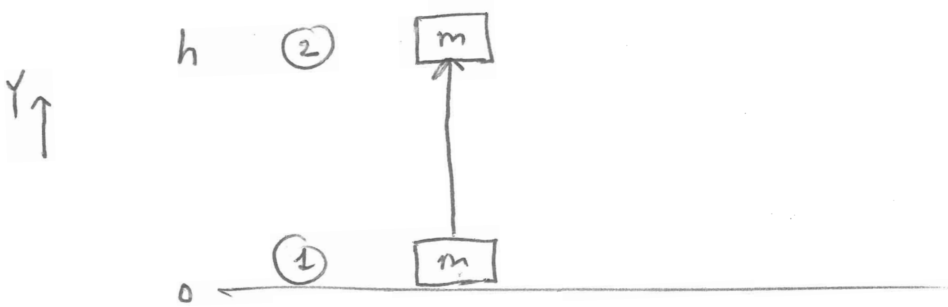
Ch 7. Conservation of Energy

In connection with forces $\begin{cases} \text{Conservative : } \underline{\text{gravity}} \rightarrow \text{lost \& gain} \\ \text{Non-conservative : } \underline{\text{friction}} \text{ (always against motion)} \rightarrow \text{lost} \end{cases}$

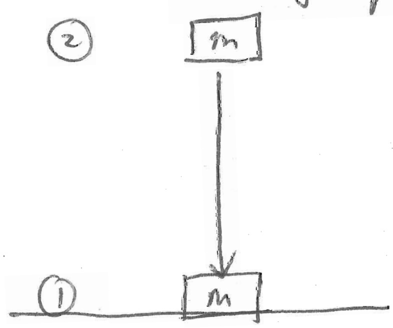
Work done by $\begin{cases} \text{Conservative forces : is conserved} \\ \text{Non-conservation : is not conserved.} \end{cases}$

$\boxed{\text{Gravity:}}$

lift



drop. back to original position



<u>Work done by gravity on the box:</u>

$$Work_{12} = \vec{F}_{grav} \cdot \Delta\vec{r}_{12} = mg(-\hat{j}) \cdot h\hat{j}$$

$$= (-) mgh$$

Constant force (h small wrt $R_E = 6.37 \times 10^6 m$)

gravity received work in this case

<u>Work done by gravity:</u>

$$Work_{21} = \vec{F}_{grav} \cdot \Delta\vec{r}_{21}$$

$$= mg(-\hat{j}) \cdot h(-\hat{j})$$

$$= (+) mgh$$

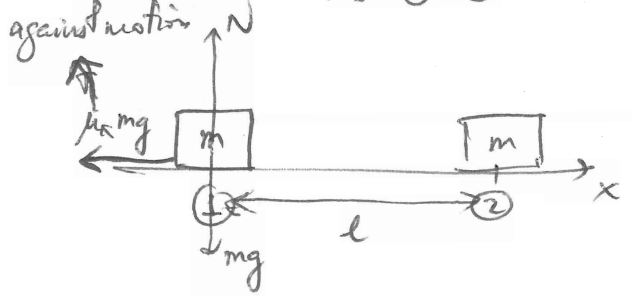
gravity did the work here.

In the end: total work done by gravity ① → ② → ① is 0 (conserved!).

It makes sense that the gravitational <u>potential</u> energy is conserved (it did not change over time!)

Friction: pushing a box of mass m on rough surface

Push ① → ②



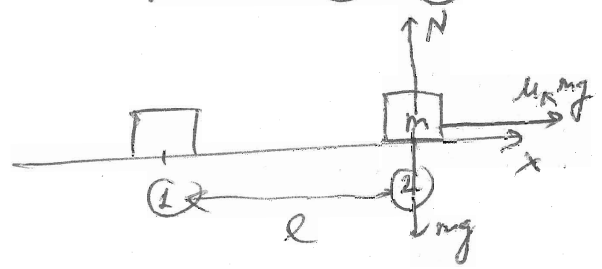
Work done by friction:

$$Work_{12} = \vec{F}_f \cdot \Delta\vec{r}_{12}$$

$$= -\mu_k mg\,\hat{i} \cdot l\,\hat{i}$$

$$= -\mu_k mgl$$

Push back ② → ①



Work done by friction:

$$Work_{21} = \vec{F}_f \cdot \Delta\vec{r}_{21}$$

$$= \mu_k mg\,\hat{i} \cdot l(-\hat{i})$$

$$= -\mu_k mgl$$

Total work done by friction ① → ② → ① : $-2\mu_k mgl$

Total work done by friction → not zero → non-conservative force.

$-2\mu_k mgl$ → friction received net work → we always loose energy for friction.

Conservation of Mechanical Energy:

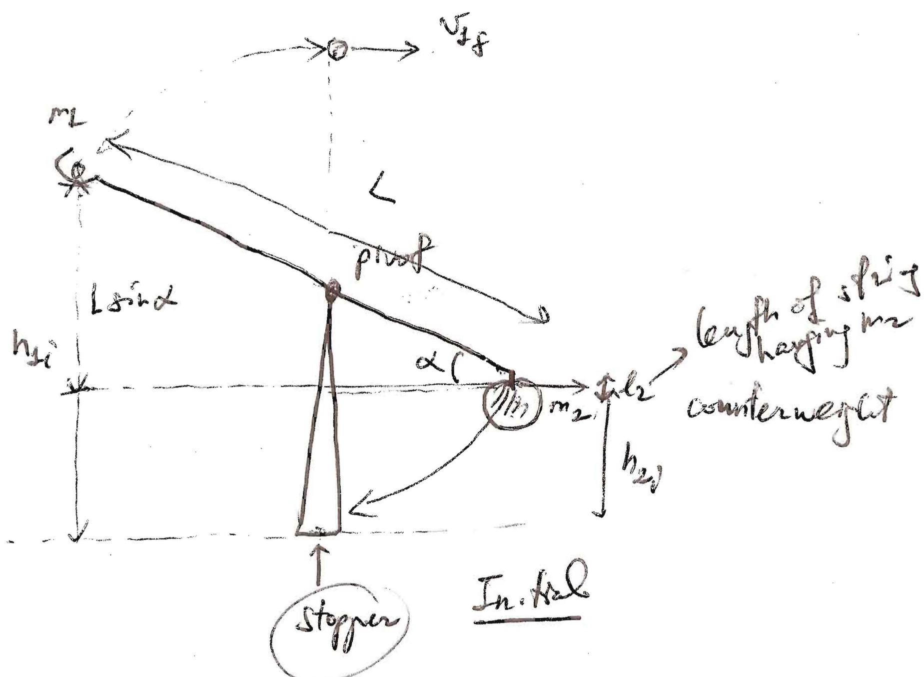
Mechanical Energy → Kinetic & Gravitational Potential Energy

$\frac{1}{2}mv^2 \quad + \quad mgh$

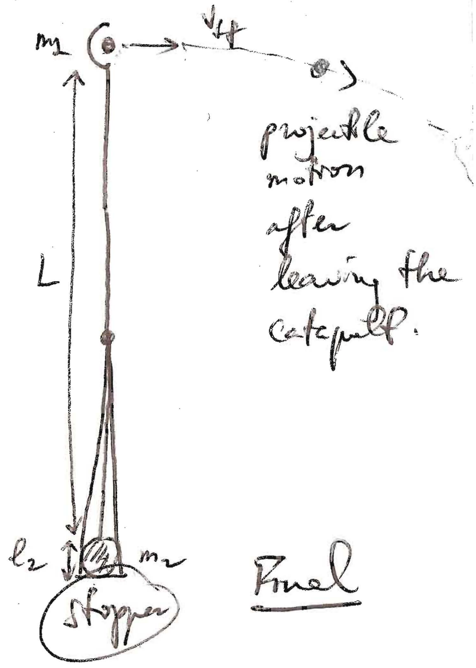
Conservation → Initial = final.

$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f$$

P.P. 7.1 Conservation of mechanical energy: Catapult or Trebuchet:



Initial



Final

- Use conservation of energy to find v_{1f} (with this information we can use projectile motion to find where it will land)

Need to identify initial & final situation

- Limitations: friction at pivot, air resistance, weights of arms $\ll m_2$,

Mech. Energy: Initial (m_1 & m_2) (only grav. potential energy) — final (m_1 & m_2) (only on m_1 → can go as far as possible)

$$\boxed{\frac{1}{2}m_1 \underbrace{\cancel{v_{1i}^2}}_{0} + m_1 g h_{1i} + \frac{1}{2}m_2 \underbrace{\cancel{v_{2i}^2}}_{0} + m_2 g \underbrace{h_{2i}}_{=h_2} = \frac{1}{2}m_1 \overset{?}{v_{1f}^2} + m_1 g(L+l_2) + \frac{1}{2}m_1 \underbrace{\cancel{v_{2f}^2}}_{0} + m_2 g \underbrace{\cancel{h_{2f}}}_{0}}$$

(Reason for stopper!)

$$h_{1i} = h_{2i} + L\sin\alpha + l_2$$

$$m_1 g\left(h_{2i} + L\sin\alpha + l_2\right) + m_2 g h_{2i} = \frac{1}{2}m_1 v_{1f}^2 + m_1 g(L+l_2)$$

$$(m_1+m_2) g(h_{2i}) = \frac{1}{2}m_1 v_{1f}^2 + m_1 g\left(L+\cancel{l_2} - L\sin\alpha - \cancel{l_2}\right)$$

$$(m_1+m_2) g h_{2i} = \frac{1}{2}m_1 v_{1f}^2 + m_1 g L(1-\sin\alpha)$$

→ can find v_{1f}!

Where will m_f land? → adjust the catapult to hit certain target!

