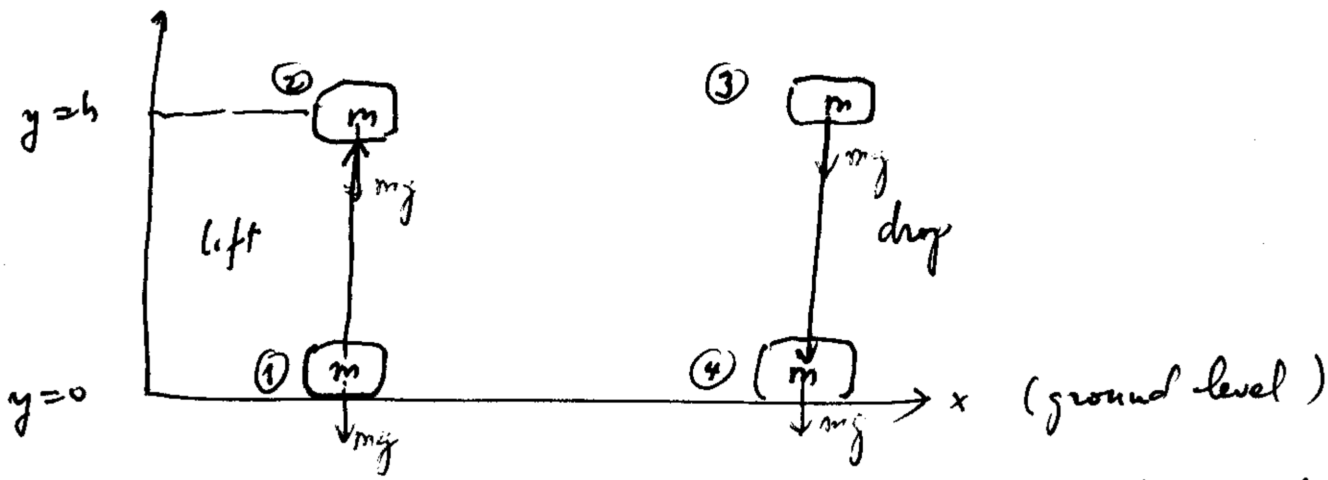
Ch7: Conservation of Energy

- Work done by a conservative force (e.g: gravitational force) is conserved
- Work done by a non-conservative force (e.g: frictional force) is not conserved



Work done by gravity ① → ②

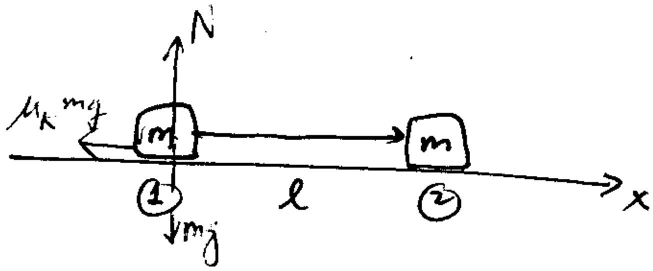
$$Work_{(12)} = \vec{F}_{grav} \cdot \Delta\vec{y}_{12}$$
$$= -mg\,\hat{j} \cdot h\,\hat{j} = -mgh\,\underbrace{\hat{j}\cdot\hat{j}}_{1}$$
$$= \ominus mgh$$

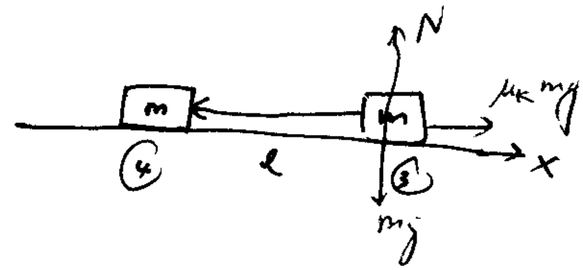
Work done by gravity ③ → ④

$$Work_{34} = \vec{F}_{grav} \cdot \Delta\vec{y}_{34}$$
$$= -mg\,\hat{j} \cdot (-h\,\hat{j})$$
$$= mgh\,\underbrace{\hat{j}\cdot\hat{j}}_{1} = \oplus mgh$$

→ Total work done by gravity ① → ② → ③ → ④ is 0!

- Work done by gravity or gravitational potential energy is conserved (gravity is a conservative force)





Work done by friction:

$$\text{Work}_{12} = \vec{F}_f \cdot \Delta\vec{x}_{12} = -\mu_K mg\,\hat{i} \cdot l\,\hat{i} = -\mu_K mgl\,\underbrace{\hat{i}\cdot\hat{i}}_{1} = -\mu_K mgl$$

$$\text{Work}_{34} = \vec{F}_f \cdot \Delta\vec{x}_{34} = +\mu_K mg\,\hat{i} \cdot (-l\,\hat{i}) = -\mu_K mgl\,\underbrace{\hat{i}\cdot\hat{i}}_{1} = -\mu_K mgl$$

Total work done by friction: 1 → 2 → 3 → 4 is $-2\mu_K mgl \neq 0$

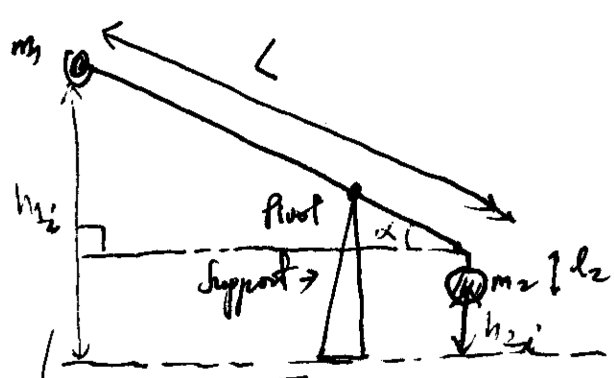
- Work done by friction is <u>not</u> conserved. Friction is not a conservative force.

<u>Conservation of Mechanical Energy:</u>

Sum of Kinetic Energy & Grav. Potential Energy is conserved.

