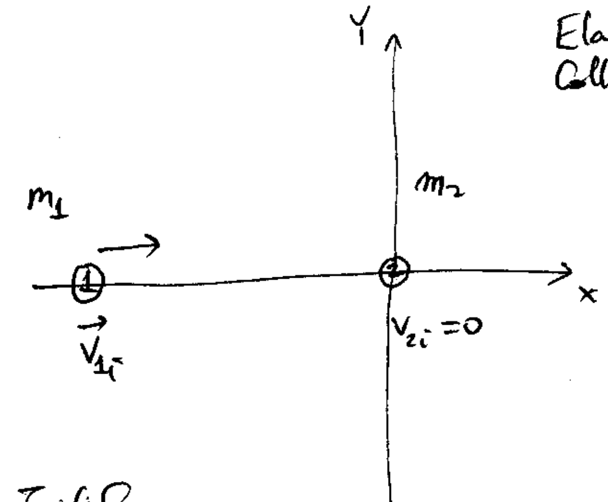
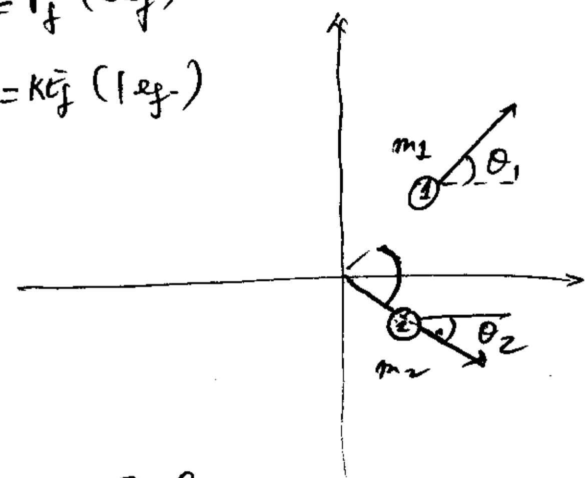
Elastic collisions in 2D: final directions after the collision form an angle of 90°.

Proof:

Elastic Collision
$$\begin{cases} \vec{P}_i = \vec{P}_f \ (2 \text{ eqs}) \\ KE_i = KE_f \ (1 \text{ eq.}) \end{cases}$$





Initial:
$\vec{v}_{1i} = v_{1i}\,\hat{\imath}$ (known)
m_1, m_2 also known

Final:
θ_1 is known
Use equations to solve for v_{1f}, v_{2f}, θ_2

① $$P_{ix} = P_{fx}$$
$$m_1 v_{1ix} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$$

② $$P_{iy} = P_{fy}$$
$$0 = m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$$ (θ_2 negative since it's below x-axis)

③ $$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$① ^2 \quad V_{fx}^2 = \left(V_{1f}\cos\theta_1 + \frac{m_2}{m_1}V_{2f}\cos\theta_2\right)^2$$

$$V_{fx}^2 = \underbrace{V_{1f}^2\cos^2\theta_1}\; + \;\underbrace{\frac{m_2^2}{m_1^2}V_{2f}^2\cos^2\theta_2} + 2\frac{m_2}{m_1}V_{1f}V_{2f}\cos\theta_1\cos\theta_2$$

$$② ^2 \quad 0 = m_1^2\left(V_{1f}\sin\theta_1 + \frac{m_2}{m_1}V_{2f}\sin\theta_2\right)^2$$

$$0 = \underbrace{V_{1f}^2\sin^2\theta_1} + \underbrace{\frac{m_2^2}{m_1^2}V_{2f}^2\sin^2\theta_2} + 2\frac{m_2}{m_1}V_{1f}V_{2f}\sin\theta_1\sin\theta_2$$

$$①^2 + ②^2 : \quad V_{1ix}^2 = V_{1f}^2\underbrace{(\cos^2\theta_1 + \sin^2\theta_1)}_{1} + \frac{m_2^2}{m_1^2}V_{2f}^2\underbrace{(\cos^2\theta_2 + \sin^2\theta_2)}_{1} +$$

$$+ 2\frac{m_2}{m_1}V_{1f}V_{2f}\Big[\underbrace{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}_{\cos(\theta_1-\theta_2) = \cos(\theta_2-\theta_1)}\Big]$$

$$\boxed{V_{1ix}^2 = V_{1f}^2 + \frac{m_2^2}{m_1^2}V_{2f}^2 + 2\frac{m_2}{m_1}V_{1f}V_{2f}\cos(\theta_1-\theta_2)} \quad ⓐ$$

Observation:

Eq ③: divide both sides by $\frac{m_1}{2}$:

$$\boxed{V_{1i}^2 = V_{1f}^2 + \frac{m_2}{m_1}V_{2f}^2} \quad ⓑ$$

ⓐ − ⓑ $\quad V_{1i}^2 = V_{1ix}^2$ (there was no y-component since we lined up x-axis with $\vec{V}_{1i}$)

$$\boxed{0 = \frac{m_2}{m_1}\left(\frac{m_2}{m_1} - 1\right)V_{2f}^2 + 2\frac{m_2}{m_1}V_{1f}V_{2f}\cos(\theta_1-\theta_2)} \quad ⓒ$$

Divide both sides by $\frac{m_2}{m_1}V_{2f}$:

$$\boxed{0 = \left(\frac{m_2}{m_1} - 1\right)V_{2f} + 2V_{1f}\cos(\theta_1-\theta_2)}$$

2D Elastic Collision (derived from $\vec{P}_i = \vec{P}_f$ & $KE_i = KE_f$)

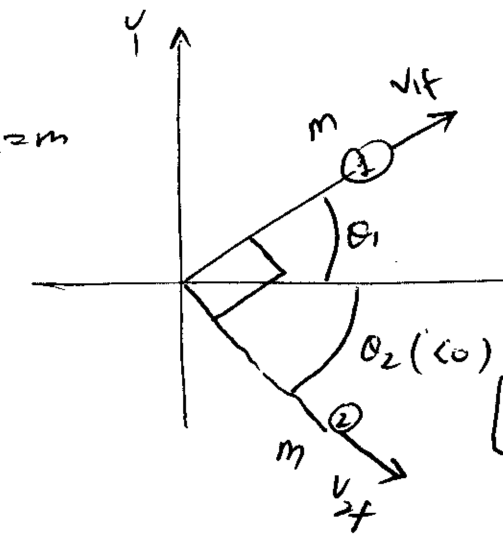
If $\boxed{m_1 = m_2} \rightarrow \left(\frac{m_2}{m_1} - 1\right) = 0$

$$0 = 0 + 2\underbrace{v_{1f}}_{\text{Not a zero in general}} \cos(\theta_1 - \theta_2)$$

Not a zero in general $\rightarrow \cos(\theta_1 - \theta_2) = 0$

$$\boxed{\theta_1 - \theta_2 = 90^\circ}$$

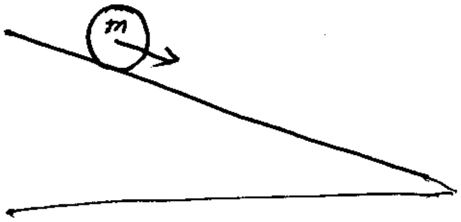
$m_1 = m_2 = m$



$$\boxed{\theta_1 - \theta_2 = 90^\circ}$$

Ch10 Rotational Motion

So far:

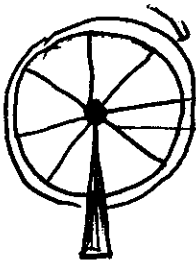


→ Sliding ball: slides downhill without any rotation. Unlikely if there friction on the slope.

More realistically:

→ Rolling ball goes downhill by a rolling motion: which involves both translational & rotational motion

Translation & rotation can happen separately as well.



→ Axis of rotation
Center of rotation
or pivot point

Bike wheel on a support: only has rotational motion

This same bike wheel can slide on an ice surface with only translational motion.