Projectile motion:

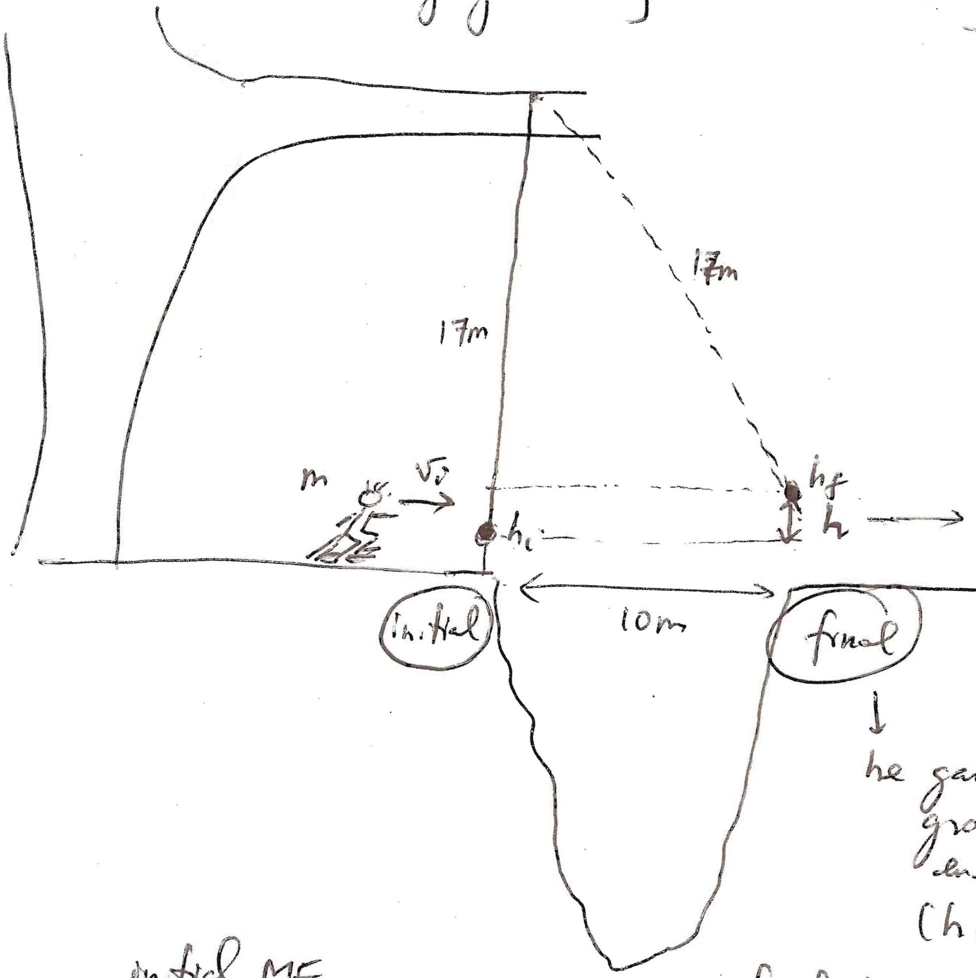
y-direction: constant acceleration:
$$L+l_2 = \frac{1}{2}gt^2 \rightarrow t = \sqrt{\frac{2(L+l_2)}{g}}$$
↓ same as in x-direction

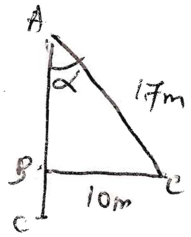
x-direction:
$$x_{landing} = v_{+f}\sqrt{\frac{2(L+l_2)}{g}}$$

$$x_{Landing} = \sqrt{\frac{(m_f+m_c)g\,h_{2i} - m_f gL(1-\sin\alpha)}{\frac{1}{2}m_1}}\sqrt{\frac{2(L+l_2)}{g}}$$

7.64

Go over a gorge using a vine : using conservation of mechanical energy.





Alternative #1:

$AB = AC\cos\alpha$

$h = BC = AC - AB$

$= AC(1-\cos\alpha)$

$\alpha = \sin^{-1}\frac{10}{17}$

$h = 17\left(1-\cos\left(\sin^{-1}\frac{10}{17}\right)\right)$

$= 3.25m$

Alternative #2: $AB^2 + 10^2 = 17^2$

$AB = 13.75m$

$\rightarrow h = 17 - 13.75 = 3.25m$

final ↓ he gained some gravitational potential energy mgh

$(h \equiv h_f - h_i)$

initial ME = final ME

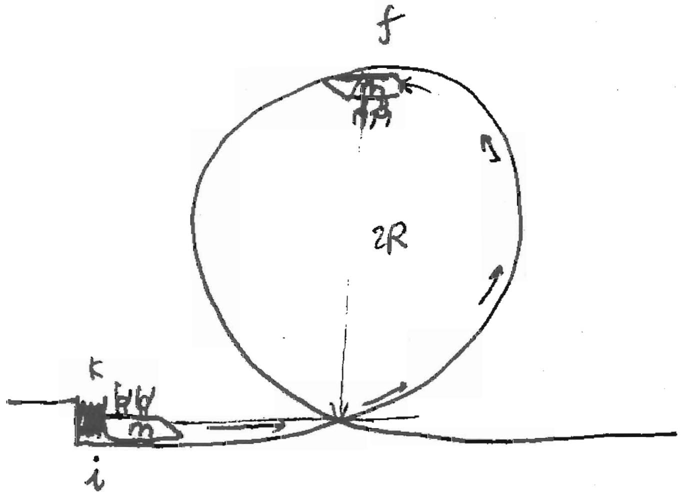
$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}m\underbrace{v_f^2}_{0} + mgh_f$$

• Initial minimum speed. → $v_f = 0$ → $\frac{1}{2}\cancel{m}v_{i_{min}}^2 = \cancel{m}g(h_f - h_i) = \cancel{m}gh$

$$v_{i_{min}} = \sqrt{2gh} = \sqrt{2\cdot 9.81\cdot 3.25} = 7.98\,\frac{m}{s}$$

$$v_{i_{min}} = \sqrt{2 \cdot 9.81 \cdot 3.25} = 7.98 \frac{m}{s}$$

7.59



$m = 840 kg$
$k = 31 \frac{kN}{m}$
$R = 6.2 m$

Min compression of spring for car to make top of loop?

Conservation of mechanical energy:

$$\frac{1}{2} m v_i^2 + mgh_i = \frac{1}{2} m v_f^2 + mgh_f$$

Important: $v_f \neq 0$ (otherwise it will not make it!)
At f: if we look at min initial speed $v_{i_{min}}$, the only acceleration to change direction of motion in a circular motion is g. In another words: the only agent to change direction of motion at f is mg:

$$\cancel{m} \frac{v_f^2}{R} = \cancel{m} g \rightarrow v_f^2 = gR$$

$$\frac{1}{2} m v_{i_{min}}^2 = \underbrace{\frac{1}{2} m v_f^2}_{\frac{1}{2} mgR} + \underbrace{mg(\underbrace{h_f - h_i}_{2R})}_{mg\,2R} = mg\frac{5}{2}R$$

$$v_{i_{min}}^2 = 5gR.$$

what is $x_{i_{min}}^2$ (min compression of spring?)
When a spring is compressed a distance x, elastic potential energy is stored in the amount of: $\frac{1}{2}kx^2$. This will go into the initial

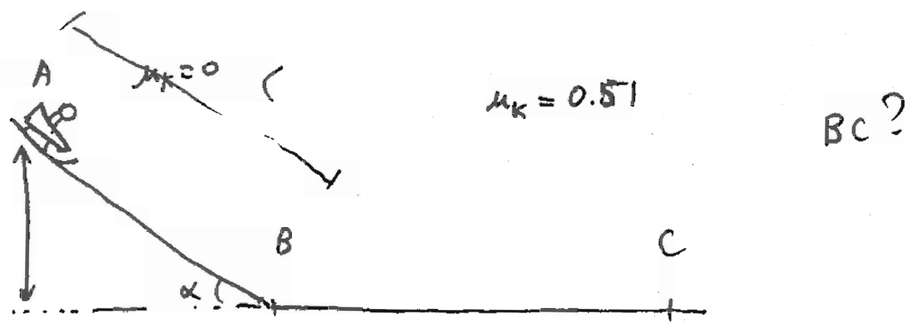
KE of the roller-coaster car:

$$\tfrac{1}{2} k x_{min}^2 = \tfrac{1}{2} m v_{i_{min}}^2 \rightarrow x_{min} = \sqrt{\frac{m v_{i_{min}}^2}{k}}$$

$$= \sqrt{\frac{m 5 g R}{k}}$$

$$= \sqrt{\frac{840 \cdot 5 \cdot 9.81 \cdot 6.2}{31000}} \simeq 2.87 m$$

(7.61)



AB: constant acceleration
BC: constant deceleration

When we solved a similar problem using kinematic equations:

$v_A = 0 \rightarrow a_{AB} \rightarrow v_B \rightarrow a_{BC}, \; v_C = 0 \rightarrow BC$.

Note: $\left\{\begin{matrix}\alpha \\ L \\ m\end{matrix}\right.$ were also given!

Conservation of mechanical energy: only applies to AB (since friction b/w B & C is not conservative!). But can find v_B, then use kinematic equation to find BC.

initial = @ A
final = @ B

$$\tfrac{1}{2} m \cancel{v_A^2} + m g h_A = \tfrac{1}{2} m v_B^2 + m g h_B$$

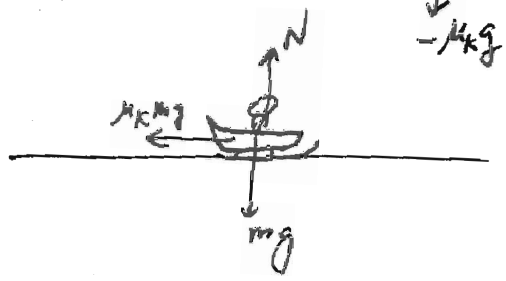
$$\cancel{m} g (h_A - h_B) = \tfrac{1}{2} \cancel{m} v_B^2$$

$$v_B = \sqrt{2 g \underbrace{(h_A - h_B)}_{7.2m}} = \sqrt{2 \cdot 9.81 \cdot 7.2}$$

$$\simeq 11.9 \, m/s$$

BC: Kinematic equation #3:

$$\frac{\overset{0}{\cancel{v_C}}^2 - v_B^2}{x_{BC}} = 2\cdot \underbrace{a_{BC}}_{-\mu_k g} \rightarrow \frac{-v_B^2}{x_{BC}} = -2\mu_k g$$



$$x_{BC} = \frac{v_B^2}{2\mu_k\cdot g}$$

$$= \frac{11.9^2}{2\cdot 0.51\cdot 9.81}$$

$$= 14.2 \text{ m.}$$

Ch. 8: Gravitation

Universal Law of Gravitation

→ also on Moon, other planets, galaxies, the universe.