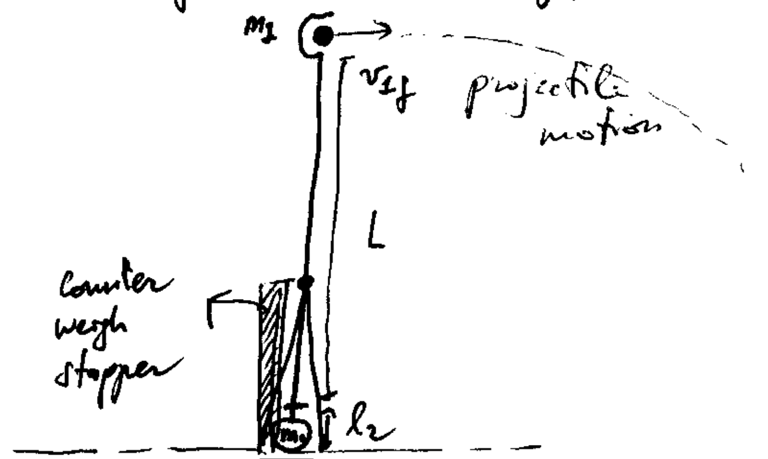
$$\underbrace{\tfrac{1}{2}mv_i^2 + mgh_i}_{\text{initial mechanical energy}} = \underbrace{\tfrac{1}{2}mv_f^2 + mgh_f}_{\text{final mechanical energy}}$$

PP3 : 7.1 Catapult or trebuchet: uses a counter weight to launch an object forward (at a higher or lower speed depending on the initial height of the counter weight) → Conservation of Mechanical Energy.



initial situation

$h_{1i} = L \sin\alpha + l_2 + h_{2i}$ (*)



final situation
(m_1 is leaving the catapult at speed v_{1f})

- If we could calculate v_{1f} → can find out where m_1 will land based on initial parameters such as h_2
- Limitations: friction at pivot, weight of arm; air resistance.

With initial & final situations defined → let's apply conservation of mechanical energy:

$$\underbrace{\frac{1}{2}m_1 \overset{0}{v_{1i}^2} + m_1 g h_{1i} + \frac{1}{2} m_2 \overset{0}{v_{2i}^2} + m_2 g h_{2i}}_{\text{only grav. potential energy at initial}} = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 \underset{\substack{0 \\ \text{due to stopper}}}{v_{2f}^2} + m_1 g (L + l_2)$$

$$m_1 g h_{1i} + m_2 g h_{2i} = \frac{1}{2} m_1 v_{1f}^2 + m_1 g (L + l_2)$$

(*) $$m_1 g h_{2i} + m_2 g h_{2i} = \frac{1}{2} m_1 v_{1f}^2 + m_1 g (L + \cancel{l_2} - L \sin\alpha - \cancel{l_2})$$

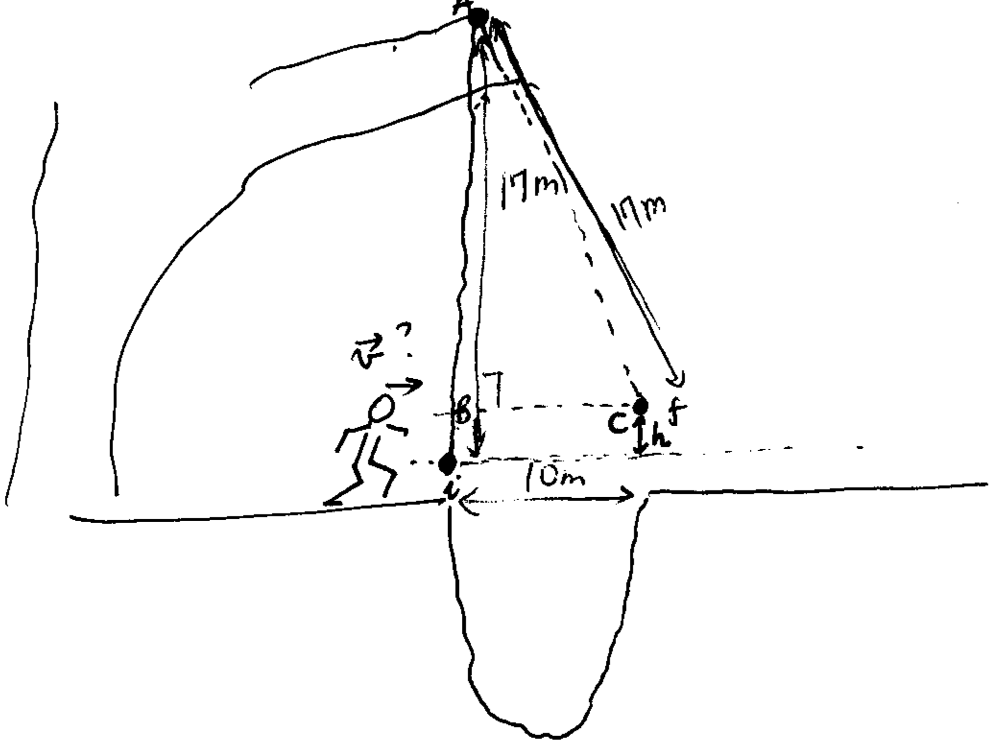
$$\boxed{(m_1 + m_2) g h_{2i} = \frac{1}{2} m_1 v_{1f}^2 + m_1 g L (1 - \sin\alpha)}$$

→ This allows us to calculate v_{1f}! from m_1, m_2, L, α

→ Projectile motion for m_1 : $\begin{cases} \text{vertical: constant acceleration:} \\ L + l_2 = \frac{1}{2} g t^2 \rightarrow t = \sqrt{\frac{2(L+l_2)}{g}} \\ \text{horizontal: } x_{Range} = v_{1f} \cdot t \end{cases}$

$$x_{Range} = \sqrt{\frac{2\left((m_1+m_2) g h_{2,i} - m_1 g L (1-\sin\alpha)\right)}{m_1}} \cdot \sqrt{\frac{2(L+l_2)}{g}}$$

7.64



Tarzan's minimum speed: just enough to land at the other side with no final kinetic energy. Using the initial height (where he grabs on the vine above ground level) as the reference level (zero height), he lands at the other side at height h (see sketch): this gives a final grav. potential energy into which the initial kinetic energy has converted.

Conservation of ME:

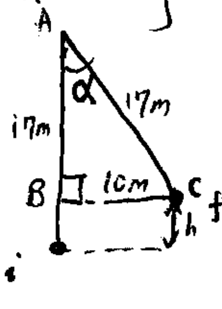
$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f$$

v_f: 0 at minimum

$$\frac{1}{2}\cancel{m}v_{i_{min}}^2 = mg(h_f - h_i) = \cancel{m}gh$$

$$v_{i_{min}} = \sqrt{2gh}$$

Calculate h using geometry (the sketch is essential!)



