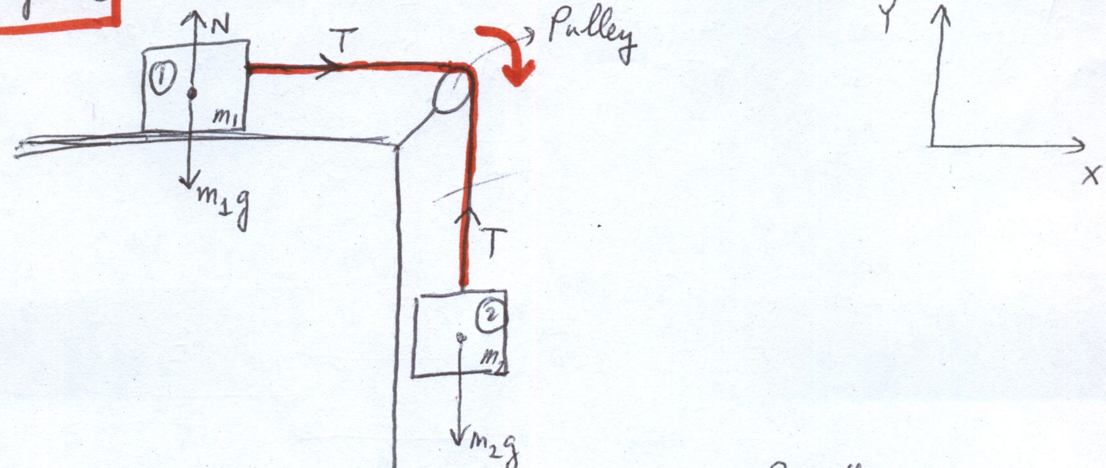
2) Two boxes connected by a massless string/rope (no friction)
Net force on each box.



2) Two boxes connected by a $\boxed{\text{massless string/rope}}$ (no friction)

sufficiently small compared to m_1 & m_2

$\boxed{\text{Net force}}$ on each box.



Box #1

Forces acting on this box:

$\begin{cases} \text{weight} & m_1 g \\ \text{tension} & T \\ \text{normal} & N \text{ (by table)} \end{cases}$

Net force = F_{net_1}

on #1 $\begin{cases} \rightarrow \text{ in } x: & T \\ \rightarrow \text{ in } y: & N - m_1 g \end{cases}$

2nd Newton's Law:

$$\vec{F}_{net_1} = m_1 \cdot \boxed{\vec{a}} \begin{cases} T = m_1 \cdot a \quad (1) \\ N - m_1 g = 0 \end{cases}$$

Box # 2

$\begin{cases} \text{weight} & m_2 g \\ \text{tension} & T \text{ (same throughout due to massless rope)} \end{cases}$

Net force: F_{net_2}

on #2 $\begin{cases} \text{in } x = & 0 \\ \text{in } y = & T - m_2 g \end{cases}$

2nd Newton's Law:

$$\vec{F}_{net_2} = m_2 \cdot \boxed{\vec{a}} \begin{cases} x = \text{no motion} \\ T - m_2 g = - m_2 a \quad (2) \end{cases}$$

not in the same directions
but same magnitude since boxes
are connected by the rope
(box #1 goes right & box #2 goes $\boxed{\text{down}}$)

These equations allow us to solve ~~for~~ any situation:

Example: given m_1 & m_2 find a & T:

→ To calculate a: plug eq (1) into eq (2)

$$m_1 a - m_2 g = - m_2 a \Rightarrow (m_1 + m_2) a - m_2 g = 0$$

$$\Rightarrow \boxed{a = \frac{m_2}{m_1 + m_2} g}$$

$\boxed{\text{Check:}}$ acceleration for m_2 is $\underbrace{\frac{m_2}{m_1+m_2}}_{<1} g < g$ slower than free fall why?