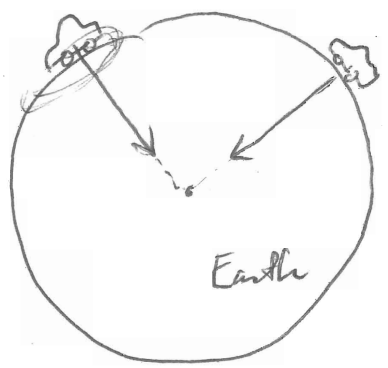
$$F = G\frac{m_1 \cdot m_2}{r^2}$$

Force of grav. attraction b/w two masses m_1 & m_2

G: Universal Grav. Constant $= 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

r = separation b/w m_1 & m_2

Force is a vector: direction of grav. attraction is center to center toward the more massive mass



On smaller scale on surface of Planet where we live & work → the attraction by gravity is vertical & downward

mg ($g = 9.81\ m/s^2$)

$m_1 = M_{Earth} = 5.97 \times 10^{24}$ kg

$R_E = 6.37 \times 10^6$ m $= r$ (if we stick on the surface of Planet)

→ Universal Law of Grav. on a mass m on surface of Planet:

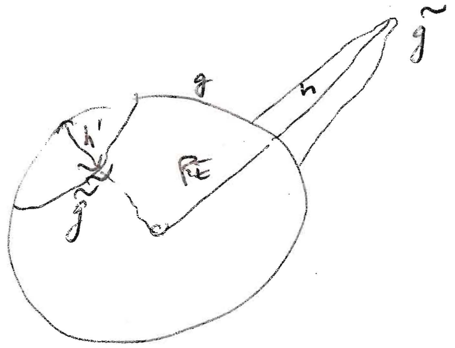
$$F = G\frac{M_E}{R_E^2} \cdot m = \underbrace{6.67 \times 10^{-11} \times \frac{5.97 \times 10^{24}}{(6.37 \times 10^6)^2}}_{9.81\ m/s^2} \cdot m$$

$F = m \cdot g$ where $\boxed{g = G\frac{M_E}{R_E^2}}$

Observation: $g = G\frac{M_E}{R_E^2}$ at ground level!

1) Top of mountain: of height h $\quad \tilde{g} = G\frac{M_E}{(R_E+h)^2} < g$ (→ Why can't boil an egg on top of Mt Everest)
↓
Do it w/ XC #3

2) Bottom of ocean: of depth h' $\quad \tilde{\tilde{g}} = G\frac{M_E}{(R_E-h')^2} > g$

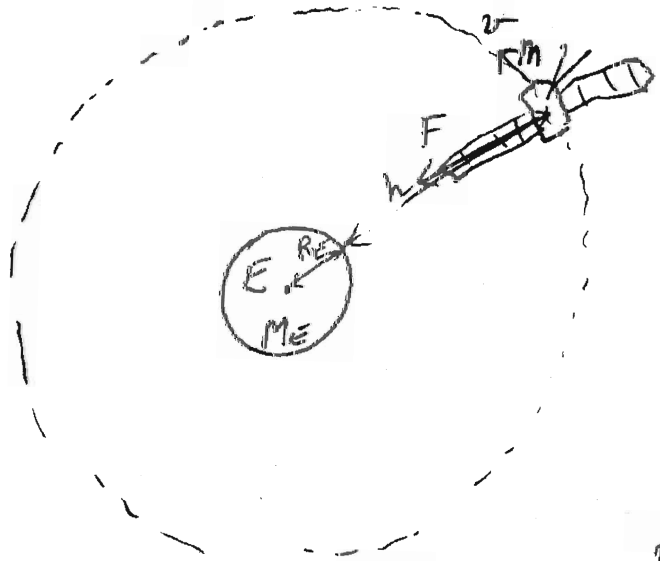


Orbital Motion:

Circular orbital motion: satellite in UCM (constant speed v)

→ No energy cost to run it in orbital motion. Energy is only needed for operation (signal communication with the planet)

→ Energy: from grav. attraction of the Earth:



Force of grav. attraction on satellite is F, which is essential to keep the satellite in UCM: Why? Because it is the agent that change direction of motion to keep it in orbit and not to go off in a straight forward direction:

$$F = G\frac{M_E \cdot \cancel{m}}{(R_E+h)^2} = \cancel{m}\cdot\frac{v^2}{R_E+h}$$

→ Univ. Law of Grav.

→ 2nd Newton's Law for satellite under UCM (w/ radial acceleration)

→ Orbital speed → $v = \sqrt{\dfrac{G \cdot M_E}{R_E+h}}$

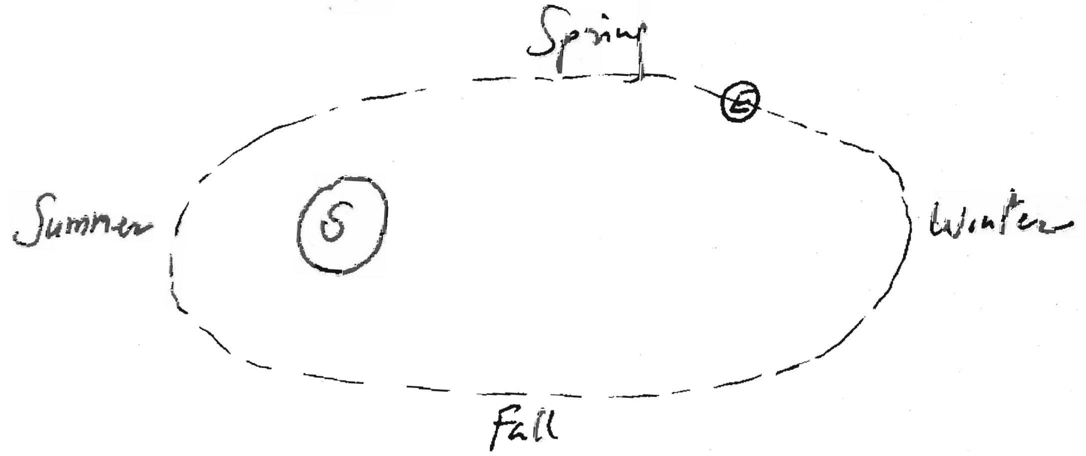
→ Orbital period: time to complete one full orbit:

$$\hookrightarrow T = \frac{2\pi(R_E+h)}{\sqrt{\dfrac{G\cdot M_E}{(R_E+h)}}} = \underbrace{\frac{2\pi}{\sqrt{G\cdot M_E}}}_{\text{constant}}(R_E+h)^{3/2} \rightarrow T^2 = \frac{4\pi^2}{GM_E}\underbrace{(R_E+h)^3}_{\equiv R}$$

If we extend this relationship b/w period & radius ($T^2 \propto R^3$) to elliptical orbits →

<u>Kepler's 3rd Law</u>: "the period squared is proportional to the semimajor axis cubed"

↳ Earth is in elliptical orbit around the Sun:



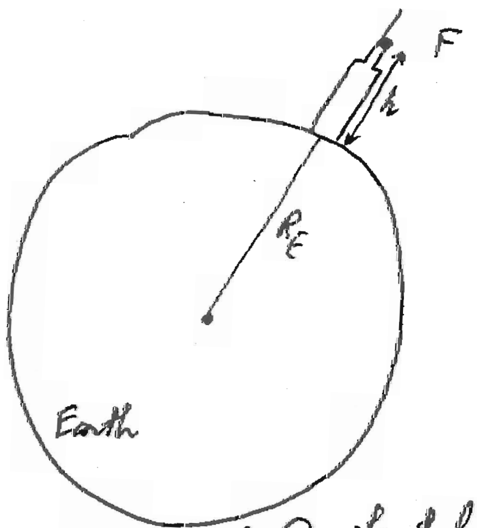
<u>Cell phone satellite</u> @ $h = 250\,km$

Orbital period for this satellite:

$$T = \frac{2\pi}{\sqrt{GM_E}}(R_E+h)^{3/2} = \frac{2\pi}{\sqrt{6.67\times10^{-11}\cdot 5.97\times10^{24}}}\left[(6370+250)\cdot10^{3}\right]^{3/2}$$

$$= 5400\,s = 1.5\,h$$

8.18 Height of Chicago Sears Tower : h $\longleftrightarrow$ $\Delta g = \underset{\substack{\downarrow \\ \text{street} \\ \text{Level}}}{g} - \underset{\substack{\downarrow \\ \text{Top of} \\ \text{building} \\ \text{of height } h}}{g_h} = 0.00136 \frac{m}{s^2}$



$F = \underbrace{G \frac{M_E}{r^2}}_{g} m$

@ street level: $g = G \frac{M_E}{R_E^2}$

@ Top of building: $g_h = G \frac{M_E}{(R_E+h)^2}$

r: center-to-center separation.

$$\Delta g = g - g_h = GM_E \left[\frac{1}{R_E^2} - \frac{1}{(R_E+h)^2} \right]$$

$$= GM_E \left[\frac{(R_E+h)^2 - R_E^2}{R_E^2 (R_E+h)^2} \right] = GM_E \left[\frac{2R_E h + h^2}{R_E^2 (R_E+h)^2} \right]$$

$$= \underbrace{\frac{GM_E}{R_E^2}}_{\text{no } h} \cdot \underbrace{\frac{2R_E h + h^2}{(R_E+h)^2}}_{\text{with } h}$$

What is the order of magnitude of h = 10m, 100m, 1000m, 10000m

R_E = 6 370 000 m

A very good approximation : $\begin{cases} R_E + h \simeq R_E \\ 2R_E + h \simeq 2R_E \end{cases}$

$$\Delta g = \frac{GM_E}{R_E^2}\,\frac{(2R_E+h)h}{(R_E+h)^2} \simeq \underbrace{\frac{GM_E}{R_E^2}}_{=g}\,\frac{2\cancel{R_E}\cdot h}{R_E^{\cancel{2}}}$$

$$\rightarrow h = \frac{\Delta g}{g}\,\frac{R_E}{2} = \frac{0.00136}{9.81}\cdot\frac{6\,370\,000}{2} = 442\,m$$

8.42

Kepler's Law: "period squared is proportional to the semimajor axis (r) cubed": $T^2 \propto r^3$

Asteroid Passachoff: $\begin{cases} T_p = 1417 \text{ days} \\ r_p\ ? \end{cases}$ (time to complete one full elliptical orbit around Sun) (r_p in units of $r_E \leftrightarrow \frac{r_p}{r_E}$?)