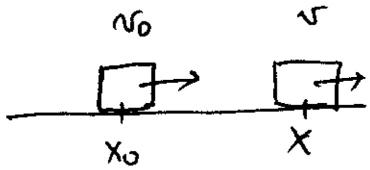
Other situations: 1) car downhill into a curve ⇒ wheels
- → translation
- → rotation (if we apply gas)
- → Normally no rotation (we brake)

2) car stuck in sand: → wheels:
- rotation
- no translation

→ car
- no translation

3) ABS: braking system: avoid sudden blocking of the wheels: using remaining kinetic energy in rotational motion to slow it down without skidding (loosing control)

Translational Motion
(change of position)



$$\bar{v} = \frac{v_0 + v}{2}$$

$$v = v_0 + a \cdot t \quad (1)$$

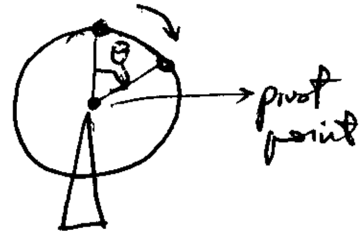
$$x = x_0 + v_0 \cdot t + \frac{1}{2} a t^2 \quad (2)$$

$$\frac{v^2 - v_0^2}{x - x_0} = 2a \quad (3)$$

$$F_{net} = m \cdot a$$

mass or inertia

Rotational Motion
(change of angle)



$$\bar{\omega} = \frac{\omega_0 + \omega}{2}$$

$$\omega = \omega_0 + \alpha \cdot t$$

$$\theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2} \alpha \cdot t^2$$

$$\frac{\omega^2 - \omega_0^2}{\theta - \theta_0} = 2\alpha$$

$$\tau_{net} = I \cdot \alpha$$

depends on location of pivot point.

ω: "omega" : angular velocity
α : "alpha" : angular acceleration
τ : "tau" : torque
I : moment of inertia

Angular velocity ω

→ Average angular velocity:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

change of angle $\Delta\theta$ over change of time Δt

$$\left(\overline{v} = \frac{\Delta x}{\Delta t} \right)$$

change of position Δx over change of time Δt

SI unit : $\frac{rad}{s} = \frac{1}{s} = s^{-1}$

SI unit: $\frac{m}{s}$

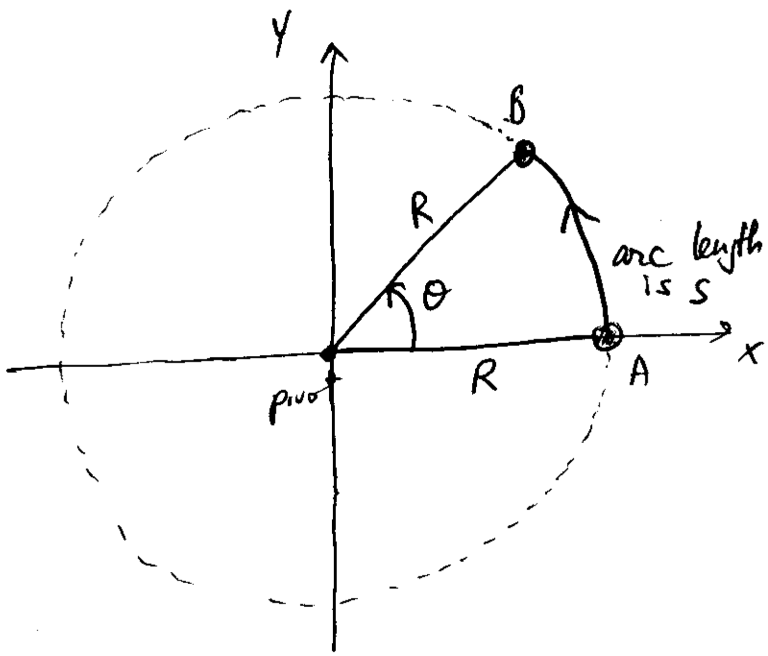
→ Instantaneous angular velocity

$$\omega = \frac{d\theta}{dt}$$

$$\left(v = \frac{dx}{dt} \right)$$

Same SI unit

→ Any connection between ω & v ? $\boxed{\text{Yes.}}$



By moving from A to B:

1) Object changed its angle from 0 to θ

2) Object changed its position along the circular trajectory by an amount equal to the length of arc AB, which is S

3) $\theta = \frac{S}{R}$

θ ↓ rotation

S → translation along circular trajectory

4) $\frac{d}{dt}\left[\theta = \frac{S}{R}\right] \rightarrow \boxed{\omega = \frac{1}{R}\frac{ds}{dt} = \frac{v}{R}}$

$\frac{1}{R}$ ↓ R is constant during rotation

$$\omega = \frac{v}{R}$$

$$\downarrow \qquad \downarrow$$

→ Unit $\quad s^{-1} \qquad \dfrac{m/s}{m} = \dfrac{1}{s} = s^{-1}$

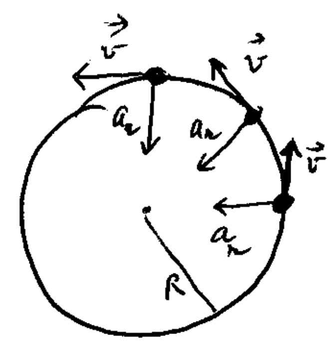
→ Other unit : rpm : revolution per minute
↓
a full rotation $\begin{cases} \text{in length: circumference of wheel} \\ \text{in angle: } 2\pi \text{ or } 360° \end{cases}$

Angular Acceleration α :

$$\bar{\alpha} = \frac{\Delta \omega}{\Delta t} \qquad \text{Average angular acceleration}$$

$$\alpha = \frac{d\omega}{dt} \qquad \text{Instantaneous angular acceleration}$$

SI Unit: $\dfrac{rad}{s^2}$ or $\dfrac{1}{s^2}$ or s^{-2}



1) Constant speed
2) Constant radial acceleration: a_r → to change direction of motion, not the linear speed

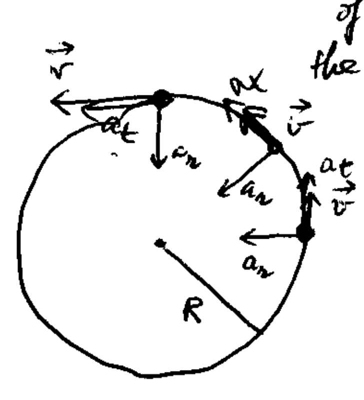
UCM : Uniform circular motion :

↳ $\boxed{a_r = \dfrac{v^2}{R} = \dfrac{\omega^2 R^2}{R} = \omega^2 R}$

$\omega = \dfrac{v}{R} \rightarrow \boxed{v = \omega R}$

3) Only radial acceleration, no tangential acceleration

$$\text{UCM} \begin{cases} a_r = \omega^2 R \\ a_t = 0 \end{cases}$$

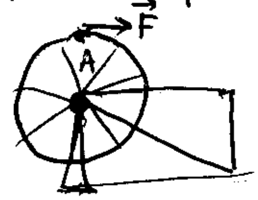


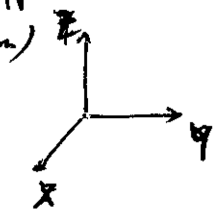
Non-UCM : Non-uniform circular motion

In addition to a_r we have a_t (tangential acceleration)

$$\text{Non-UCM} \begin{cases} a_r = R\omega^2 \\ a_t = \dfrac{dv}{dt} = \dfrac{d(\omega R)}{dt} = R\dfrac{d\omega}{dt} = R\alpha \end{cases}$$

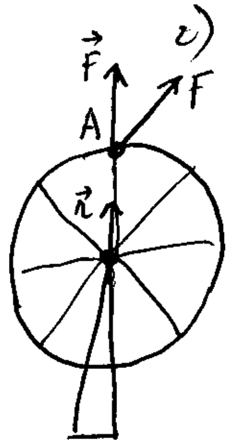
Torque $\vec{\tau}$: 1) it is a vector (direction matters): to start a bike wheel (bike up side down) to a good speed with the least effort we apply a force at A in +y (CW rotation)

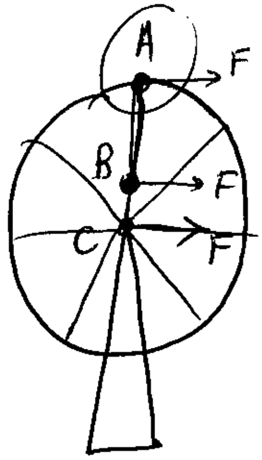




(out of page)

2)





Not only direction of force by also distance of application point to pivot point does matter!