→ To calculate T: eq (1) : $\boxed{T = m_1 a = \frac{m_1 m_2}{m_1 + m_2} g}$

$\boxed{\text{Check:}}$ $a = \frac{m_2}{m_1 + m_2} g$

1) If we double the masses $\left.\begin{array}{l} m_1 \to 2m_1 \\ m_2 \to 2m_2 \end{array}\right\} \Rightarrow$ same acceleration

2) If we double m_2 only : $a' = \frac{2m_2}{m_1 + 2m_2} g > a$

$$a = \frac{2m_2}{2m_1 + 2m_2} g$$

Spring forces:

Hooke's Law:

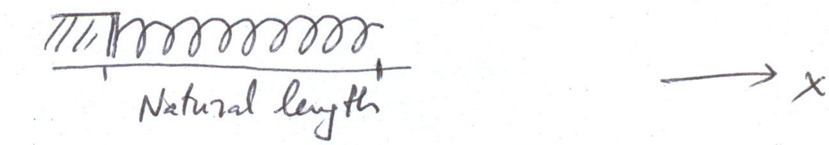
$$F_s = - k \underbrace{\Delta x}$$

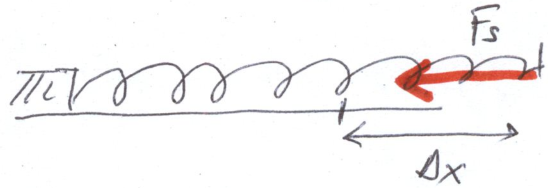
change of length from the natural length } or stretch or compression

$-$: resistance to stretch/compression

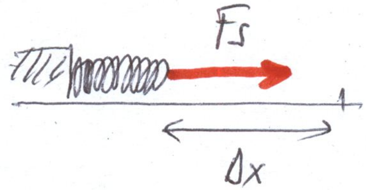
k: spring constant ($\frac{N}{m}$ in S.I.)

Horizontal:

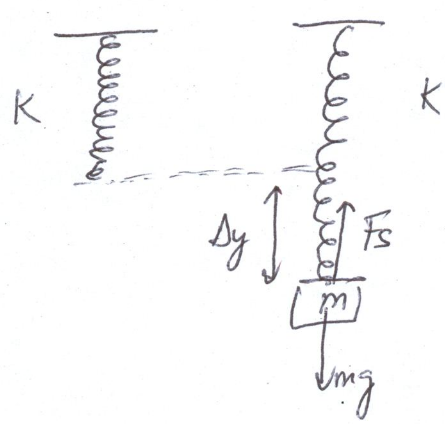




$F_s = -k\Delta x$



Vertical:



If m is static:

$F_s - mg = m \cdot 0$

$F_s = mg$

$+k\Delta y = mg$

$$\boxed{\Delta y = \frac{mg}{k}}$$

F_s in $+y$

<u>Friction forces</u> : are present whenever an object is in contact with a surface

↳ 2 types. { static friction : in contact but not in relative motion
$F_s = \mu_s N$

Kinetic friction : in contact & in relative motion
$F_k = \mu_k N$

For a same object & surface:
- → μ_s : coeff. of static friction (a number w/o units)
- N : normal force by surface on object
- → μ_k : coeff of kinetic friction

<u>Microscopically : b/w bottom of object & the surface :</u>

static

if we look close enough : roughness on any surface

kinetic

$\mu_s > \mu_k$

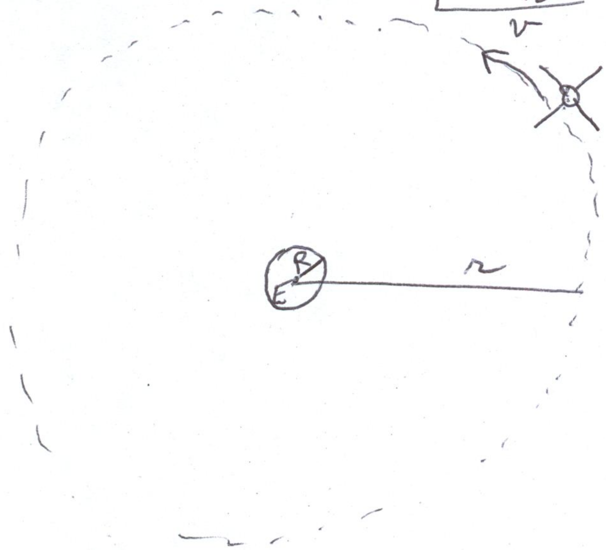
When we ~~move~~ push heavy boxes, after we overcome the static friction : the box acquires an acceleration $F_s - F_k = m \cdot a$