Alternative #1: $AB^2 + 10^2 = 17^2 \rightarrow AB = \sqrt{17^2 - 100} = 13.75$ m

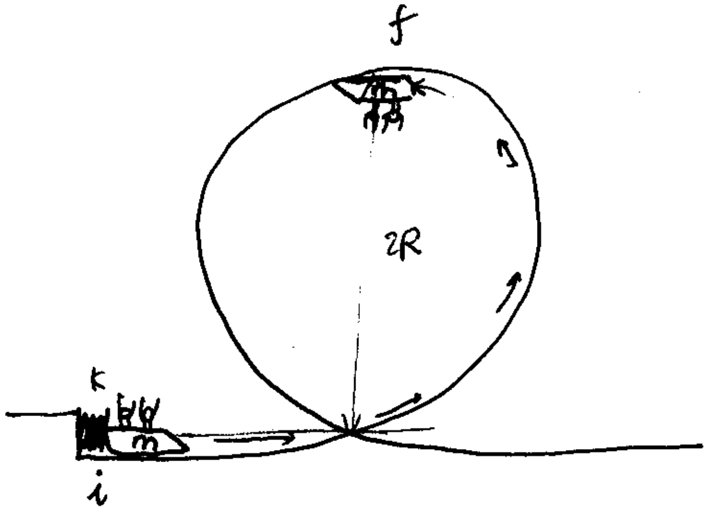
$\rightarrow h = 17 - AB = 3.25$ m

Alternative #2: $\sin\alpha = \frac{10}{17} \rightarrow \alpha = \sin^{-1}\frac{10}{17} = 36°$ | $AB = 17\cos\alpha \rightarrow h = 17 - 17\cos\alpha$

$h = 17(1-\cos\alpha) = 17\left(1 - \cos\left(\sin^{-1}\frac{10}{17}\right)\right) \approx 3.25$ m

$$v_{i_{min}} = \sqrt{2 \cdot 9.81 \cdot 3.75} = 7.98 \frac{m}{s}$$

7.59



$m = 840 kg$
$k = 31 \, k\frac{N}{m}$
$R = 6.2 \, m$

Min compression of spring for car to make top of loop?

Conservation of mechanical energy:

$$\frac{1}{2} m v_i^2 + mgh_i = \frac{1}{2} m v_f^2 + mgh_f$$

Important: $v_f \neq 0$ (otherwise it will not make it!)
At f: if we look at min initial speed $v_{i_{min}}$, the only acceleration to change direction of motion in a circular motion is g. In another words: the only agent to change direction of motion at f is mg:

$$\cancel{m} \frac{v_f^2}{R} = \cancel{m} g \rightarrow v_f^2 = gR$$

$$\frac{1}{2} m v_{i_{min}}^2 = \underbrace{\frac{1}{2} m v_f^2}_{\frac{1}{2} mgR} + \underbrace{mg \underbrace{(h_f - h_i)}_{2R}}_{mg\,2R} = mg \frac{5}{2} R$$

$$v_{i_{min}}^2 = 5gR .$$

what is $x_{i_{min}}^2$ (min compression of spring?)
When a spring is compressed a distance x, elastic potential energy is stored in the amount of: $\frac{1}{2} k x^2$. This will go into the initial

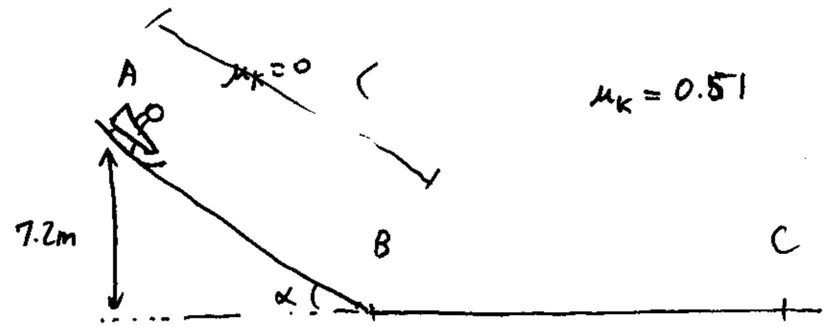
KE of the roller-coaster car:

$$\frac{1}{2} k x_{min}^2 = \frac{1}{2} m v_{i_{min}}^2 \rightarrow x_{min} = \sqrt{\frac{m v_{i_{min}}^2}{k}}$$

$$= \sqrt{\frac{m 5 g R}{k}}$$

$$= \sqrt{\frac{840 \cdot 5 \cdot 9.81 \cdot 6.2}{31000}} \simeq 2.87 m$$

7.61



BC ?

AB: constant acceleration
BC: constant deceleration

When we solved a similar problem using kinematic equations:

$v_A = 0 \rightarrow a_{AB} \rightarrow v_B \rightarrow a_{BC}, \; v_C = 0 \rightarrow BC$.

Note: $\left\{ \begin{matrix} \alpha \\ L \\ m \end{matrix} \right.$ were also given!

Conservation of mechanical energy: only applies to AB (since friction b/w B & C is not conservative!). But can find v_B, then use kinematic equations to find BC.

$\left\{ \begin{array}{l} \text{initial} = @ A \\ \text{final} = @ B \end{array} \right.$

$$\frac{1}{2} m \overset{0}{v_A^2} + mgh_A = \frac{1}{2} m v_B^2 + mgh_B$$

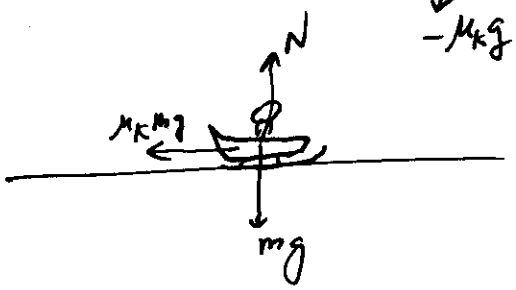
$$\cancel{m} g (h_A - h_B) = \frac{1}{2} \cancel{m} v_B^2$$

$$V_B = \sqrt{2g(\underbrace{h_A - h_B}_{7.2m})} = \sqrt{2 \cdot 9.81 \cdot 7.2}$$

$$= 11.9 \text{ m/s}$$

BC: Kinematic equation #3:

$$\frac{\overset{0}{\cancel{v_C}}^{2} - v_B^{2}}{x_{BC}} = 2\cdot \underbrace{a_{BC}}_{-\mu_k g} \rightarrow \frac{-v_B^{2}}{x_{BC}} = -2\mu_k g$$



$$x_{BC} = \frac{v_B^{2}}{2\mu_k \cdot g} = \frac{11.9^{2}}{2\cdot 0.51\cdot 9.81} = 14.2\ m.$$