3)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

cross product b/w two vectors

$\vec{r}$: position vector : from pivot (or center of rotation) to the force application point

$\vec{F}$: force applied

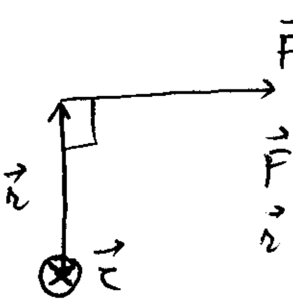
$$= rF\sin\theta \ \hat{\tau}$$

θ = angle b/w $\vec{r}$ & $\vec{F}$

$\hat{\tau}$ = unit vector that is perpendicular to the plane formed by $\vec{r}$ & $\vec{F}$

<u>Torque</u>: is vector whose magnitude is the product of the magnitudes of the position vector & the force vector times the sine of the angle θ b/w them. The torque direction is perpendicular to both $\vec{r}$ & $\vec{F}$ by right-hand rule (RHR). Unit SI: Nm (will not use J to distinguish torque from work)

Cross-Product & Right Hand Rule:



$\vec{F} = F\hat{j}$

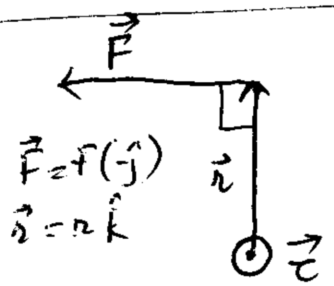
$\vec{r} = r\hat{k}$

$\vec{\tau} = \vec{r} \times \vec{F}$: perpendicular to both $\vec{r}$ & $\vec{F}$.

RHR: point RH fingers along 1st vector of cross product ($\vec{r}$), then turn these fingers towards 2nd vector of cross product ($\vec{F}$). Thumb will indicate direction of the cross product ($\vec{\tau}$)

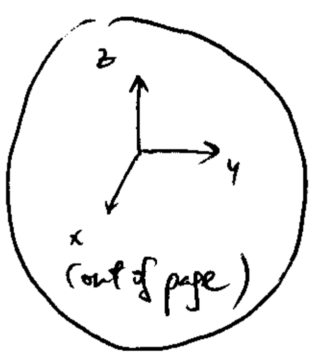
In the above example : $\vec{\tau}$ is <u>into</u> the screen ⊗

$$\boxed{\vec{\tau} = rF\sin\theta\,(-\hat{i}) = rF\,(-\hat{i})}$$

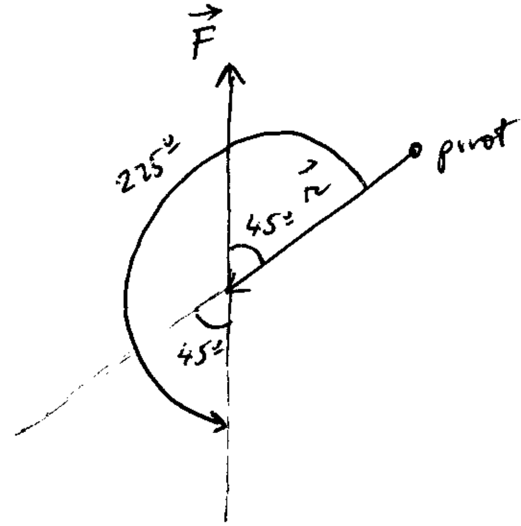


Here, by the RHR, $\vec{\tau}$ is pointing out of screen ⊙

$$\boxed{\vec{\tau} = rF\,\hat{i}}$$



Angles:

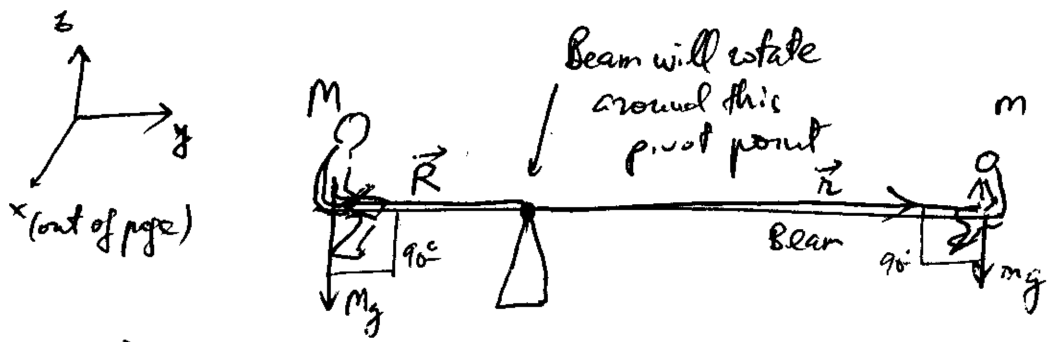


If we look at magnitudes: (non negative)

$$\tau = rF\sin\theta \begin{cases} 1)\ 0 < \theta < 90° \\ \quad \sin\theta > 0 \\ 2)\ 180 < \theta < 270° \\ \quad \sin\theta < 0 \end{cases}$$

take $|\sin\theta|$

Net torque $\vec{\tau}_{net}$ (if more than one are involved):



$$\vec{\tau}_{net} = \underset{\substack{\downarrow \\ +\hat{i} \\ (RHR)}}{\vec{\tau}_M} + \underset{\substack{\downarrow \\ -\hat{i} \\ (RHR)}}{\vec{\tau}_m} = RMg\,\underline{\hat{i}} + rmg\,\underline{(-\hat{i})} = (RM - rm)g\,\hat{i}$$

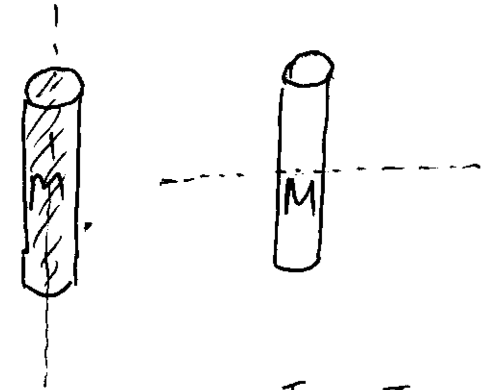
$$\Rightarrow \vec{\tau}_{net} = 0 \quad \text{if} \quad RM = rm \quad \text{or} \quad \boxed{\frac{R}{r} = \frac{m}{M}}$$

Analog of 2nd Newton's Law:

$$F_{net} = m \cdot a$$

$$\underset{\text{Net torque}}{\tau_{net}} = \underset{\text{moment of inertia}}{I} \cdot \underset{\text{angular acceleration}}{\alpha}$$

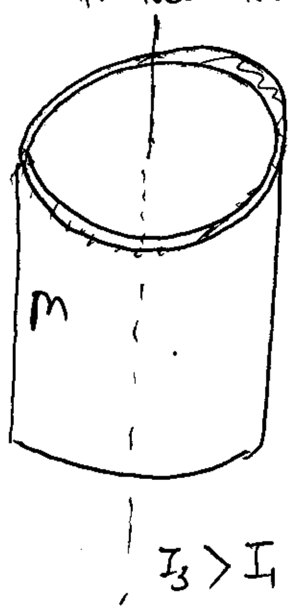
Moment of Inertia: 1) depends ~~on~~ location of axis of rotation



I_1 $\qquad$ $I_2 \neq I_1$

$$\begin{cases} I = \sum_i m_i r_i^2 \\ I = \int dm\, r^2 \end{cases}$$

2) depends on separation ~~separation~~ of mass distribution to the axis of rotation



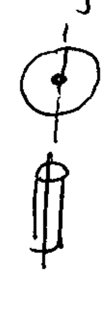
$I_3 > I_1$

Moment of inertia for round & symmetrical objects:
(disk, rod, sphere, etc...)