3.40 → Orbital period of GPS satellite @ 20,000 km above surface
$g' = 0.058g$

Statement: 1) UCM constant speed v → $\boxed{a = \frac{v^2}{r}}$

2) Separation to center of circular trajectory:

$r = 20{,}000\,km + 6{,}370\,km$
$= 26{,}370\,km$



$R_E = 6370\,km$

Orbital period: time to complete one orbit or one turn:

$$T = \frac{2\pi r}{v}$$

$$g' = \frac{v^2}{r}$$ (g' allows satellite to follow circular orbit)

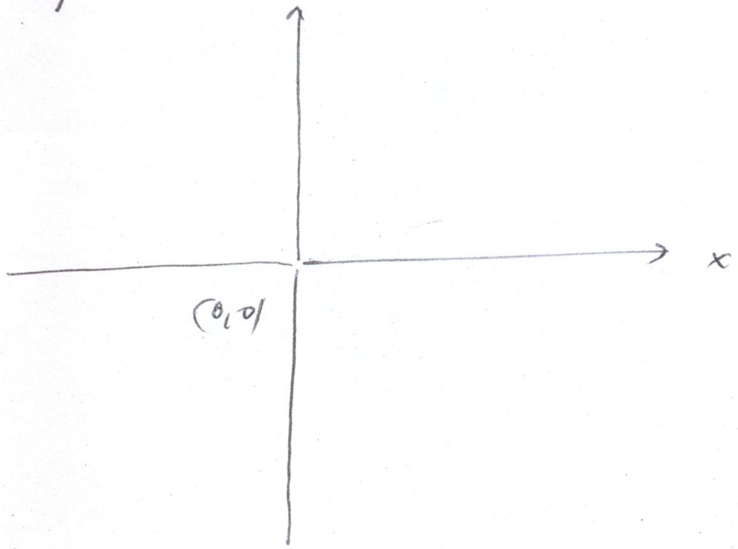
$$v = \sqrt{g'r}$$

$$T = \frac{2\pi r}{\sqrt{g'r}} = 2\pi\sqrt{\frac{r}{g'}} = 2\pi\sqrt{\frac{2.637\times10^7}{0.058\times9.81}} = 42774\,s$$

$$T = \frac{42774}{3600}\,hr = 11.88\,hrs \simeq 12\,hrs.$$

3.45 $\vec{r} = (ct^2 - 2dt^3)\hat{i} + (2ct^2 - dt^3)\hat{j}$ $\qquad c, d > 0$

a) Find $t > 0$ when particle will be moving in x-direction



$y = 0$

$2ct^2 - dt^3 = 0$

or $\quad 2c - dt = 0$

$$\boxed{t = 2\frac{c}{d}}$$

@ this time it will be crossing the x-axis

$$\vec{v} = \frac{d\vec{r}}{dt} = (2ct - 6dt^2)\hat{i} + (\underbrace{4ct - 3dt^2}_{v_y = 0})\hat{j}$$