Ch 8: Gravitation

Universal Law of Gravitation

↳ here on Earth, moon, other planets, other galaxies, to the universe

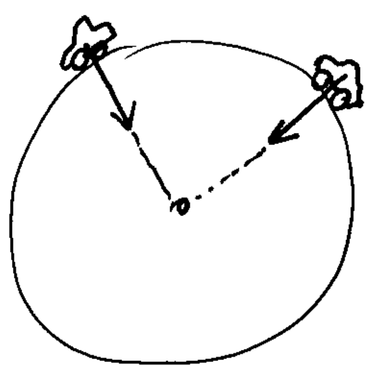
$$F = G \frac{m_1 \cdot m_2}{r^2}$$

Force of grav. attraction b/w m_1 & m_2

Universal Grav. Constant: $6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

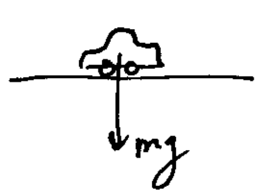
separation b/w m_1 & m_2 ; (larger separation → lower grav. attraction)

Force is a vector; the above is its magnitude. Direction is center to center toward the more massive object.



At our scale: these attractions are vertical & downward (for small distances, the curvature of the Earth is not noticeable)

All along we have been using a force of gravitational attraction of mg: what is it compared to the Univ. Law of Grav.?



$(g = 9.81 \frac{m}{s^2})$

$$M_{Earth} = 5.97 \times 10^{24}\ \text{kg} \rightarrow m_1$$

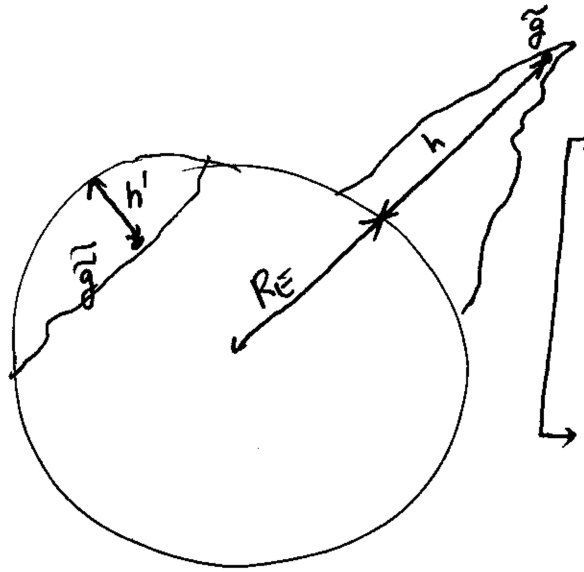
$$R_E = 6.37 \times 10^{6}\ \text{m} \rightarrow r$$

For an object of mass $m \rightarrow m_2$ on the surface of the Earth: Univ. Law of Grav. says:

$$F = G\frac{M_E \cdot m}{R_E^2} = \underbrace{6.67\times10^{-11} \cdot \frac{5.97\times10^{24}}{(6.37\times10^{6})^2}}_{g = 9.81\ \frac{m}{s^2}} \cdot m$$

mass of object

$$\boxed{\begin{aligned} F &= m \cdot g \\ \text{where } g &= G\frac{M_E}{R_E^2} \end{aligned}}$$



if h is not negligible compared to R_E $\rightarrow$ $\tilde{g} = G \cdot \frac{M_E}{(R_E + h)^2} < g$

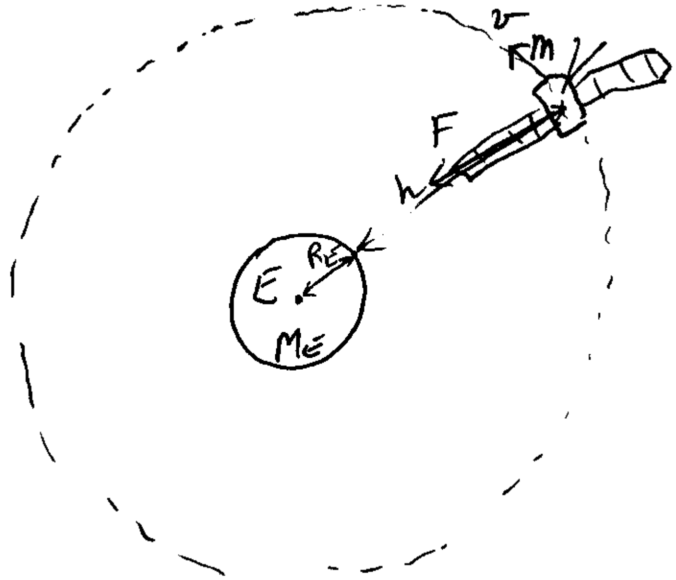
at bottom of ocean: $\tilde{\tilde{g}} = G\frac{M_E}{(R_E - h')^2} > g$

Orbital Motion:

<u>Circular orbital motion:</u> satellite in UCM (constant speed v)

→ No energy cost to run it in orbital motion. Energy is only needed for operation (signal communication with the planet)

→ Energy: from grav. attraction of the Earth;



Force of grav. attraction on satellite is F, which is essential to keep the satellite in UCM: Why? Because it is the agent that change direction of motion to keep it in orbit and not to go off in a straight forward direction:

$$F = G\frac{M_E \cdot \cancel{m}}{(R_E+h)^2} = \cancel{m}\cdot\frac{v^2}{R_E+h}$$

→ <u>Orbital speed</u> → $v = \sqrt{\dfrac{G \cdot M_E}{R_E+h}}$

→ Univ. Law of Grav.
→ 2nd Newton's Law for satellite under UCM (w/ radial acceleration)