Earth: $\begin{cases} T_E \\ r_E \end{cases}$

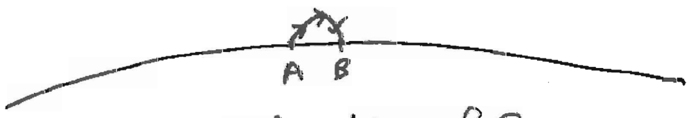
$$\left.\begin{array}{l} T_p^2 \propto r_p^3 \\ T_E^2 \propto r_E^3 \end{array}\right| \left(\frac{T_p}{T_E}\right)^2 = \left(\frac{r_p}{r_E}\right)^3 \leftrightarrow \left[\frac{r_p}{r_E} = \left(\frac{T_p}{T_E}\right)^{2/3}\right.$$

$$= \left(\frac{1417}{365}\right)^{2/3}$$

$$= (3.88)^{2/3} = 2.47$$

Projectile Motion (in part: vertical component was due to gravitational attraction)

→ So far its trajectory was a parabola (provided the ground is "flat": balls, bullets, short-range missile)



surface b/w A & B is flat to a very good approximation

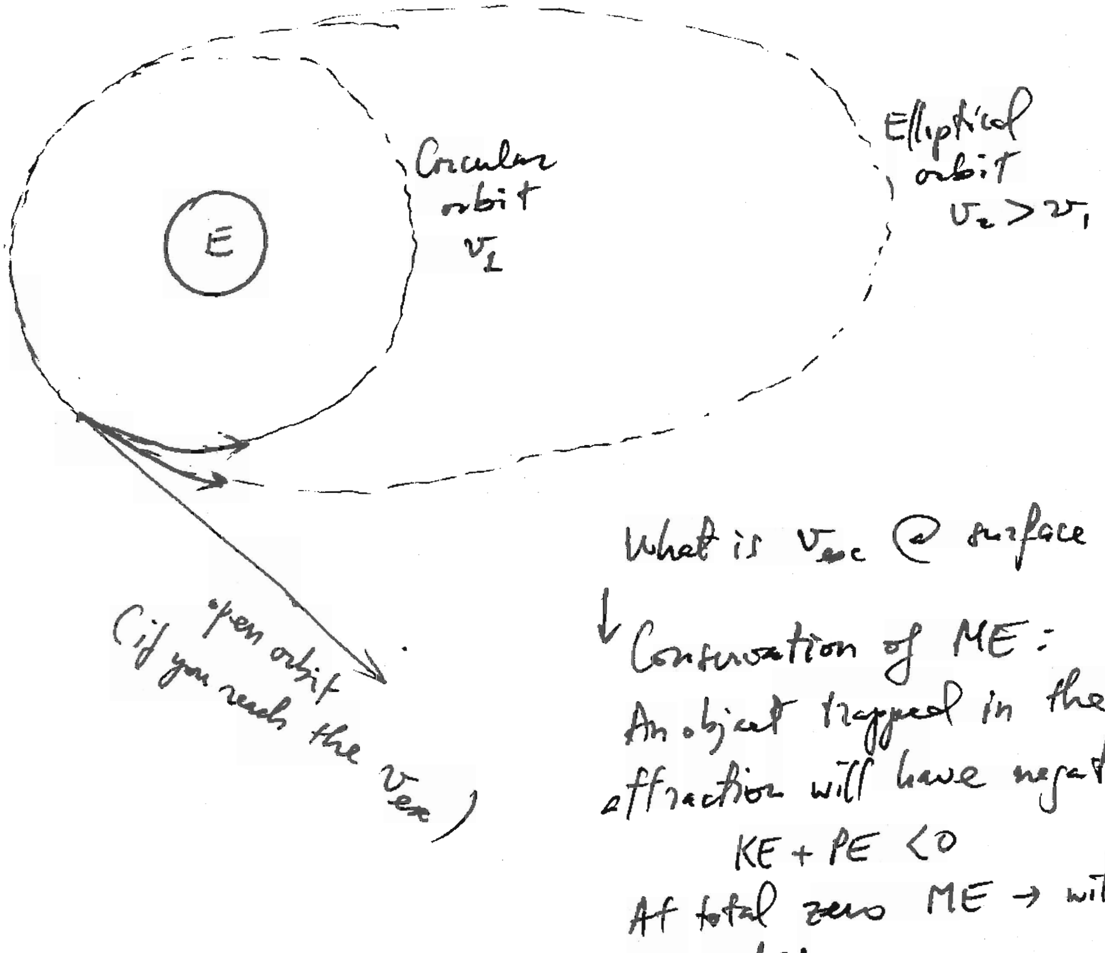
→ Trajectory is a parabola

surface b/w A & B is no longer flat

→ Trajectory is part of an elliptical orbit

→ Long-range missile, etc.

<u>Escape Speed</u>: if you don't have at least the escape speed → you are trapped in the gravitational attraction of the planet: stuck on surface or in an orbit around the planet. If $v \geq v_{esc}$ → object will follow an "open orbit" away from the gravitational attraction → can travel to the outer space: interplanetary travel



What is v_{esc} @ surface of Earth?

↓ Conservation of ME:
An object trapped in the grav. attraction will have negative ME

$$KE + PE < 0$$

At total zero ME → will start open orbit:

$v_{esc} \rightarrow \quad ME = 0$

More general expression for the gravitational potential energy:

Definition of work:

$$\Delta U_{AB} = -\int_A^B \vec{F}\cdot d\vec{r} = -\int_A^B G\frac{M_E \cdot m}{r^2}\, dr$$

Change of potential energy b/w points A & B

Univ. Law of Gravitation: (radial direction)

$$= -GM_E m \underbrace{\int_A^B \frac{dr}{r^2}}_{\left[-\frac{1}{r}\right]_A^B} = GM_E m\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$U \equiv -\frac{GM_E m}{r} \longrightarrow \; = U_A - U_B$$

. Reference of zero potential energy: $B \to \infty$ (Normally assumed)

$$\hookrightarrow \boxed{U = \Delta U_{A\infty} = U_A - \underset{0}{U_\infty} = U_A = -\frac{GM_E m}{r}}$$

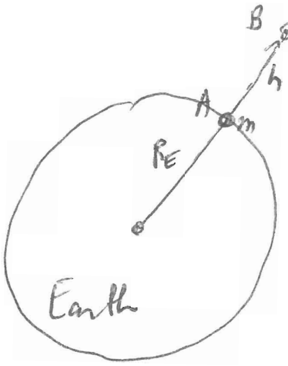
center to center separation b/w M_E & m

. Observation: 1) if object of mass m on surface of Earth:

$$r = R_E \rightarrow U = -\frac{GM_E m}{R_E}\cdot\frac{R_E}{R_E} = -gmR_E$$

yesterday: $g = G\frac{M_E}{R_E^2}$

2)



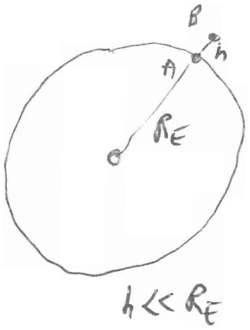
$$\Delta U_{AB} = U_B - U_A$$

$$= -\frac{G M_E m}{R_E + h} + \frac{G M_E m}{R_E}$$

$$= G M_E m \left[-\frac{1}{R_E+h} + \frac{1}{R_E} \right]$$

$$= G M_E m \left[\frac{-\cancel{R_E} + \cancel{R_E} + h}{(R_E+h) R_E} \right]$$

$$\Delta U_{AB} = G M_E m \frac{h}{(R_E+h) R_E}$$



$\boxed{h \ll R_E} \Rightarrow R_E + h \approx R_E$

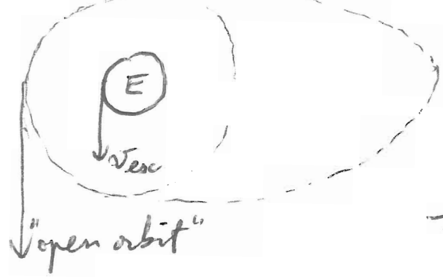
$$\Delta U_{AB} = G M_E m \frac{h}{R_E^2} = \boxed{mgh}$$

yesterday: $g = \frac{G M_E}{R_E^2}$

$R_E = 6370000\,m$

$h \leq 1000\,m \rightarrow h \ll R_E$

$\boxed{\text{Escape speed}}$: for a planet or moon: minimum speed to follow an open orbit:



Under normal (attached) condition: total ME < 0

At zero ME → open orbit.

→ find v_{esc}:

$$KE + PE = 0 \;:\; \frac{1}{2}\cancel{m} v_{esc}^2 - \frac{G M_E \cancel{m}}{r} = 0 \rightarrow v_{esc} = \sqrt{\frac{2 G M_E}{r}}$$

space travel: $r = R_E \rightarrow v_{esc} = \sqrt{\frac{2 G M_E}{R_E}} = \sqrt{\frac{2 \cdot 6.67\times10^{-11} \cdot 5.97\times10^{24}}{6.37\times10^{6}}} = 11.2\,\frac{km}{s}$

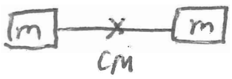
(From surface)

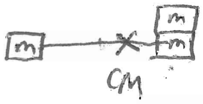
$= 40320\,\frac{km}{h}$

Ch9. System of Particles :

So far, for simplification we looked at objects as point-like particles with mass m located at its center of mass (FBD's)

Center of Mass: average position of all components of a system weighted by their masses.





Same positions to average, but the second has double weight due to its double mass.

Formulas:

$\vec{R}$: position vector for CM.

· Discrete systems: $\vec{R} \equiv \dfrac{\sum_i m_i \vec{r}_i}{M}$

m_i: mass of component i

$\vec{r}_i$: position vector of component i

$M = \sum_i m_i$: Total mass of system

· Continuous systems: $\vec{R} \equiv \dfrac{\int \vec{r}\, dm}{M}$

dm: infinitesimal mass

$\vec{r}$: position vector of dm

$M = \int dm$: Total mass of system

Implications for 2nd Newton's Law: when we look at an object as a system of particles:

$$\underbrace{\vec{F}_{net}}_{\text{Net force on system}} = \underbrace{M}_{\text{Total mass of system}} \cdot \underbrace{\frac{d\vec{\vec{R}}}{dt^2}}_{\text{acceleration of CM}}$$

→ No change! All we did so far was correct (Ch4 & 5)

→ Subtlety: the interactions b/w components or particles of the object are "internal forces": they are present in pairs by the principle of action & reaction (3rd Newton's Law) and cancel at the object level

Sum of all internal forces = 0

$\vec{F}_{net}$ = only involves external force on system.

$\boxed{\text{Linear Momentum of a System :}}$ $\vec{P}$

$$\vec{P} \equiv M \cdot \vec{V} = M \cdot \frac{d\vec{R}}{dt} = M \frac{d}{dt} \frac{\sum_i m_i \vec{r}_i}{M} = \sum_i m_i \underbrace{\frac{d\vec{r}_i}{dt}}_{\text{Velocity of Component } i}$$

M: Total mass

$\vec{V}$: velocity of CM or time derivative of position of CM

$$\vec{P} = \sum_i \underbrace{m_i \vec{v}_i}_{\substack{\text{linear momentum} \\ \text{of component } i \\ \vec{P}_i}} = \sum_i \vec{P}_i$$

2nd Newton's Law of a system of particles using total linear momentum:

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

"Net external force on a system equals the change of its total linear momentum over time"

$\vec{F}_{net} = 0 \longrightarrow \frac{d\vec{P}}{dt} = 0 \rightarrow$ Total linear momentum of the system is conserved : $\boxed{\vec{P}_i = \vec{P}_f}$

So far conservation laws : 2

→ Conservation of Mechanical Energy : $\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f$
(only conservative forces are involved) $(h \ll R_E)$

→ Conservation of total linear momentum : $\vec{P}_i = \vec{P}_f$
(Net external force on system is 0)

Collisions

↓

$\boxed{\text{Net external force is 0: } \vec{F}_{net} = 0}$

↓

$\vec{P}_i = \vec{P}_f$

Inelastic: if colliding components stick together after collision
$\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f$

$\vec{F}_{net,\,external} = 0 \rightarrow$ Conservation of total linear momentum:
2 components: $\underbrace{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}_{\text{before collision}} = \underbrace{(m_1 + m_2)\vec{v}_f}_{\text{after collision}}$

$ME = KE$: $\rightarrow$ Total KE is not conserved: $KE_i \neq KE_f$
$KE_i > KE_f$: $KE_i = KE_f +$ energy into internal structural deformation or damage of the colliding components.