$4ct - 3dt^2 = 0$

$4c - 3dt = 0$

$$\boxed{t = \frac{4}{3}\frac{c}{d}}$$ ✓

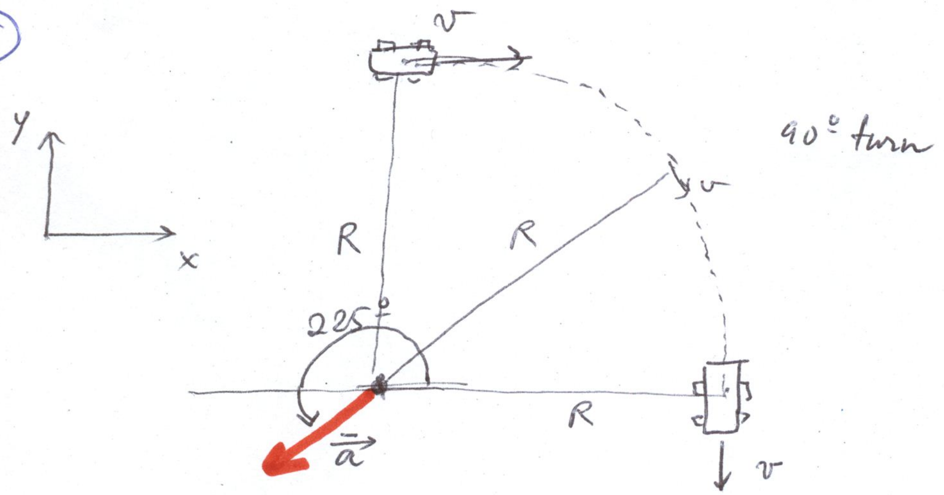
@ this time it will be moving in the x-direction.

b) It will be moving in y-direction:

$v_x = 2ct - 6dt^2 = 0$

$2c - 6dt = 0$ or $\boxed{t = \frac{1}{3}\frac{c}{d}}$

3.22



speedometer reading constant :→ UCM
↕
$a = \frac{v^2}{R}$

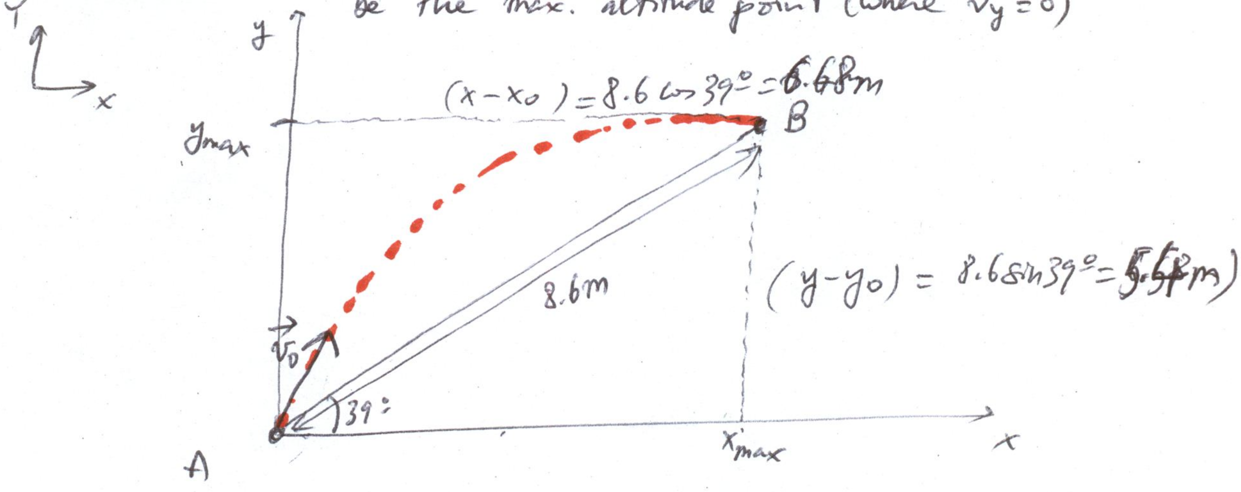
Direction of car average acceleration vector?

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{-v\hat{j} - v\hat{i}}{\Delta t} = \frac{1}{\Delta t}(-v\hat{i} - v\hat{j}) \quad \boxed{3^{rd} \text{ quad.}}$$

$$\theta_{\vec{a}} = \tan^{-1}\left(\frac{-\frac{v}{\Delta t}}{-\frac{v}{\Delta t}}\right) = \tan^{-1}\left(\frac{-1}{-1}\right) = 45° + 180°$$
$$= 225°$$

3.62

Statement: projectile motion for protein bar; B needs to be the max. altitude point (where $v_y = 0$)



We need to find $\vec{v_0}$ for bar as it leaves A. θ_{v_0} has to be larger than 39°!