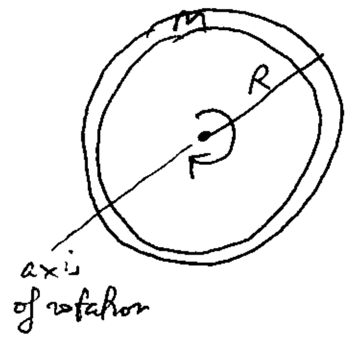
$I = \alpha M R^2$ (M: total mass of object, R: radius or length (depending on the center of rotation))

1) sphere wrt center axis: $\alpha = \frac{2}{5}$
2) cylinder wrt center axis $\alpha = \frac{1}{2}$
3) thin rod of length L $\alpha = \frac{1}{12}$ $\quad I = \frac{1}{12} M L^2$

(axis ⊥ rod through its center)

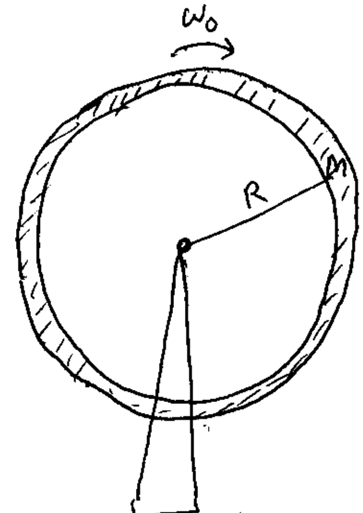


4) Ring of mass M & radius R $\rightarrow \alpha = 1$

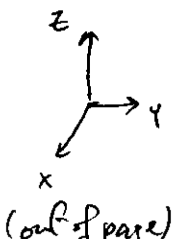


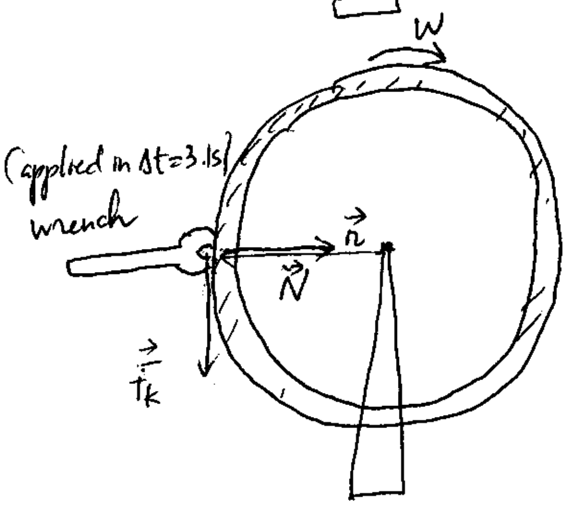
$I = MR^2$

(10.58) Bike Wheel spinning @ $\omega_0 = 230$ rpm



$M = 1.9$ kg
$R = 0.33$ m
$\Big\}$ Ring $\rightarrow I = MR^2$





$N = 2.7$ N $\rightarrow$ (Note: this force is not directly changing the rotational motion: $\vec{\tau}_N = \vec{r} \times \vec{N} = 0$ (angle b/w $\vec{r}$ & $\vec{N}$ is 180°)

$$\begin{cases} \vec{N} = 2.7\hat{j} \\ \vec{r} = 0.33(\hat{j}) \\ \vec{\tau}_N = 0 \end{cases}$$

$\mu_k = 0.46 \rightarrow F_k = \mu_k N$

$$\begin{cases} \vec{F}_k = 0.46 \cdot 2.7 \, (-\hat{k}) \text{ N} \\ \vec{r} = -0.33\hat{j} \text{ m} \\ \vec{\tau}_F = \vec{r} \times \vec{F}_k = 0.33 \cdot 0.46 \cdot 2.7 \, \hat{i} \text{ Nm} \end{cases}$$

$\odot$ (RHR)

This torque will change the rotational motion (slowing it down since friction is always opposing motion) $\boxed{\text{so } \omega < \omega_0}$

$\rightarrow \tau_{net} = I \cdot \alpha \longrightarrow \underbrace{0.33 \cdot 0.46 \cdot 2.7}_{R\mu N} = MR^2 \cdot \dfrac{\omega_0 - \omega}{\Delta t}$

Here all information are given except for ω :

$$\omega = \omega_0 - \underbrace{\frac{\tau_{net}}{MR^2}}_{\alpha} \cdot \Delta t = \underbrace{\omega_0}_{230\,rpm} - \underbrace{\underbrace{\frac{\mu N \cancel{R} \cdot \Delta t}{M R^{\cancel{2}}}}_{\frac{rad}{s}\ (\text{in SI})}}_{\Delta\omega}$$

angular deceleration due to torque of friction

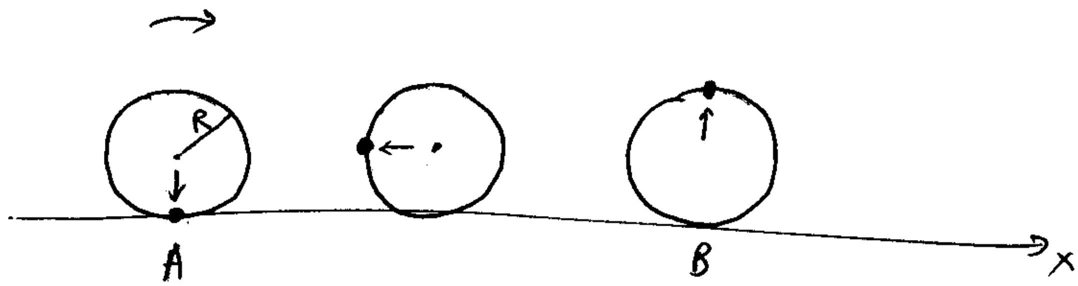
$$\Delta\omega = \frac{0.46 \cdot 2.7 \cdot 3.1}{1.9 \cdot 0.33} = 6.14 \frac{\cancel{rad}}{\cancel{s}} \cdot \frac{1\,rev}{2\pi\,\cancel{rad}} \cdot \frac{60\,\cancel{s}}{1\,min} = 58.6\ rpm$$

(Change of angular speed due to angular decceleration because of the torque of friction.)

$$\boxed{\omega = \omega_0 - 58.6\,rpm = 230\,rpm - 58.6\,rpm = 171\,rpm}$$

Rolling motion:

↳ not skidding:



1) CM of disk went from point A to point B
2) Disk in contact with surface at all time (non skidding)
AB is the same as the length of half the circumference of disk or πR

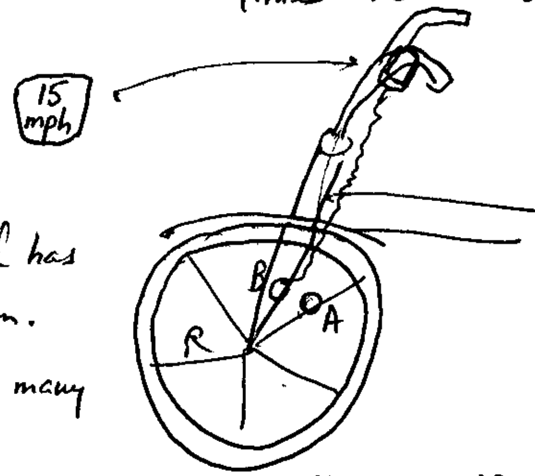
$$v_{cm} = \frac{\Delta x}{\Delta t} = \frac{\pi R}{\Delta t}$$

$$\omega = \frac{\Delta \theta}{\Delta t} = \frac{\pi}{\Delta t}$$

In a rolling motion $\boxed{v_{cm} = \omega R}$

linear speed is angular speed times radius of the disk.