Elastic: colliding components do not stick together after collision neither are deformed: $\rightarrow$ Total KE is conserved.

$\vec{F}_{net,\,external} = 0 \rightarrow$ Conservation of total linear momentum.
2 component collision: $m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$
(one extra unknown)

KE is conserved $\rightarrow \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

Simplifications: 1) $m_1 = m_2 \equiv m$
2) collisions 1D but we will also do 2D collisions.

Additional discussion on elastic collision: (hard ball collisions)

1) **1D Elastic Collision:** $\begin{cases} \vec{P}_i = \vec{P}_f \leftrightarrow P_i = P_f \\ KE_i = KE_f \end{cases}$ 2 equations → can solve up to 2 unknowns: e.g.: $\boxed{v_{1f} \text{ \& } v_{2f}}$

$$\left.\begin{aligned} &a) \quad v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\ &b) \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \\ &c) \quad v_{1i} + v_{1f} = v_{2i} + v_{2f} \end{aligned}\right\}$$

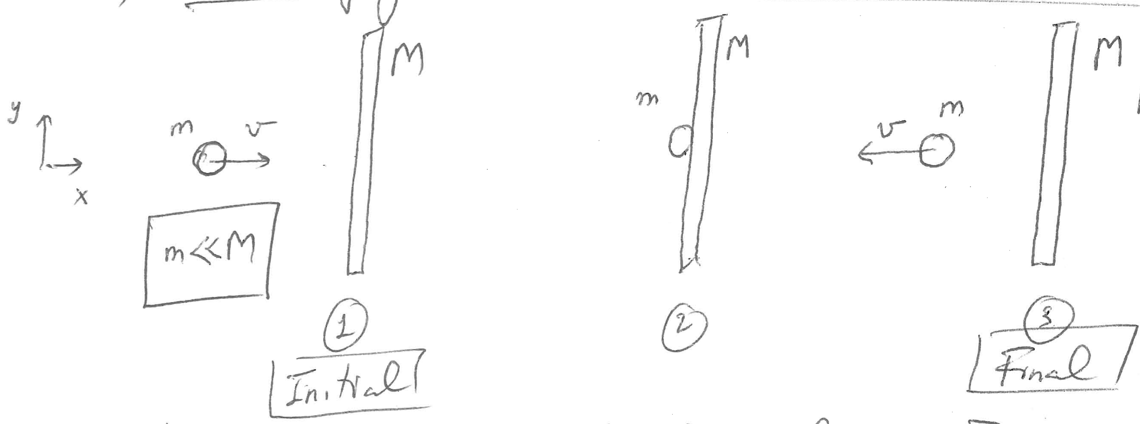
are derived from conservation laws.

2) **2D elastic collision** $\begin{cases} \vec{P}_i = \vec{P}_f \leftrightarrow \begin{cases} P_{ix} = P_{fx} \\ P_{iy} = P_{fy} \end{cases} \\ KE_i = KE_f \end{cases}$ 3 equations → can solve up to 3 unknowns.

→ Note: we need at least one information after collision!

Typical problem: asks for v_{1f}, v_{2f}, θ (angle b/w two final velocities)

3) **Collision of gas molecules (hard balls) with the container wall:**



No deformation → comes back with same KE! ($m \ll M$)

a) Looking at motion in x-direction only: → $F_{net,x} = 0$ → $\boxed{P_{i,x} = P_{f,x}}$

on system of gas molecule & wall

$$m v \hat{i} = m v (-\hat{i}) + \boxed{2mv(\hat{i})}$$

Momentum acquired by wall after collision with gas molecule

$2mv = MV$

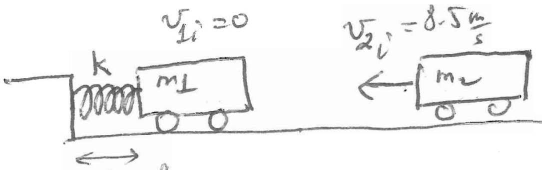
$V = \frac{2mv}{M} \approx 0$

9.41

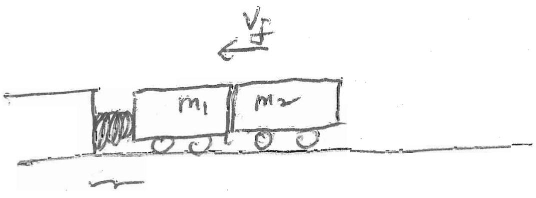
a) $\boxed{\text{Inelastic Collision}}$ ("two cars couple together")

b) Define initial & final situations

$\boxed{\text{Before (initial)}}$ | After (final)



natural length (m₁ only resting against spring)



some compression

m_1 & m_2 with a same final velocity v_f:
$\vec{v}_f = v_f(-\hat{i})$

$k = 3.2 \times 10^5 \frac{N}{m}$

$m_1 = 11{,}000\, kg$, $\vec{v}_{1i} = 0$

$m_2 = 9400\, kg$, $\vec{v}_{2i} = 8.5 \frac{m}{s}(-\hat{i})$

c) Maximum compression of spring: how? spring gets compressed the most when $\boxed{\text{total KE of two coupled cars}}$ has been transferred to the spring. After this point spring will return this energy to the cars which will rebound (question 6)

$$\frac{1}{2}(m_1+m_2)v_f^2 = \frac{1}{2}k\underbrace{(\Delta x)}_{\text{max compression}}^2 \rightarrow \Delta x_{max} = v_f\sqrt{\frac{m_1+m_2}{k}}$$

We need v_f: from conservation law (total momentum) for inelastic collision:

$$\vec{P}_i = \vec{P}_f$$

$$m_2 v_{2i}(-\hat{i}) = (m_1+m_2)v_f(-\hat{i})$$

$$\hookrightarrow m_2 v_{2i} = (m_1+m_2)v_f$$

$$v_f = \frac{m_2}{m_1+m_2}v_{2i} = \frac{9400}{20400}8.5$$

$$\boxed{v_f = 3.92 \frac{m}{s}} \rightarrow \vec{v}_f = -3.92\,\hat{i}\ \frac{m}{s}$$

$$\boxed{\Delta x_{max} = 3.92\sqrt{\frac{20400}{3.2\times10^5}} = 0.989\, m}$$

b) Rebound speed of two cars: when spring returns all of $\frac{1}{2}k(\Delta x_{max})^2 \rightarrow \vec{v}_{rebound} = +3.92\hat{i}\ \frac{m}{s}$

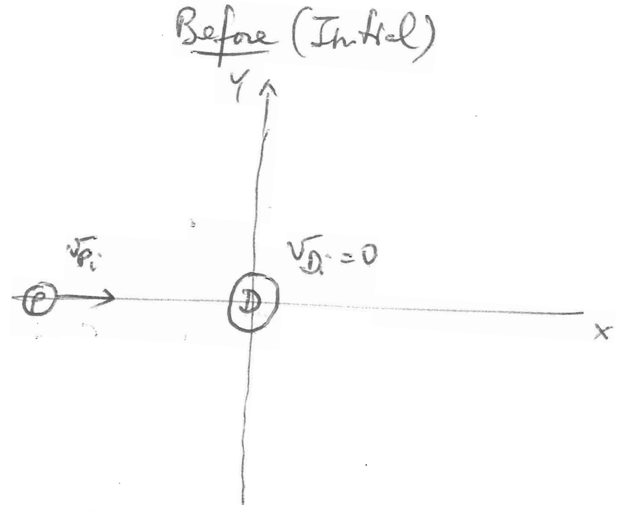
9.65

a) Proton $\boxed{\text{elastically}}$ collides with a Deuteron @ rest.

$\boxed{\text{2D}} \rightarrow \boxed{\begin{cases} P_{ix} = P_{fx} \\ P_{iy} = P_{fy} \\ KE_i = KE_f \end{cases}}$

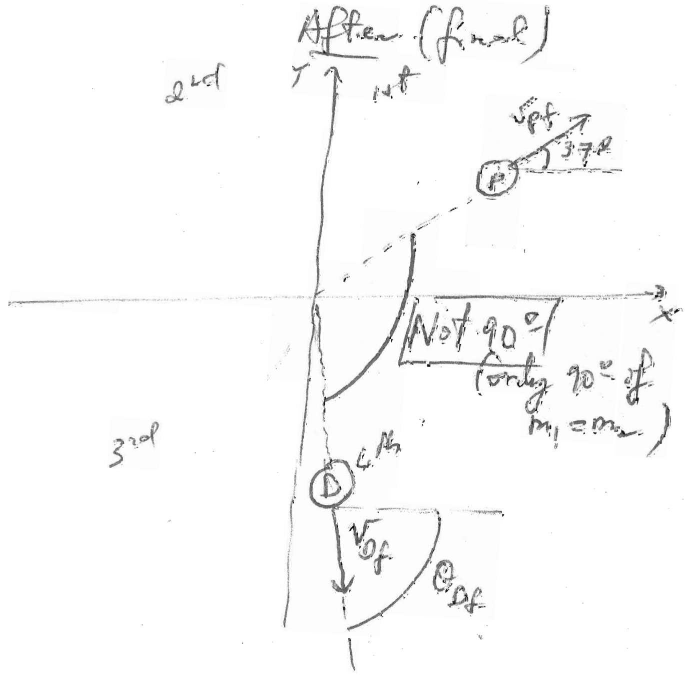
b) Define before (initial) & after (final) situations.