Alternative #1:

$$\begin{cases} x_{max} = \dfrac{v_0^2 \sin 2\theta_{v_0}}{2g} = 8.6 \cos 39° = 6.68\,m \\ y_{max} = \dfrac{v_0^2 \sin^2 \theta_{v_0}}{2g} = 8.6 \sin 39° = 5.4\,m \end{cases}$$

Two eqs with 2 unknowns v_0 & θ_{v_0} → can solve.

Alternative #2: remember eqs for x_{max} & y_{max} were derived from kinematic eqs for constant acceleration in 2D!

$$\text{Eq 3:} \begin{cases} 3a)\quad \dfrac{v_x^2 - v_{0x}^2}{(x-x_0)} = 2 \cdot a_x \quad (a_x = 0) \\ 3b)\quad \dfrac{v_y^2 - v_{0y}^2}{(y-y_0)} = 2 \cdot a_y \quad (a_y = -g \text{, 1st half of parabola: upward motion}) \end{cases}$$

$v_y = 0$ (@ B)

$$3b)\quad \frac{0 - v_{0y}^2}{5.4} = -2 \times 9.81 \Rightarrow v_{0y} = \sqrt{2 \times 9.81 \times 5.4} = 10.3\,\frac{m}{s}$$

Now find v_{0x} (initial vel. in x) $= v_x$ (uniform motion in x!)

Note: ball needs to go 6.68m in x-direction in same time

Statement: it needs to go 5.4m in y-direction!

↓

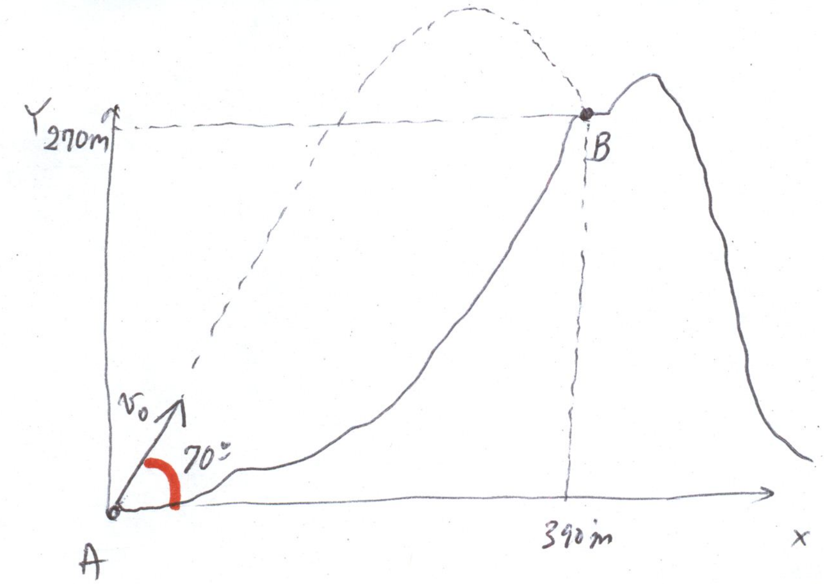
$$v_y = 0 = v_{0y} - g \cdot t \Rightarrow t = \frac{v_{0y}}{g} = \frac{10.3}{9.81}$$

$=$

$$\Rightarrow v_{0x} = \frac{6.68}{\frac{10.3}{9.81}} = 6.36\ m/s$$

$$\Rightarrow \vec{v}_0 = 6.36\,\hat{i} + 10.3\,\hat{j}\ \frac{m}{s} \xrightarrow{\text{Polar}} \begin{cases} v_0 = \sqrt{6.36^2 + 10.3^2} = 12.1\ m/s \\ \theta_{v_0} = \tan^{-1} \dfrac{10.3}{6.36} = 58.3° \quad > 39° \checkmark \end{cases}$$