"Bike computer"



Every pin A passes by pin B (sensor) the wheel has completed one revolution. Computer counts how many passes per minute →
ω (rpm) → v_{cm} using rolling motion connection

Kinetic energy:

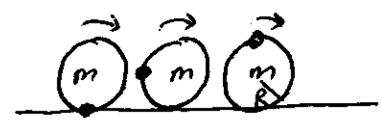
① linear motion	② Rotational motion	③ Rolling motion
$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

No friction

Disk skidding w/o rotation on a frictionless surface

$I = \frac{1}{2}MR^2$

Disk rotating freely on a support (no translational motion!)



Disk rolling on a surface with friction: translation of CM & rotation wrt CM

Rolling motion since $v_{cm} = \omega \cdot R \rightarrow \omega = \frac{v_{cm}}{R}$

$\hookrightarrow KE = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$

comparing ③ with ①: $m \longrightarrow m + \frac{I}{R^2} \underset{\text{disk}}{=} m + \frac{\frac{1}{2}m\cancel{R^2}}{\cancel{R^2}} = m + \frac{m}{2}$

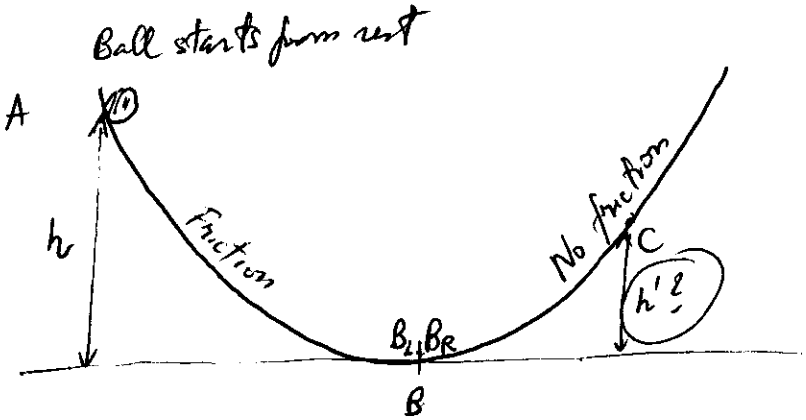
$\rightarrow$ For a disk if there is a rolling motion $\rightarrow$ mass increases by 50%

Consequence: ABS braking $\rightarrow$ goal was to brake with rolling wheels instead of blocked wheels

Rolling wheels $= \rightarrow$ non-skidding car ③ — m is increased by 50% $\rightarrow$ lower speed

Blocked wheels $= \rightarrow$ skidding car ①

10.64



Ball: { mass m, radius R }

AB : rolling motion (translation & rotation)

BC : skidding motion (only translation)

<u>Conservation of Mechanical energy</u> :

$$(A) \xleftrightarrow{\text{Rolling motion}} (B_L)$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \qquad (I : \text{sphere wrt center} : I = \frac{2}{5}mR^2)$$

$$= \frac{1}{2}mv^2 + \frac{1}{\cancel{2}}\frac{\cancel{2}}{5}m\cancel{R^2}\cdot\frac{v^2}{\cancel{R^2}}$$

ω^2 in rolling motion! $v = \omega R$

$$= \frac{1}{2}m\left(1 + \frac{2}{5}\right)v^2$$

$$\rightarrow v_B^2 = \frac{2gh}{1+\frac{2}{5}} = \frac{10}{7}gh \quad \text{or} \quad v_B = \sqrt{\frac{10}{7}gh}$$

$$(B_R) \xleftrightarrow{\text{skidding motion}} (C) \quad \text{highest point} \rightarrow \text{no KE}$$

$$\frac{1}{2}\cancel{m}v_B^2 = \cancel{m}gh' \quad \rightarrow h' = \frac{v_B^2}{2g} = \frac{\frac{10gh}{7}}{2g} = \frac{10}{14}h$$

$$\boxed{h' = \frac{5}{7}h}$$

Why $h' < h$? Since some initial grav. PE was due to friction (or the rotational motion of ball b/w A & B_L)

Difference b/w linear & rotational motion:

Linear

1) Agent to change motion:
   Force

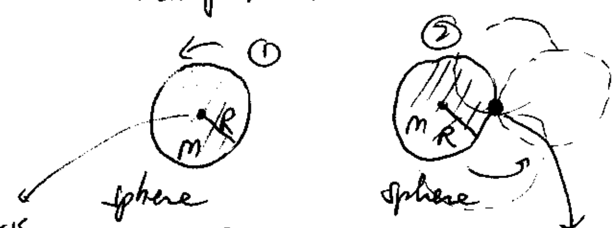
2) Same mass → same inertia

Rotational

1) Agent to change motion:
   Torque

2) a) Same shape same mass
   → may have different inertia depending on axis of rotation:

view from above



sphere — axis of rotation — $I_1 = \frac{2}{5}MR^2$

sphere — axis of rotation

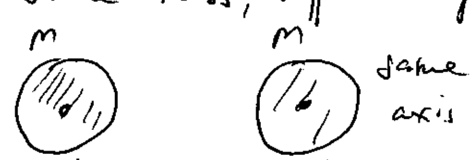
$= \frac{7}{5}MR^2$

$I_2 = I_1 + MR^2$

displacement of axis from center to the edge

$\boxed{\text{"Parallel Axis Theorem"}}$

b) Same mass, different shape:

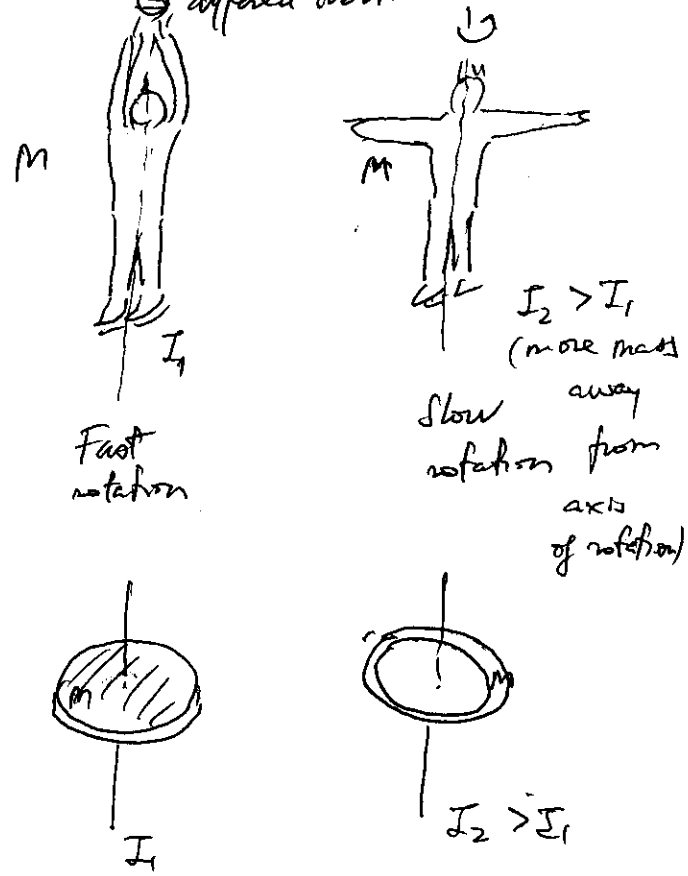


sphere: $I = \frac{2}{5}MR^2$

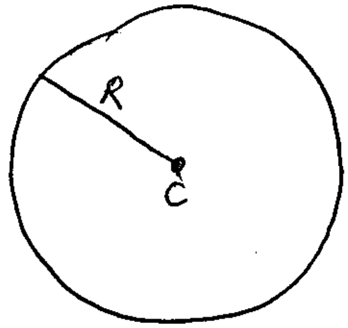
disk: $I = \frac{1}{2}MR^2$

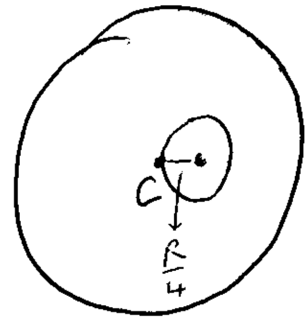
3) Same mass, any distribution
→ same inertia