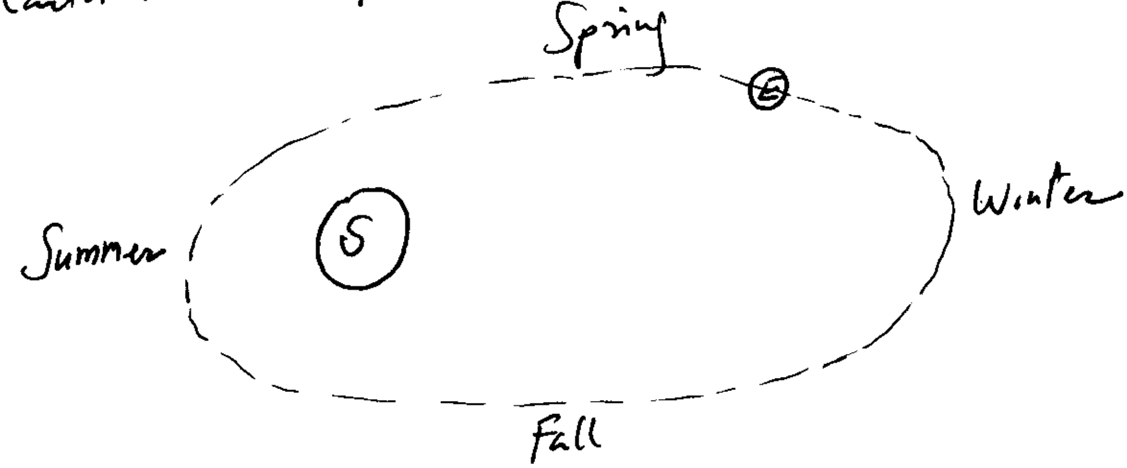
→ <u>Orbital period</u>: time to complete one full orbit:

↳ $T = \dfrac{2\pi(R_E+h)}{\sqrt{\dfrac{G\cdot M_E}{(R_E+h)}}} = \underbrace{\dfrac{2\pi}{\sqrt{G\cdot M_E}}}_{\text{constant}}(R_E+h)^{3/2} \rightarrow T^2 = \dfrac{4\pi^2}{GM_E}\underbrace{(R_E+h)^3}_{\equiv R}$

If we extend this relationship b/w period & radius ($T^2 \propto R^3$) to elliptical orbits →

<u>Kepler's 3rd Law</u>: " the period squared is proportional to the semimajor axis cubed"

↳ Earth is in elliptical orbit around the Sun:



<u>Cell phone satellite</u> @ $h = 250\,km$

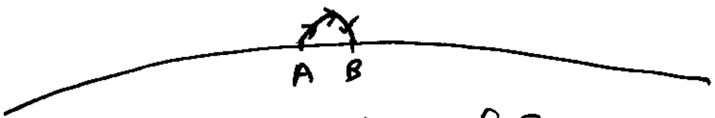
Orbital period for this satellite:

$$T = \frac{2\pi}{\sqrt{GM_E}}(R_E + h)^{3/2} = \frac{2\pi}{\sqrt{6.67\times10^{-11}\cdot 5.97\times10^{24}}}\left[(6370+250)\cdot10^3\right]^{3/2}$$

$$= 5400\,s = 1.5\,h$$

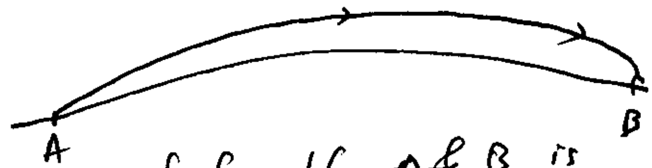
<u>Projectile Motion</u> (in part: vertical component was due to gravitational attraction)

↳ So far its trajectory was a parabola (provided the ground is "flat": balls, bullets, short-range missile)



Surface b/w A & B is flat to a very good approximation

→ Trajectory is a parabola

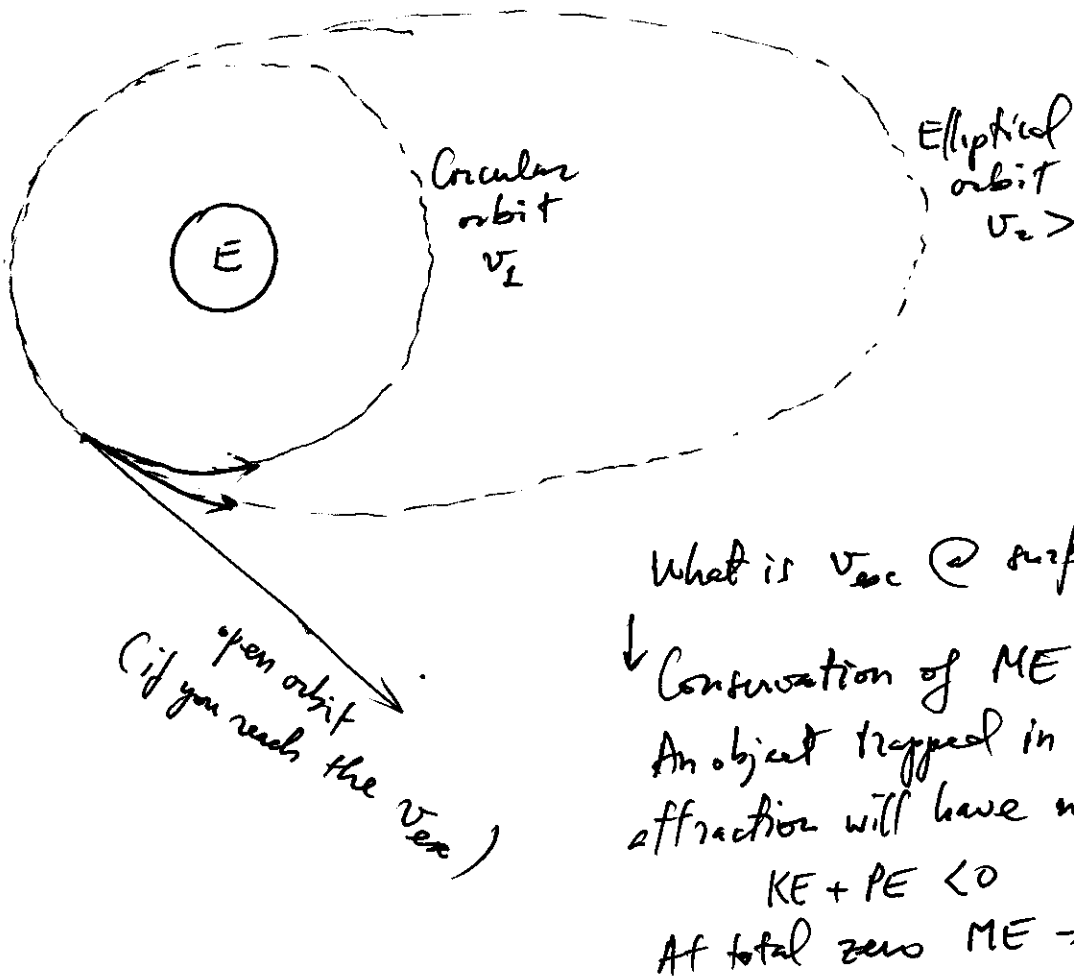


Surface b/w A & B is no longer flat

→ Trajectory is part of an elliptical orbit

→ Long-range missile, etc.