3.70



Statement: projectile motion for medical packet; B is a point on parabola (being A the initial point)

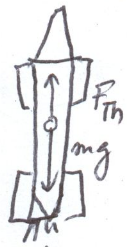
Trajectory eq:
$$y_B = x_B \tan\theta_{v_0} - \frac{g}{2}\frac{x_B^2}{v_0^2 \cos^2\theta_{v_0}}$$

Solve for v_0:
$$v_0^2 = \frac{g}{2}\frac{x_B^2}{(x_B \tan\theta_{v_0} - y_B)\cos^2\theta_{v_0}}$$

$$v_0 = \sqrt{\frac{9.81}{2}\frac{390^2}{(390\tan 70° - 270)\cos^2 70}} = 89.2\,\frac{m}{s}$$

4.55 Statement: Application of Newton's Law.

a)



$y \uparrow$

$\downarrow a = 1.4g$

$$F_{net} = F_{th} - mg = -m \times 1.4g$$

$$F_{th} = (-1.4 + 1)mg$$

$$= -0.4mg$$

b)

$\uparrow a = 1.4g.$

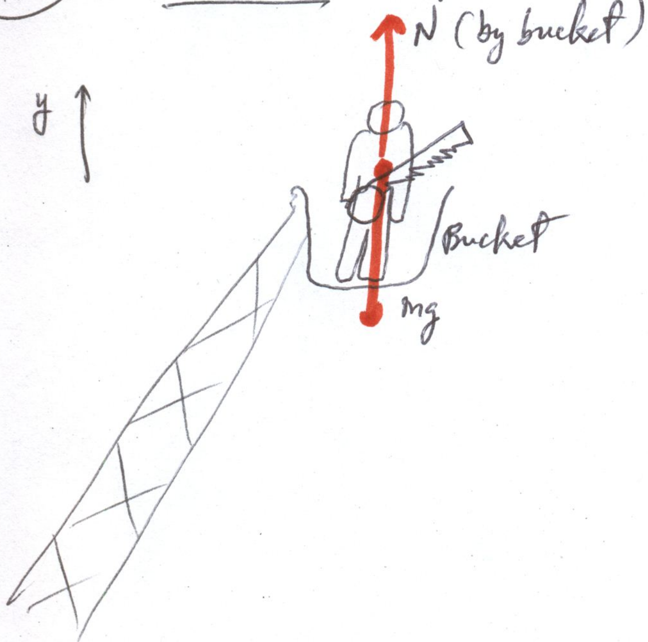
$$F_{th} - mg = +1.4mg.$$

$$F_{th} = 2.4mg$$

c) interstellar space far from any planet (no weight)

$$F_{th} = 1.4m.g$$

4.40 Statement: Application of second Newton's Law: $F_{net} = m \cdot a$



a) Bucket @ rest: $v = 0 = a$

$$F_{net} = N - mg = 0 \Rightarrow N = mg$$
$$= 74 \times 9.81$$
$$\boxed{N = 725\,N}$$

b) Bucket moving up @ steady $v = 2.4\,m/s$ $\Downarrow$ $a = 0$

$$F_{net} = 0 \Rightarrow \boxed{N = 725\,N}$$

c) Bucket moving down @ steady $v = 2.4\,m/s$ $\Downarrow$ $a = 0$

$$F_{net} = 0 \Rightarrow \boxed{N = 725\,N}$$

d) Bucket accelerating up @ $1.7\,m/s^2 = a$

$$F_{net} = m \cdot a$$
$$N - mg = m \cdot a$$
$$N = m(g + a)$$
$$= 74(9.81 + 1.7)$$
$$\boxed{N = 851\,N}$$

(Feels heavier)

e) Bucket accelerating down @ $1.7\,m/s^2$

$$F_{net} = - m \cdot a$$
$$N - mg = -ma \Rightarrow N = m(g - a) = 74(9.81 - 1.7)$$
$$\boxed{N = 599\,N}$$

(Feels lighter)