Before (Initial)



$\vec{P}_i = m_p v_{P_i}\hat{i}$

After (final)



$\vec{P}_f = (m_p v_{pf}\cos 37° + m_D v_{Df}\cos\theta_{Df})\hat{i} + (m_p v_{pf}\sin 37° - m_D v_{Df}\sin\theta_{Df})\hat{j}$

c) Fraction of KE transferred to deuteron: Since D had no KE_i ($KE_{D_i} = 0$) $\rightarrow$

$$\frac{KE_{Df}}{KE_{P_i}} = \frac{KE_{P_i} - KE_{Pf}}{KE_{P_i}} = 1 - \frac{KE_{Pf}}{KE_{P_i}} = 1 - \frac{\frac{1}{2}\cancel{m_p} v_{pf}^2}{\frac{1}{2}\cancel{m_p} v_{pi}^2} = 1 - \frac{v_{pf}^2}{v_{pi}^2}$$

~~→ Only need to find v_{pf}~~

Conservation Laws:

$\boxed{\vec{P}_i = \vec{P}_f} \rightarrow \begin{cases} x\text{-direction}: & m_p v_{pi} = m_p v_{pf}\cos 37° + m_D v_{Df}\cos\theta_{Df} \\ y\text{-direction}: & 0 = m_p v_{pf}\sin 37° - m_D v_{Df}\sin\theta_{Df} \end{cases}$

$\boxed{m_D = 2m_p} \rightarrow \begin{cases} v_{pi} = v_{pf}\cos 37° + 2v_{Df}\cos\theta_{Df} & (1)\checkmark \\ 0 = v_{pf}\sin 37° - 2v_{Df}\sin\theta_{Df} & (2)\checkmark \end{cases}$

unknowns: (v_{pi}), v_{pf}, v_{Df}, θ_{Df}

$\boxed{KE_i = KE_f}$: $\frac{1}{2}m_p v_{pi}^2 = \frac{1}{2}m_p v_{pf}^2 + \frac{1}{2}m_D v_{Df}^2$

$\boxed{m_D = 2m_p} \rightarrow \quad v_{pi}^2 = v_{pf}^2 + 2v_{Df}^2 \quad (3)\checkmark$

Need v_{pf} in term of v_{pi} (v_{pi} is not an unknown)
for fraction of energy transferred to deuteron: $1 - \frac{v_{pf}^2}{v_{pi}^2}$

Algebra manipulations: 1) 2) 3)

a) $\boxed{\text{Eliminate } \theta_{Df}}$: $\boxed{\cos^2\theta_{Df} + \sin^2\theta_{Df} = 1}$

(1) $\cos\theta_{Df} = \frac{v_{pi} - v_{pf}\cos 37°}{2v_{Df}}$

(2) $\sin\theta_{Df} = \frac{v_{pf}\sin 37°}{2v_{Df}}$

$1 = \frac{1}{4v_{Df}^2}\left[v_{pi}^2 - 2v_{pi}v_{pf}\cos 37° + \underbrace{v_{pf}^2\cos^2 37° + v_{pf}^2\sin^2 37°}_{v_{pf}^2}\right]$

$1 = \frac{1}{4v_{Df}^2}\left[v_{pi}^2 + v_{pf}^2 - 2v_{pi}v_{pf}\cos 37°\right] \quad (1\&2)$

b) Eliminate v_{Df}: using of (3) $\boxed{v_{Df}^2 = \frac{v_{pi}^2 - v_{pf}^2}{2}}$ (3)

Plug (3) into (1&2): $2(v_{pi}^2 - v_{pf}^2) = v_{pi}^2 + v_{pf}^2 - 2v_{pi}v_{pf}\cos 37°$

$\boxed{3v_{pf}^2 - (2v_{pi}\cos 37°)v_{pf} - v_{pi}^2 = 0}$

Solving for v_{pf} using this quadratic eq: $\begin{cases} x = v_{pf} \\ a = 3 \\ b = -2v_{pi}\cos 37^\circ \\ c = -v_{pi}^2 \end{cases}$

$$v_{pf} = \frac{2v_{pi}\cos 37^\circ \pm \sqrt{4v_{pi}^2\cos^2 37 + 12v_{pi}^2}}{6}$$

$$= v_{pi}\underbrace{\frac{2\cos 37^\circ \pm \sqrt{4\cos^2 37 + 12}}{6}}_{0.902}$$

→ Fraction of energy transferred to deuteron:

$$1 - \frac{v_{pf}^2}{v_{pi}^2} = 1 - \frac{(0.902\,v_{pi})^2}{v_{pi}^2} = 1 - 0.902^2 = 0.186$$

or 18.6 %

Solving for v_{pf} using this quadratic eq:
$$\begin{cases} x = v_{pf} \\ a = 3 \\ b = -2v_{pi}\cos 37^\circ \\ c = -v_{pi}^2 \end{cases}$$

$$v_{pf} = \frac{2v_{pi}\cos 37^\circ \pm \sqrt{4v_{pi}^2\cos^2 37 + 12\,v_{pi}^2}}{6}$$

$$= v_{pi}\underbrace{\frac{2\cos 37^\circ \pm \sqrt{4\cos^2 37 + 12}}{6}}_{0.902}$$

→ Fraction of energy transferred to deuteron:

$$1 - \frac{v_{pf}^2}{v_{pi}^2} = 1 - \frac{(0.902\,v_{pi})^2}{v_{pi}^2} = 1 - 0.902^2 = 0.186$$

or 18.6 %

Let's also solve for v_{df} in term of v_{pi}:

Eq(3) or CKE: $$v_{Df}^2 = \frac{v_{pi}^2 - v_{pf}^2}{2} = \frac{v_{pi}^2}{2}\Big(1 - \underbrace{\frac{v_{pf}^2}{v_{pi}^2}}_{\left(\frac{0.902\,\cancel{v_{pi}}}{\cancel{v_{pi}}}\right)^2}\Big) = \frac{v_{pi}^2}{2}\,\underbrace{0.186}$$

$$\boxed{v_{Df} = v_{pi}\sqrt{\frac{0.186}{2}} = 0.305\,v_{pi}}$$

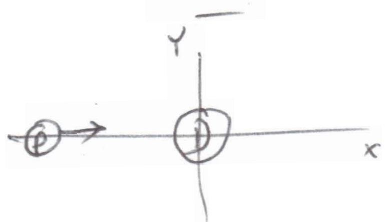
Let's find the third unknown: θ_{Df}

Eq(1) or CLM-x: $$\cancel{v_{pi}} = \cancel{v_{pi}}\,0.902\cos 37^\circ + 2\cdot 0.305\,\cancel{v_{pi}}\cos\theta_{Df}$$

$$1 = 0.902\cos 37^\circ + 0.61\cos\theta_{Df}$$

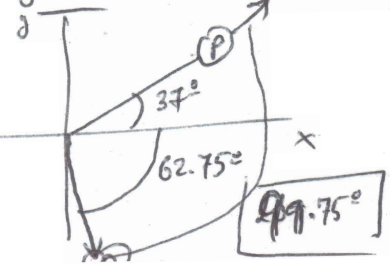
$$\theta_{Df} = \cos^{-1}\underbrace{\left[\frac{1 - 0.902\cos 37^\circ}{0.61}\right]}_{0.458} = 62.75^\circ$$

initial



Note: Final directions do not form a 90° angle because the masses of colliding particles are NOT equal!

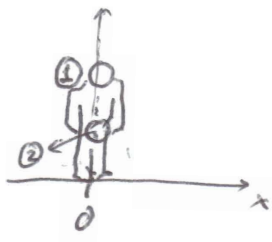
final



9.52 Tossing a rock standing on ice (no friction)

Conservation law: → Initial

1) System: you + rock
2) $F_{net} = 0$
$\hookrightarrow \vec{P}_i = \vec{P}_f$

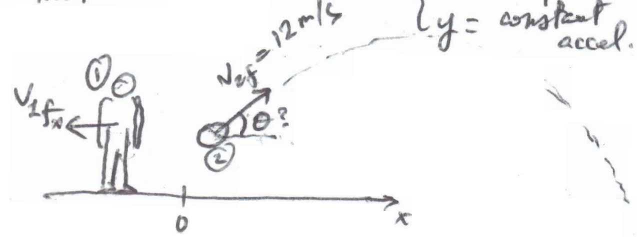


$\vec{P}_i = 0$

Final

① Uniform motion.

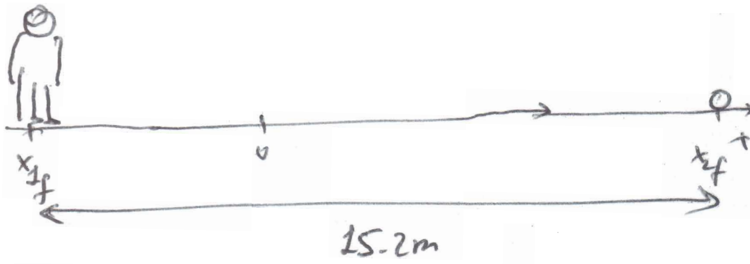
② → Projectile motion. $\begin{cases} x = \text{uniform} \\ y = \text{constant accel.} \end{cases}$



Find θ.

Kinematic equation:

$m_1 = 65\,kg$
$m_2 = 4.5\,kg$



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x - direction: $\boxed{x_{2f} - x_{1f} = 15.2\,m}$ $\begin{cases} \text{rock: } x_{2f} = v_{2f}\cos\theta \cdot 2t_{up} & (a) \\ \text{you: } x_{1f} = v_{1fx} \cdot 2t_{up} & (b) \end{cases}$

t_{up}: time for rock to go to max altitude point: final speed in y-direction of 0:

$$v_{2fy} = v_{2foy} - g \cdot t$$

$$0 = v_{2f}\sin\theta - g \cdot t_{up} \rightarrow t_{up} = \frac{v_{2f}\sin\theta}{g}$$

→ Plug back into (a) $x_{2f} = v_{2f}\cos\theta \cdot 2 \cdot \frac{v_{2f}\sin\theta}{g}$

$\left(x_{Range} = \frac{v_0^2 \sin 2\theta}{g}\right)$ $= \frac{v_{2f}^2\, 2\cos\theta\sin\theta}{g} = \frac{v_{2f}^2 \sin(2\theta)}{g}$

$$\boxed{x_{2f} = \frac{12^2 \sin(2\theta)}{9.81}}$$ (x_{2f} in term of θ)

(b) Writing $\underbrace{x_{1f}}_{\text{person}}$ in term of $\underbrace{\theta}_{\text{rock}}$: person & rock are related in conservation of Total momentum! $\vec{P}_i = \vec{P}_f$

$$\vec{P}_i = \vec{P}_f$$

$$0 = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

Rock follows 2D motion
Person acquired $\vec{v}_{1f}$ but only v_{1fx} showed
(v_{1fy} just adds pressure on his feet!)