3) Same mass, different distribution
→ different inertia



10.65





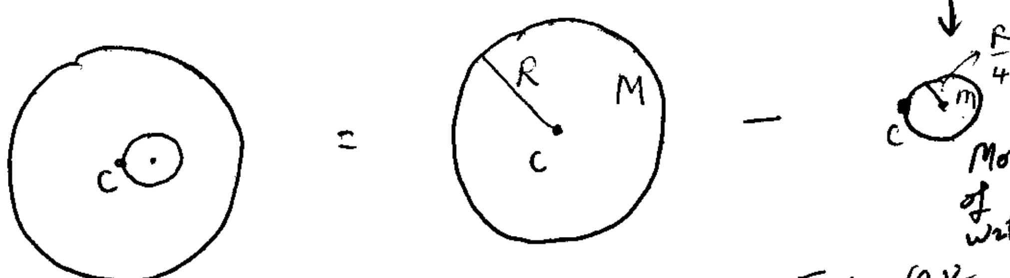
Disk of radius R and mass M

$I_1 = \frac{1}{2} M R^2$

I_2 ? with the hole wrt C

$I_2 < I_1$

Parallel Axis Theorem: where do we apply it ? here



Moment of inertia of disk of radius $\frac{R}{4}$ wrt. a pivot at edge

$$I_2 = I_1 - \Big[\underbrace{\tfrac{1}{2} m \left(\tfrac{R}{4}\right)^2}_{\text{Moment of inertia of little disk wrt its own center}} + \underbrace{m \left(\tfrac{R}{4}\right)^2}_{\text{Due to Parallel Axis Theorem}} \Big]$$

$$I_2 = \frac{1}{2} M R^2 - \frac{3}{2} m \left(\frac{R}{4}\right)^2$$

$$\frac{m}{M} = \frac{\pi \left(\frac{R}{4}\right)^2}{\pi R^2} = \frac{1}{16}$$

$$I_2 = \frac{1}{2} M R^2 - \frac{3}{2} \frac{M}{16} \frac{R^2}{16} = \frac{1}{2} M R^2 \underbrace{\left[1 - \frac{3}{16^2} \right]}_{\frac{253}{256}}$$

Reduction factor due to the hole

10.37 Translational & rotation KE

Translational (v) & rotation (ω) KE

Fraction of rotational in the total KE?

$m = 0.15\,kg$
$R = 0.037\,m$

$v = 33\,\frac{m}{s}$

$\omega = 42\,\frac{rad}{s}$

Baseball → sphere : $I = \frac{2}{5} M R^2$

$$\frac{KE_{Rot.}}{KE_{Trans} + KE_{rot}} = \frac{\frac{1}{2} I \omega^2}{\frac{1}{2} M v^2 + \frac{1}{2} I \omega^2}$$

$$\rightarrow = \frac{\cancel{\frac{1}{2}} \frac{2}{5} \cancel{M} R^2 \omega^2}{\cancel{\frac{1}{2}} \cancel{M} v^2 + \cancel{\frac{1}{2}} \frac{2}{5} \cancel{M} R^2 \omega^2}$$

$$= \frac{\frac{2}{5} R^2 \omega^2}{v^2 + \frac{2}{5} R^2 \omega^2}$$

$$= \frac{\frac{2}{5}\, 0.037^2 \cdot 42^2}{33^2 + \frac{2}{5}\, 0.037^2 \cdot 42^2}$$

$$= 8.86 \cdot 10^{-4} \text{ or } 0.0886\%$$

(very small)

Note:

$\omega R = 42 \cdot 0.037 = 1.554\,\frac{m}{s}$ → This is the linear speed of a point on the surface of baseball

$v = 33\,\frac{m}{s}$

→ It's not v (velocity of CM of ball) because this is not a rolling motion!

→ It's very small so most KE is in the translational motion

Ch 11 Rotational Vectors & Angular Momentum:

Linear Motion

$\vec{F}_{net} = m\vec{a}$

$\uparrow$

$\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$

$\vec{p}$: linear momentum

$\vec{p} = m\vec{v}$

Rotational Motion

$\vec{\tau}_{net} = I \cdot \vec{\alpha}$

$\uparrow$

$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$

$\vec{L}$: angular momentum

$\vec{L} = \vec{r} \times \vec{p}$

(cross-product between position vector $\vec{r}$ & the linear momentum $\vec{p}$)

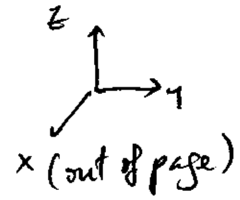
From what we discussed about the vector cross-product:

$\vec{L}$: its direction is perpendicular to both $\vec{r}$ & $\vec{p}$ (RHR)

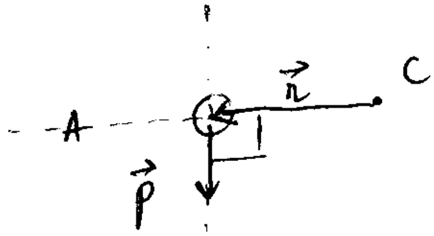
Note: Same as with the torque $\vec{\tau}$, the angular momentum is defined wrt a center of rotation:

$$\rightarrow \begin{cases} \vec{\tau} = \vec{r} \times \vec{F} & : \vec{r} \text{ position vector: from the pivot point or center of rotation to the force application point.} \\ \vec{L} = \vec{r} \times \vec{p} & : \vec{r} \text{ position vector: from the center of rotation to the position of the mass } m \text{ that has linear momentum } \vec{p} \end{cases}$$

Example:

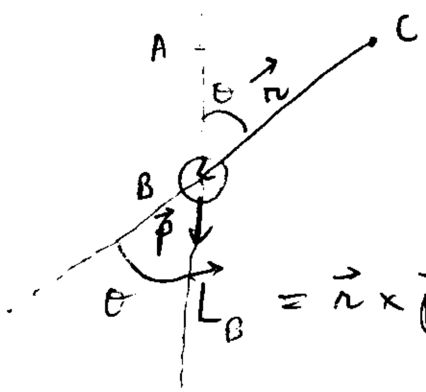


Object moving along $-z$ or $(-\hat{k})$ away from point C, with linear momentum $\vec{p}$



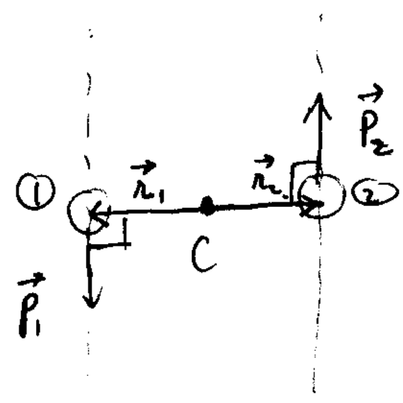
What is the angular momentum for this object at A?

$$\vec{L}_A = \vec{r} \times \vec{p} = rp \sin 90^\circ \underset{\uparrow \, RHR}{(\hat{\imath})}$$



What is the angular momentum for this object at B?

$$\vec{L}_B = \vec{r} \times \vec{p} = \underbrace{rp\sin\theta}_{>0} \underset{\uparrow \, RHR}{(\hat{\imath})}$$