Escape Speed: if you don't have at least the escape speed → you are trapped in the gravitational attraction of the planet: stuck on surface or in an orbit around the planet. If $v \geq v_{esc}$ → object will follow an "open orbit" away from the gravitational attraction → can travel to the outer space: interplanetary travel



What is v_{esc} @ surface of Earth?

↓ Conservation of ME:

An object trapped in the grav. attraction will have negative ME

$$KE + PE < 0$$

At total zero ME → will start open orbit:

v_{esc} → $ME = 0$

$$KE + G.P.E = 0$$

$$\frac{1}{2}\cancel{m}v_{esc}^2 \boxed{- G\frac{M_E \cdot \cancel{m}}{r}} = 0 \rightarrow v_{esc} = \sqrt{\frac{2G \cdot M_E}{r}}$$

at surface $r = R_E \rightarrow v_{esc} = \sqrt{\frac{2 \cdot 6.67\times10^{-11} \cdot 5.97\times10^{24}}{6.37\times10^{6}}}$

$$= 11.2\,\frac{km}{s} = 40320\,\frac{km}{h}$$

Why the grav. potential energy is $-G\frac{M_E \cdot m}{r}$?

$$\underbrace{\Delta U}_{\text{ch. 6}} \overset{\text{work}}{=} -\int_A^B \vec{F} \cdot d\vec{r} \underset{\text{Ch. 8 Grav. Force}}{=} -\int_A^B \frac{GM_E m}{r^2}\,dr$$

$$= -GM_E m \underbrace{\int_A^B \frac{dr}{r^2}}_{\left[-\frac{1}{r}\right]_A^B} = GM_E m \left(\frac{1}{r}\right)_A^B$$

$\Delta U \propto \frac{1}{r} \rightarrow$ Zero of potential energy @ $B = \infty$
(far apart $\rightarrow$ no attraction) $\rightarrow$ Ref is at ∞

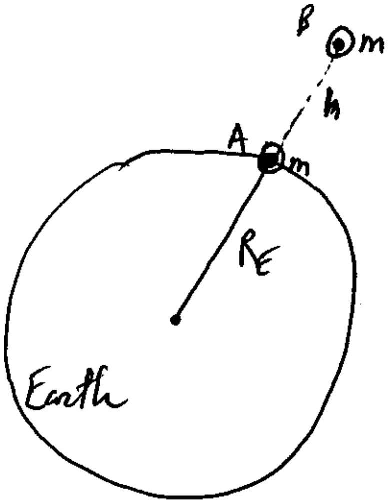
$$\Delta U = GM_E m \left(\frac{1}{r}\right)_A^\infty = GM_E m \left(\underbrace{\frac{1}{\infty}}_{0} - \frac{1}{r_A}\right)$$

generic point

$$\boxed{\Delta U = -\frac{GM_E m}{r}} \rightarrow \boxed{\begin{aligned} ME &= KE + \Delta U \\ &= \frac{1}{2}mv^2 - \frac{GM_E m}{r} \end{aligned}}$$

higher $r \rightarrow$ lower potential.

Any connection b/w $PE = -\frac{GM_E m}{r}$ and $PE = mgh$?



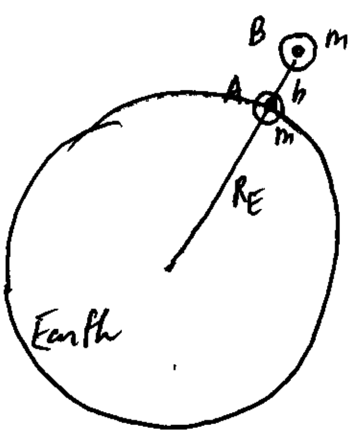
$$\Delta U_{AB} = \Delta PE_{AB} = U_B - U_A$$

$$= -\frac{GM_E m}{R_E + h} + \frac{GM_E m}{R_E}$$

$$= GM_E m\left[-\frac{1}{R_E+h} + \frac{1}{R_E}\right]$$

$$= GM_E m\left[\frac{-R_E + R_E + h}{(R_E+h)R_E}\right]$$

$$\boxed{\Delta U_{AB} = GM_E m \frac{h}{(R_E+h)R_E}}$$



$h \ll R_E$
$(h \le 1000\,m)$

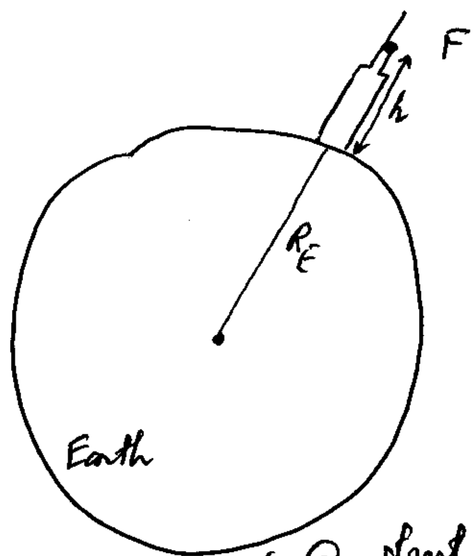
A very good approx: $R_E + h \simeq R_E$

$$\boxed{\Delta U_{AB} = GM_E m \frac{h}{R_E^2} = \underbrace{\frac{GM_E}{R_E^2}}_{g} mh = mgh}$$

works for smaller heights above surface

8.18 Height of Chicago Sears Tower : $h \longleftrightarrow \Delta g = g - g_h = 0.00136 \frac{m}{s^2}$

g: street Level; g_h: Top of building of height h



$$F = G\frac{M_E m}{r^2}$$

$\underbrace{G\frac{M_E}{r^2}}_{g}$

r: center-to-center separation.