x-direction : $\boxed{0 = m_1 v_{1fx} + m_2 v_{2fx}}$

y-direction $0 = m_1 v_{1fy} + m_2 v_{2fy}$

$$0 = m_1 v_{1fx} + m_2 v_{2f} \cos\theta \rightarrow \boxed{v_{1fx} = -\frac{m_2}{m_1} v_{2f} \cos\theta}$$

$$x_{1f} = v_{1fx} \cdot 2 t_{up} = -\frac{m_2}{m_1} v_{2f} \cos\theta \cdot 2 \frac{v_{2f} \sin\theta}{g}$$

$$\boxed{x_{1f} = -\frac{m_2}{m_1} \frac{v_{2f}^2 \sin(2\theta)}{g}}$$

$$x_{2f} - x_{1f} = 15.2\,m$$

$$\frac{12^2 \sin(2\theta)}{9.81} + \frac{4.5}{65} \frac{12^2 \sin(2\theta)}{9.81} = 15.2\,m$$

$$\sin(2\theta) = \frac{15.2}{\left[\frac{144}{9.81} \left(1 + \frac{4.5}{65} \right) \right]}$$

$$\theta = \frac{1}{2} \sin^{-1} \left[0.974 \right]$$

$$\theta = 38.5^\circ$$

9.28

Neutron striking a Deuteron and $\boxed{\text{combine}}$ (inelastic) to form Tritium (90)

(1u) (2u) (3u)

$\vec{v}_{N_i} = 28\hat{i} + 17\hat{j}\ \frac{Mm}{s}$

$\vec{v}_{D_i} = ?$

$\vec{v}_f = 12\hat{i} + 20\hat{j}\ \frac{Mm}{s}$

System: N+D $\rightarrow \vec{F}_{net,external} = 0 \rightarrow \vec{P}_i = \vec{P}_f$

initial final

$\boxed{\begin{matrix} m_N = m \\ m_D = 2m \end{matrix}}$

$\boxed{\vec{P}_i = m(28\hat{i} + 17\hat{j})10^6 + 2m\vec{V}_{Di}}$

$\boxed{\vec{P}_f = 3m(12\hat{i} + 20\hat{j})10^6}$

$\vec{P}_i = \vec{P}_f \rightarrow (28\hat{i} + 17\hat{j})10^6 + 2\vec{V}_{D_i} = (36\hat{i} + 60\hat{j})10^6$

$\vec{V}_{D_i} = \frac{(36\hat{i} + 60\hat{j})10^6 - (28\hat{i} + 17\hat{j})10^6}{2}$

$\vec{V}_{D_i} = \frac{10^6}{2}(8\hat{i} + 43\hat{j})$

$\boxed{\vec{V}_{D_i} = 4\hat{i} + 21.5\hat{j}\ \frac{Mm}{s}}$

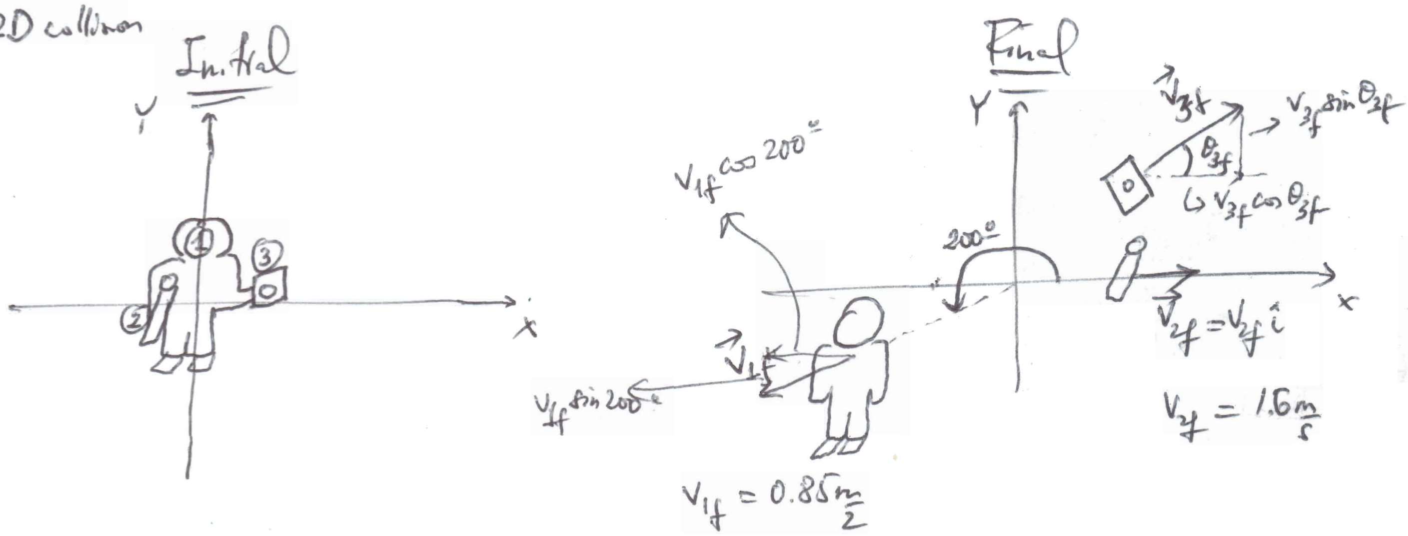
9.48

$m_1 = 60\,kg \quad m_2 = 14\,kg \quad m_3 = 5.8\,kg$

3 component system : astronaut + O_2 tank + camera

~~m_1~~ recoils at 200° ccw from x-axis → 2D collision

– In space
→ $\vec{F}_{net,\,external} = 0 \rightarrow \vec{P}_i = \vec{P}_f$



$$\vec{P}_i = 0 \quad = \quad \vec{P}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} + m_3\vec{v}_{3f}$$

$$0 = \left(m_1 v_{1f}\cos 200° + m_2 v_{2f} + m_3 v_{3f}\cos\theta_{3f}\right)\hat{i} + \left(m_1 v_{1f}\sin 200° + m_3 v_{3f}\sin\theta_{3f}\right)\hat{j}$$

$0\hat{i} + 0\hat{j}$

x-direction : $0 = 60\cdot 0.85\cos 200° + 14\cdot 1.6 + 5.8\,\underbrace{\boxed{v_{3f}\cos\theta_{3f}}}_{\equiv v_{3fx}}$

y-direction : $0 = 60\cdot 0.85\sin 200° + 5.8\,\underbrace{\boxed{v_{3f}\sin\theta_{3f}}}_{\equiv v_{3fy}}$

$$v_{3fx} = \frac{-60\cdot 0.85\cos 200° - 14\cdot 1.6}{5.8} = 4.4\,\frac{m}{s}$$

$$v_{3fy} = \frac{-60.\,0.85\sin 200°}{5.8} = 3\,\frac{m}{s}$$

1st quadrant.

Cartesian → Polar

$$v_{3f} = \sqrt{4.4^2 + 3^2} = 5.33\,\frac{m}{s}$$

$$\theta_{3f} = \tan^{-1}\frac{3}{4.4} = 34.3°$$

(ccw from axis-x 1st quadrant)

9.57

$m_1 = 1200\,kg \quad m_2 = 2200\,kg$ $\boxed{\text{inelastic}}$

a) System: Toyota + Buick ; $\boxed{\text{lock together}}$ & then skid 22m

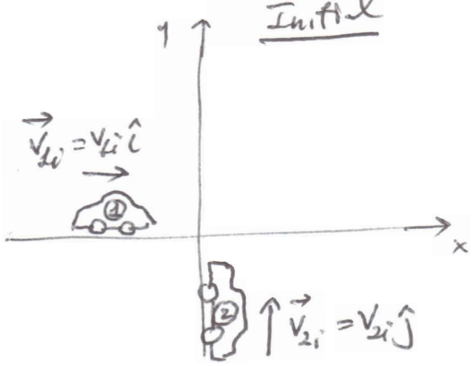
$\boxed{\text{colliding}}$ @ right angles. $\rightarrow$ $\boxed{2D}$

$\mu_K = 0.91$

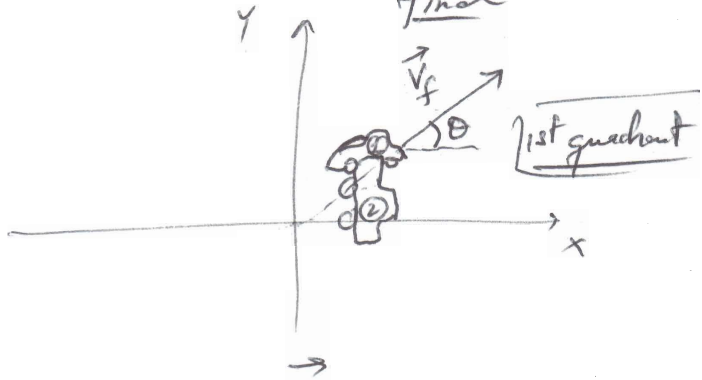
Show at least one car exceeded $25\frac{km}{h}$ speed limit.

CLM $\vec{P}_i = \vec{P}_f$ $\rightarrow$ b) Define initial & final situations with a sketch:

Initial



final



$\vec{P}_i$ $\qquad\qquad$ $\vec{P}_f$

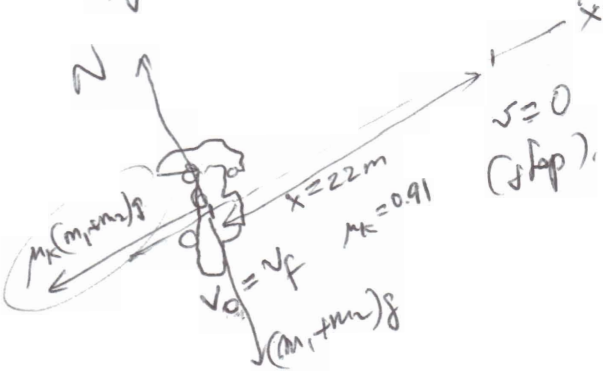
$\parallel$

1) $\quad m_1 v_{1i}\,\hat{i} + m_2 v_{2i}\,\hat{j} = (m_1 + m_2)\left(v_f \cos\theta\,\hat{i} + v_f \sin\theta\,\hat{j}\right)$

(I) $\quad m_1 v_{1i} = (m_1 + m_2) v_f \cos\theta \rightarrow 1200 v_{1i} = 3400 v_f \cos\theta$

(II) $\quad m_2 v_{2i} = (m_1 + m_2) v_f \sin\theta \rightarrow 2200 v_{2i} = 3400 v_f \sin\theta$

2) Skid together 22m to stop (b/c of friction $\mu_K = 0.91$)



They will come to a stop when all kinetic energy has been used to overcome friction:

$$\frac{1}{2}(m_1 + m_2) v_f^2 = \mu_K (m_1 + m_2) g \cdot x$$

$$v_f = \sqrt{2 \cdot \mu_K g x} = \sqrt{2 \cdot 0.91 \cdot 9.81 \cdot 22} =$$

$$\boxed{v_f = 19.82\,\frac{m}{s}}$$

(I) $1200\, V_{1i} = 3400\, v_f \cos\theta \rightarrow \cancel{\cos\theta}$

(II) $2200\, V_{2i} = 3400\, v_f \sin\theta$

$$1200^2 V_{1i}^2 + 2200^2 V_{2i}^2 = (3400 \cdot 19.82)^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$V_{1i}^2 + \underbrace{\left(\frac{2200}{1200}\right)^2}_{3.36} V_{2i}^2 = \underbrace{\left(\frac{3400 \times 19.82}{1200}\right)^2}_{3150}$$

$$V_{1i}^2 + 3.36\, V_{2i}^2 = 3150$$

Now: speed limit was $25 \frac{km}{h} = \frac{25}{3.6} \frac{m}{s} = 6.94 \frac{m}{s}$

Hypothesis: Each car was traveling @ $6.94 \frac{m}{s} \simeq 7 \frac{m}{s}$

$49 + 3.36 \cdot 49 \simeq 200$ → clearly at least one car was traveling well above $25 \frac{km}{h}$.