What is the total angular momentum for both objects at this instant

$$\vec{L}_{Total} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= rp \underbrace{\sin 90^\circ}_{1} (\hat{\imath}) + rp \underbrace{\sin 90^\circ}_{1} (\hat{\imath})$$

$$= 2rp\,\hat{\imath}$$

$p_1 = p_2 \equiv p \rightarrow \vec{p}_1 = p(-\hat{k})$; $\vec{p}_2 = p\hat{k}$

$r_1 = r_2 \equiv r \rightarrow \vec{r}_1 = r(-\hat{j})$; $\vec{r}_2 = r(\hat{j})$

<u>Note:</u> if both ① & ② go down $(-z)$ with same momentum there is no total angular momentum $L_{Total} = 0$

From previous example: We can define the angular momentum for a system of particles:

$$\vec{L} = \underbrace{\sum_i}_{\text{sum over components of the system}} \vec{L}_i = \sum_i \underbrace{\vec{r}_i \times \vec{p}_i}_{\text{each component has its own position \& linear momentum}}$$

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right) = \sum_i \frac{d}{dt}\left(\vec{r}_i \times m_i \vec{v}_i\right)$$

$$= \sum_i \left(\underbrace{m_i \underbrace{\frac{d\vec{r}_i}{dt}}_{\vec{v}_i} \times \vec{v}_i}_{\underbrace{v_i V_i \sin 0}_{0}} + \underbrace{\vec{r}_i \times \underbrace{\frac{d(m_i \vec{v}_i)}{dt}}_{\frac{d\vec{p}_i}{dt} \ (\text{2}^{\text{nd}} \text{ Newton's law}) \ = \ \vec{F}_i \ (\text{force on component } i)}}_{\vec{\tau}_i \ (\text{torque on component } i)} \right)$$

$$\frac{d\vec{L}}{dt} = \underbrace{\sum_i \vec{\tau}_i}_{\vec{\tau}_{net} \text{ on system}} \quad \Bigg\} \rightarrow \boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

Got from
1) Definition of angular momentum $\vec{L}$ $(\vec{L}_i = \vec{r}_i \times \vec{p}_i)$
2) 2nd Newton's Law
3) Definition of torque $(\vec{\tau}_i = \vec{r}_i \times \vec{F}_i)$
4) Property of cross-product $(\vec{v}_i \times \vec{v}_i = 0)$

Linear Motion

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

$$\vec{F}_{net} = 0 \rightarrow \vec{p}_i = \vec{p}_f$$

(collisions)

Conservation of Linear Momentum.

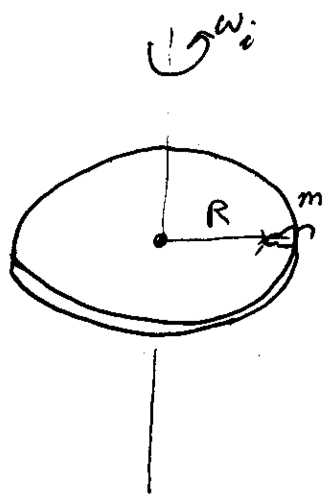
Rotational Motion

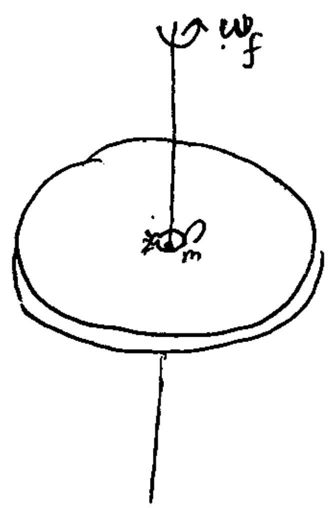
$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau}_{net} = 0 \rightarrow \vec{L}_i = \vec{L}_f$$

Conservation of Angular Momentum.

11.40





→ Disk spinning wrt center axis with I_D (rotational inertia)

→ Mouse of mass m at outer edge with $I_m = mR^2$ (R)

↳ Total inertia here is $I_D + mR^2$

→ Disk spinning wrt center axis → I_D

→ Mouse is now at center of rotation → $I_m = 0$

↳ Total inertia is I_D

---

a) System of disk & mouse: $\vec{\tau}_{net} = 0 \rightarrow \boxed{\vec{L}_i = \vec{L}_f}$

What is the connection with angular speeds?

$\tau = I \cdot \alpha$ also: $\tau = \frac{dL}{dt}$

$= I \frac{d\omega}{dt} = \frac{d(I\omega)}{dt}$

time independent

$\rightarrow \boxed{L = I\omega} \longrightarrow \boxed{I_i \omega_i = I_f \omega_f}$

$$\left(I_D + mR^2\right)\omega_i = I_D\,\omega_f \Rightarrow \omega_f = \frac{I_D + mR^2}{I_D}\,\omega_i$$

b) Rotational KE:

$$KE_i = \frac{1}{2}\left(I_D + mR^2\right)\omega_i^2 = \frac{1}{2}\left(0.0154 + 0.0195 \cdot 0.25^2\right)2.3^2 = 0.044\,J$$

$$KE_f = \frac{1}{2} I_D\,\omega_f^2 = \frac{1}{2}\,0.0154 \cdot 2.48^2 = 0.0474\,J$$

$$\omega_i = 22\,rpm \cdot \frac{2\pi}{1\,rev}\,\frac{1\,min}{60\,s} = 2.3\,\frac{rad}{s}$$

$$\omega_f = 23.7 \cdot \frac{2\pi}{60} = 2.48\,\frac{rad}{s}$$

$$KE_f - KE_i = 0.003\,J = 3\,mJ$$

↓

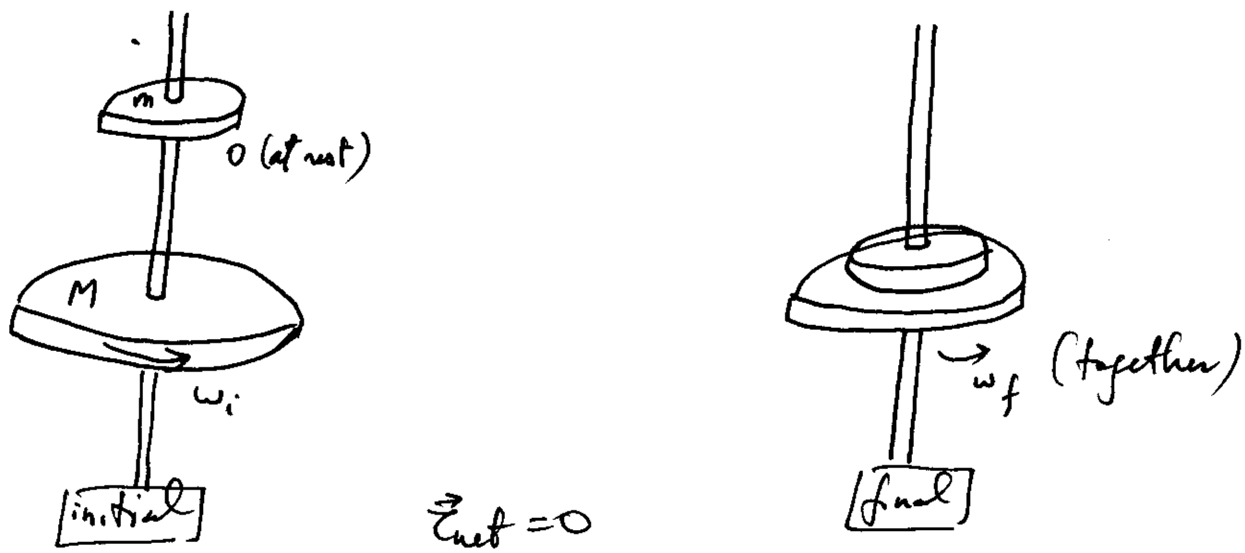
work done by mouse

$$\omega_f = \frac{0.0154 + 0.0195 \cdot 0.25^2}{0.0154}\,22\,rpm$$

$$\boxed{\omega_f = 23.7\,rpm}$$

As the rotational inertia is reduced, the angular speed is higher due to conservation of angular momentum