@ street level: $g = G\frac{M_E}{R_E^2}$

@ Top of building: $g_h = G\frac{M_E}{(R_E+h)^2}$

$$\Delta g = g - g_h = GM_E\left[\frac{1}{R_E^2} - \frac{1}{(R_E+h)^2}\right]$$

$$= GM_E\left[\frac{(R_E+h)^2 - R_E^2}{R_E^2(R_E+h)^2}\right] = GM_E\left[\frac{2R_E h + h^2}{R_E^2(R_E+h)^2}\right]$$

$$= \underbrace{\frac{GM_E}{R_E^2}}_{\text{no } h} \cdot \underbrace{\frac{2R_E h + h^2}{(R_E+h)^2}}_{\text{with } h}$$

What is the order of magnitude of h = 10m, 100m, 1000m ✓, 10000m

$R_E = 6\,370\,000$ m

A very good approximation : $\begin{cases} R_E + h \simeq R_E \\ 2R_E + h \simeq 2R_E \end{cases}$

$$\Delta g = \frac{GM_E}{R_E^2}\frac{(2R_E+h)h}{(R_E+h)^2} \simeq \underbrace{\frac{GM_E}{R_E^2}}_{\equiv g}\frac{2\cancel{R_E}\cdot h}{R_E^{\cancel{2}}}$$

$$\rightarrow h = \frac{\Delta g}{g}\frac{R_E}{2} = \frac{0.00136}{9.81}\cdot\frac{6370\,000}{2} = 442\,m$$

8.42 Kepler's Law: "period squared is proportional to the semimajor axis (r) cubed": $T^2 \propto r^3$

Asteroid Pasachoff: $\begin{cases} T_p = 1417 \text{ days} & \text{(time to complete one full elliptical orbit around Sun)} \\ r_p \,? & \left(r_p \text{ in units of } r_E \leftrightarrow \frac{r_p}{r_E}\,?\right) \end{cases}$

Earth: $\begin{cases} T_E \\ r_E \end{cases}$

$$\left.\begin{matrix} T_p^2 \propto r_p^3 \\ T_E^2 \propto r_E^3 \end{matrix}\right\} \left(\frac{T_p}{T_E}\right)^2 = \left(\frac{r_p}{r_E}\right)^3 \leftrightarrow \boxed{\frac{r_p}{r_E} = \left(\frac{T_p}{T_E}\right)^{2/3} = \left(\frac{1417}{365}\right)^{2/3} = (3.88)^{2/3} = 2.47}$$

Ch 9: System of Particles

So far: we have looked at an object as a point-like object
$\Leftrightarrow$ we took an object as a point located at its center of mass (CM) $\leftrightarrow$ Recall the free-body diagrams!

$\boxed{\text{Center of Mass}}$: average position of all components of a system weighted by their masses.

There are still two positions to average, however the 1st has double weight due to its double mass

Quantitative formulas:

→ Discrete systems $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{M}$
- m_i : mass of component i
- $\vec{r}_i$: position vector of component i
- M = total mass of all components: $M = \sum_i m_i$

→ Continuous systems: $\vec{R} = \dfrac{\int \vec{r}\, dm}{M}$
- dm : infinitesimal mass (of the element of mass)
- $\vec{r}$: position vector of the element of mass
- M : total of all elements of mass $M = \int dm$

What is the implication of the components to the 2nd Newton's Law of a system (or object)? No; only a <u>subtlety</u>:

- 2nd Newton's Law for a system of particles or components:

$$\vec{F}_{net} = M \cdot \frac{d^2\vec{R}}{dt^2}$$

$\vec{F}_{net}$ → Net force on system

M → Total mass of system

$\frac{d^2\vec{R}}{dt^2}$ → acceleration of the center of mass

→ No change compared to what we have done until now

- subtlety: now that we look at components of a system there will be also interactions between there components which are called "internal forces": by 3rd Newton's law (action & reaction)

the internal forces show up in pairs of equal magnitudes & opposite directions

→ sum of all internal forces is 0!

↓

$\vec{F}_{net}$ = only involves external forces on the system (no internal forces!)

$\boxed{\text{Linear Momentum of a system}}$: $\vec{P}$

$$\vec{P} \equiv \underbrace{M}_{\substack{\text{Total}\\ \text{mass}}} \cdot \underbrace{\vec{V}}_{\substack{\text{Velocity}\\ \text{of}\\ \text{CM}}} = M \cdot \frac{d\vec{R}}{dt} \underset{\substack{\uparrow\\ \text{def. of CM}\\ \text{of a discrete}\\ \text{system}}}{=} \cancel{M} \cdot \frac{d}{dt} \frac{\sum_i m_i \vec{r}_i}{\cancel{M}} = \sum_i m_i \underbrace{\frac{d\vec{r}_i}{dt}}_{\substack{\vec{v}_i\\ \text{(velocity}\\ \text{of}\\ \text{component } i)}}$$

$$\vec{P} = \sum_i \underbrace{m_i \cdot \vec{v}_i}_{\substack{\vec{p}_i\\ \text{linear momentum}\\ \text{of component } i}} = \sum_i \vec{p}_i$$

2nd Newton's law of a system of particles using it's total linear momentum:

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

'Net external force on a system is equal to the change of the total linear momentum over time'

$$\vec{F}_{net} = 0 \rightarrow \frac{d\vec{P}}{dt} = 0 \text{ or } \vec{P} \text{ is conserved!}$$

<u>Conservation of linear momentum</u> (when there is not a net external force on system)

With this there are 2 conservation laws:

→ <u>Conservation of Mechanical Energy</u>:
(When only conservative forces are involved, not friction!)

→ <u>Conservation of Linear Momentum</u>
(When there is no net external force on the system).

<u>Collisions</u>

Inelastic: the colliding components stick together after the collision ($\vec{v}_{1f} = \vec{v}_{2f} \equiv \vec{v}_f$)

- Net external force on system: 0 → Conservation of linear Momentum:

$$\boxed{\underbrace{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}_{\text{before collision}} = \underbrace{(m_1 + m_2)\vec{v}_f}_{\text{after collision}}}$$

- ME = KE (1D or 2D) → Kinetic Energy is not conserved:

$KE_i - KE_f$ = went into deforming internal structures of either colliding components.