9.67

System: O_2 & O
$(m_1 = 32u)$ $(m_2 = 16)$

inelastic collision.
$\boxed{\text{stick}}$ together → Ozone $\vec{v}_f$?

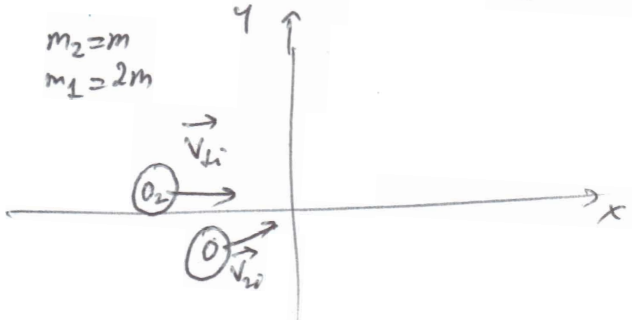
94

$\vec{v}_{1i} = v_{1i}\,\hat{i}$ $\vec{v}_{2i} = \underbrace{v_{2i}\cos 27°\,\hat{i} + v_{2i}\sin 27°\,\hat{j}}$

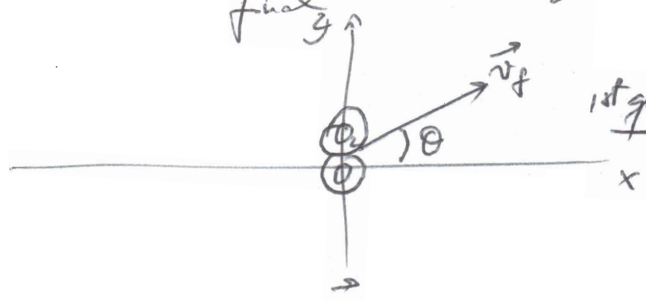
$v_{1i} = 580\,\frac{m}{s}$ $v_{2i} = 870\,\frac{m}{s}$ 2D

initial

$m_2 = m$
$m_1 = 2m$



final



1st quadrant

$\vec{F}_{net} = 0 \rightarrow \vec{P}_i = \vec{P}_f$

$$\cancel{2m}\,580\,\hat{i} + \cancel{m}\,870\left(\cos 27°\,\hat{i} + \sin 27\,\hat{j}\right) = 3\cancel{m}\left(v_{fx}\,\hat{i} + v_{fy}\,\hat{j}\right)$$

$$1160 + 870\cos 27° = 3\,v_{fx}$$
$$870\sin 27° = 3\,v_{fy}$$

$$v_{fx} = \frac{1160 + 870\cos 27°}{3} = 645.06\,\frac{m}{s}$$
$$v_{fy} = \frac{870\sin 27°}{3} = 131.66\ m/s.$$

Cartesian → polar:

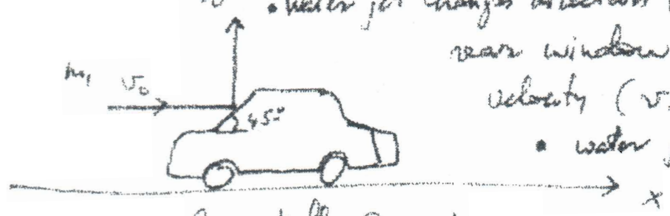
$$\boxed{\begin{array}{l} v_f = \sqrt{645.06^2 + 131.66^2} = 658.4\ m/s \\ \text{(ozone)} \\ \theta = \tan^{-1}\dfrac{131.66}{645.06} = 11.54° \end{array}}$$

$$\frac{1}{2}\cancel{m}v_{min}^2 = \cancel{m}g\,17\left(1-\cos\left(\sin^{-1}\frac{10}{17}\right)\right)$$

$$v_{min} = \sqrt{2\times 9.81\times 17\,(1-\cos 36^\circ)} = \boxed{7.98\ \text{m/s}}$$

9.43

• No friction



• water jet changes direction ($+x$ to $+y$) after hitting rear window, not magnitude of velocity (v_0)

• water jet mass rate is $\frac{dm_1}{dt}$

Car initially @ rest

M_2

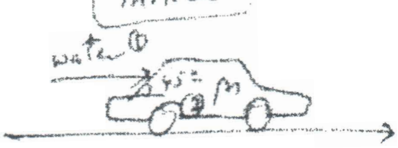
a) a_x for the car?

→ The collision b/w jet of water & car @ rest transfers some of its momentum into the car allowing it to go from zero speed to non-zero speed : acquiring an acceleration in the horizontal direction → a_x.

→ $\vec{F}_{net\,ext} = 0$ all forces are b/w components (water jet & car) of the system. (No friction).

↳ $\boxed{\vec{P}_i = \vec{P}_f}$

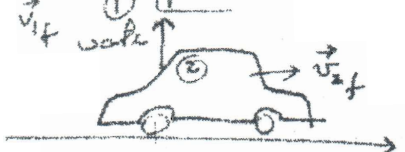
initial



• water jet $\frac{dm_1}{dt}$ @ $\vec{v}_{1i} = v_0\,\hat{i}$

• car @ rest, mass M_2

final



(vertical)

• water going up : $\vec{v}_{1f} = v_0\,\hat{j}$

• car going $+x$ @ $\vec{v}_{2f}\,\hat{i}$ (horizontal)

$$m_1\vec{v}_{1i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

Note: water & car are not sticking together after collision!

observation: $\vec{a} = \frac{d\vec{v}}{dt}$ → car $\vec{a}_2 = \frac{d\vec{v}_{2f}}{dt}$

→ $\vec{V}_{2f} = m_1 \frac{1}{m_2}(\vec{V}_{1i} - \vec{V}_{1f}) = m_1 \frac{1}{m_2}(v_0 \hat{i} - v_0 \hat{j})$

→ $\vec{a}_2 = \frac{d\vec{v}_{2f}}{dt} = \frac{dm_1}{dt} \frac{v_0}{m_2}(\hat{i} - \hat{j})$

only m_1 is changing

Observation: final acceleration of car has 2 components

$$\begin{cases} \boxed{a_x = \frac{dm_1}{dt}\frac{v_0}{m_2}} & \leftarrow \text{forward in } +x. \\ a_y = -\frac{dm_1}{dt}\frac{v_0}{m_2} \end{cases}$$

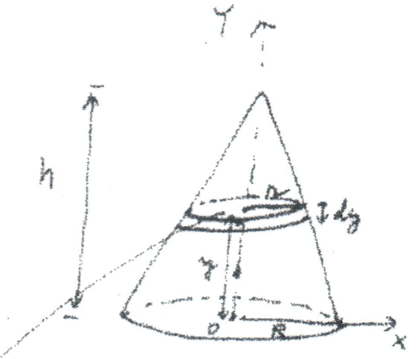
(since initially there was no momentum in the y direction, and since finally the water goes up → car gets pushed down)

b) Max. speed reached by the car?

Why max? or can we accelerate the car to ∞ speed with this water jet? No b/c when the car reaches v_0 (same speed as the water jet) no more pushing or momentum transfer is possible.

→ max speed for car is v_0!

9.39



$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm \begin{cases} x_{cm} = \frac{1}{M}\int x\, dm \\ y_{cm} = \frac{1}{M}\int y\, dm \end{cases}$$

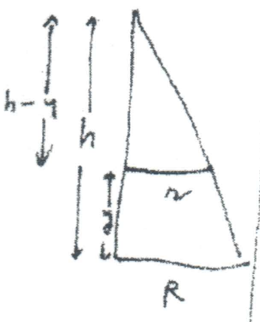
M: total mass.

$\vec{R}$: position vector of center of mass.

b/c symmetry $x_{cm} = 0$ in the coord. syst. shown. (Y axis coincide with the axis of symmetry of the cone).

infinitesimal disk of thickness dy & mass dm, radius r, located @ distance y above the base.

$$\rightarrow y_{cm} = \frac{1}{M}\int y\, dm = \frac{1}{M}\int_0^h y\, \rho\pi R^2\left(1-\frac{y}{h}\right)^2 dy =$$



$$\frac{r}{h-y} = \frac{R}{h} \rightarrow r = R\,\frac{(h-y)}{h}$$

$$\boxed{r = R\left(1-\frac{y}{h}\right)}$$

Infinitesimal disk volume is $dV = \pi r^2 dy$

Since $\rho = \frac{dm}{dV} \rightarrow dm = \rho\, dV$

$$dm = \rho\,\pi r^2\, dy = \rho\pi R^2\left(1-\frac{y}{h}\right)^2 dy$$

$$y_{cm} = \frac{\rho\pi R^2}{M}\int_0^h y\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy$$

$$= \frac{\rho\pi R^2}{M}\int_0^h \left(y - \frac{2}{h}y^2 + \frac{1}{h^2}y^3\right) dy$$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}}$$

$$\boxed{\rho = \frac{M}{\text{Vol of cone}} = \frac{M}{\pi R^2 \frac{h}{3}}}$$

$$= \frac{3}{h}\left[\frac{y^2}{2} - \frac{2}{h}\frac{y^3}{3} + \frac{1}{h^2}\frac{y^4}{4}\right]_0^h$$

$$= \frac{3}{h}\left[\frac{h^2}{2} - \frac{2}{3}h^2 + \frac{1}{4}h^2\right] = 3h\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= 3h\left[\frac{6-8+3}{12}\right] = \frac{3h}{12} = \boxed{\frac{h}{4}}$$