11.51



$M = 0.44\,kg$

$R = 0.035\,m$

$\omega_i = 180\,rpm$

$m = 0.27\,kg$

$r = 0.023\,m$

$\vec{\tau}_{net} = 0$

↓

a) Conservation of angular momentum:

$$L_i = L_f$$

$$I_i\,\omega_i = I_f\,\omega_f$$

$$\tfrac{1}{2}MR^2\,\omega_i = \left(\tfrac{1}{2}MR^2 + \tfrac{1}{2}mr^2\right)\cdot\omega_f$$

$$\omega_f = \frac{MR^2}{MR^2 + mr^2}\,\omega_i = \frac{0.44 \cdot 0.035^2}{0.44 \cdot 0.035^2 + 0.27 \cdot 0.023^2}\,180\,rpm$$

$$\boxed{\omega_f = 142\,rpm}$$

b) Fraction of energy lost:

$$\frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i}$$

$$= 1 - \frac{\cancel{\tfrac{1}{2}}\left(\tfrac{1}{2}MR^2 + \tfrac{1}{2}mr^2\right)\omega_f^2}{\cancel{\tfrac{1}{2}}\ \tfrac{1}{2}MR^2 \cdot \omega_i^2}$$

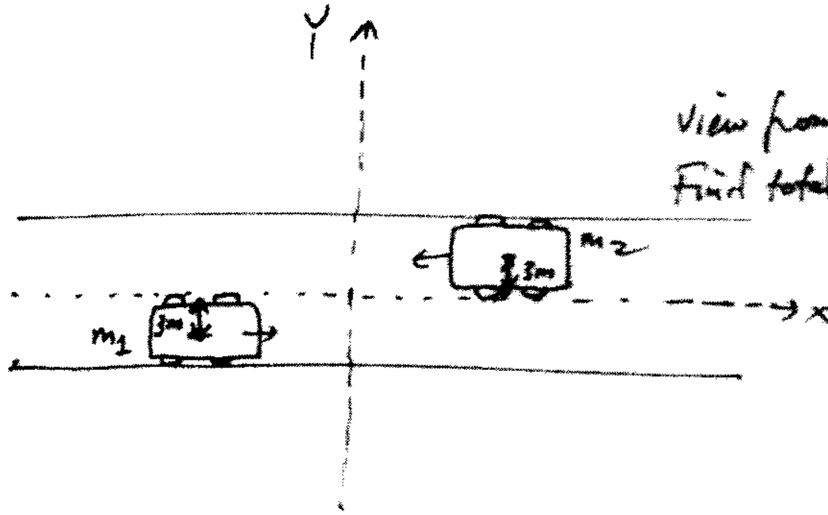
$$= 1 - \frac{MR^2 + mr^2}{MR^2}\,\frac{\omega_f^2}{\omega_i^2}$$

$$= 1 - \frac{0.44 \cdot \cancel{0.25}\,0.035^2 + 0.27 \cdot 0.023^2}{0.44 \cdot 0.035^2} \cdot \frac{142^2}{180^2}$$

$$= 0.265 \text{ or } 26.5\%$$

This lost went into friction b/w smaller & large disks.

11-37



view from above!
Find total $\vec{L} = \vec{L}_1 + \vec{L}_2$
$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times$

$m_1 = m_2 = 1800\ kg$

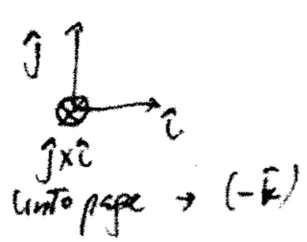
$90\frac{km}{h} = 25\frac{m}{s}$ $\quad \vec{V}_1 = 25\ \hat{\imath}\ m/s \quad ; \quad \vec{r}_1 = x_1\hat{\imath} - 3\hat{\jmath}\ (m)$

$\vec{V}_2 = 25(-\hat{\imath})\ m/s \quad ; \quad \vec{r}_2 = x_2\hat{\imath} + 3\hat{\jmath}\ (m)$

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= (x_1\hat{\imath} - 3\hat{\jmath}) \times \underbrace{25\hat{\imath} \cdot 1800}_{45000\hat{\imath}} + (x_2\hat{\imath} + 3\hat{\jmath}) \times \underbrace{25(-\hat{\imath})1800}_{-45000\hat{\imath}}$$
$$= 135000\,\hat{k} + 135000\,\hat{k} = 270000\,\hat{k}\ \ \frac{kg\,m^2}{s} = J \cdot s$$

$\hat{\imath} \times \hat{\imath} = 0$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$



Equations:

$$L_i = L_f$$

$$I_1 \omega_i + |\vec{r}_2 \times \vec{p}_2| = (I_1 + I_2)\,\omega_f$$

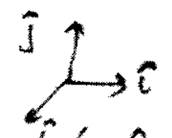
$I_2 \downarrow m_2 r_2^2 \downarrow ?$

$\boxed{\text{clay}}$

$$\vec{r}_2 \times \vec{p}_2 = (x_2\hat{i} + y_2\hat{j}) \times v_2 \hat{j}\, m_2$$

$$= x_2 v_2 m_2 \underbrace{\hat{i}\times\hat{j}}_{\hat{k}} = \underbrace{0.15 \times 1.3 \times m_2}_{0.195} \hat{k}$$

$\downarrow$ $\hat{j} \times \hat{j}$



$$\boxed{\begin{array}{l} \vec{L}_{Table} = I_1 \omega_i (-\hat{k}) \\ \vec{L}_{clay} = 0.195 m_2 (+\hat{k}) \end{array} \Big\}\ \text{opposite signs} \\ \to \text{clay was opposing rotation!}}$$

$\boxed{\text{Turn table}}$

$$I_1 \vec{\omega}_i = I_1 \omega_i (-\hat{k})$$ (Direction by RHR: fingers turning as $\vec{\omega}_i$, thumb is direction of $\vec{\omega}_i$)

$$-0.021 \times 0.29 + 0.195\, m_2 = -\left(0.021 + m_2 \underbrace{0.15^2}\right) 0.085$$

$$+0.021 \times 0.29 - 0.195\, m_2 = (0.021 + 0.0225)\, 0.085 \quad 0.0225$$

$$+\frac{0.021 \times 0.29}{0.085} - 0.021 = m_2 \left(0.0225 + \frac{0.195}{0.085}\right)$$

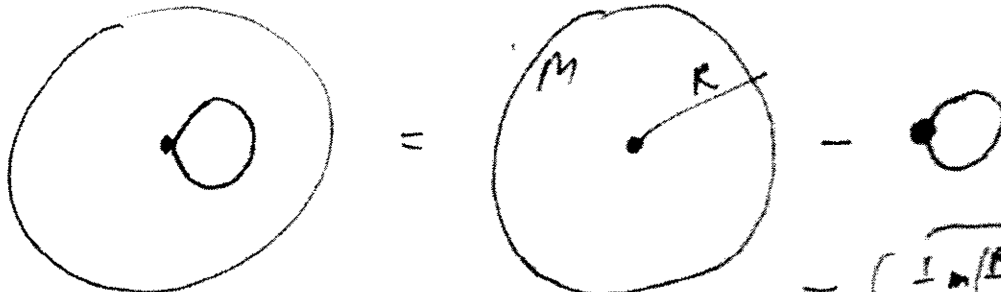
$$m_2 = \frac{\frac{0.021 \times 0.29}{0.085} - 0.021}{0.0225 + \frac{0.195}{0.085}}$$

$$\boxed{m_2 = 0.0218\ kg = 21.8\ g.}$$

$\to$ Pay attention on the direction of angular momentum!

$$m = \frac{\pi \left(\frac{R}{4}\right)^2}{\pi R^2} M = \frac{M}{16}$$

10.65



$$-\overbrace{\left(\frac{1}{2} m \left(\frac{R}{4}\right)^2 + m\left(\frac{R}{4}\right)^2\right)}^{\frac{3}{32} m R^2} = 0.496$$