Elastic: total KE of colliding components is conserved (billiard balls collisions)

- Net external force on system: 0 → Conservation of linear Momentum:

$$\boxed{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}}$$

- KE is conserved:

$$\boxed{\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2}$$

More on Elastic Collisions (hard ball collisions)

1D Elastic Collision: we have 2 equations: $\begin{cases} \text{conservation of } \vec{P} \\ \text{conservation of KE} \end{cases}$

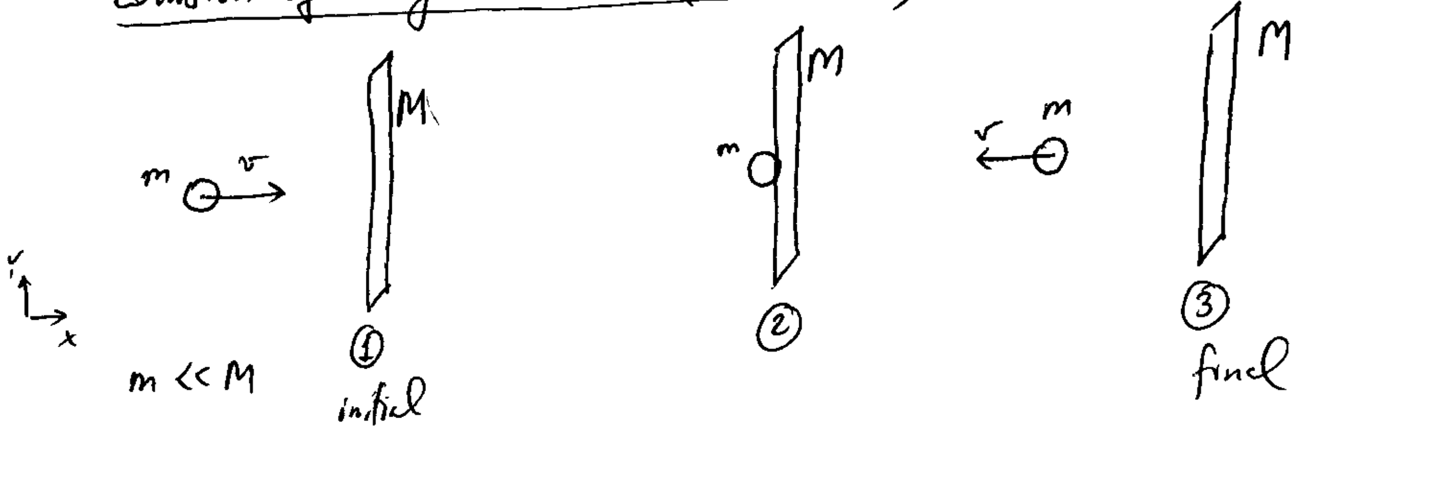
can solve up to 2 unknowns: for example given $m_1, m_2, v_{1i}, v_{2i} \rightarrow$ calculate v_{1f} & v_{2f}:

$$\left.\begin{aligned} &1)\quad v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\ &2)\quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \\ &3)\quad v_{1i} + v_{1f} = v_{2i} + v_{2f} \end{aligned}\right\} \text{Derived from Conservations of } \vec{P} \text{ \& KE in 1D}$$

2D Elastic Collision: we have 3 equations $\begin{cases} \text{conservation of } P_x \\ \text{conservation of } P_y \\ \text{conservation of KE} \end{cases}$

can solve up to 3 unknowns! $\rightarrow$ can't solve for $\vec{v}_{1f}$ & $\vec{v}_{2f}$. A typical problem will ask for v_{1f}, v_{2f}, θ (the angle b/w the two final velocities)

Collision of a gas molecule (hard ball) with a container wall



- Wall is much heavier than gas molecule : $m \ll M$:

$$\vec{F}_{net,ext} = 0 \rightarrow \vec{P}_i = \vec{P}_f$$

$$m v \hat{\imath} = m v(-\hat{\imath}) + \Delta \vec{P}_{wall} \Rightarrow \boxed{\Delta \vec{P}_{wall} = 2 m v \hat{\imath}}$$

After the gas molecule - wall collision, the wall is moving in the $+x$ direction with momentum $2mv$.

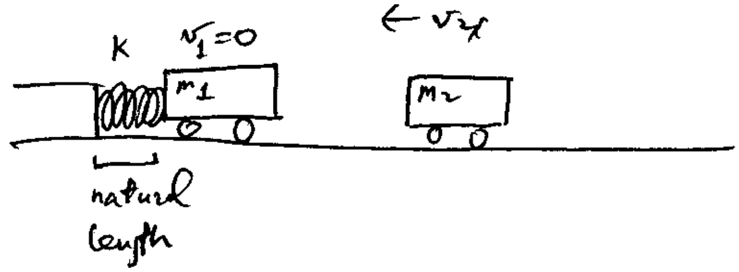
$$\Delta \vec{P}_{wall} = M \cdot \vec{v}_{wall} = 2 m v \hat{\imath} \rightarrow M v_{wall} = 2 m v$$

$$v_{wall} = \left(\frac{2m}{M}\right) v$$

Tiny fraction (negligible) of the molecule speed !

$\boxed{\text{Inelastic Collision}}$

9.41



<u>initial</u> (before collision)

$K = 3.2 \times 10^5 \frac{N}{m}$

$m_1 = 11000$ kg

$m_2 = 9400$ kg

$\vec{v}_{2i} = 8.5(-\hat{i}) \frac{m}{s}$

<u>final</u> (after collision)
two cars going together @ v_f

a) <u>C.L.M</u>: $\vec{P}_i = \vec{P}_f$

$$m_2 \vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_2}{m_1+m_2}\vec{v}_{2i} = \frac{9400}{11000+9400} 8.5(-\hat{i}) = -3.92\,\hat{i}\ \frac{m}{s}$$

Spring will slow the two cars down from $v_f = -3.92\,\hat{i}\ \frac{m}{s}$ to zero : $KE_{cars} = EPE_{spring}$

$$\frac{1}{2}(m_1+m_2)v_f^2 = \frac{1}{2}k(\Delta x)^2_{max} \rightarrow \Delta x_{max} = v_f\sqrt{\frac{m_1+m_2}{k}}$$

$$= 3.92\sqrt{\frac{11000+9400}{3.2\times10^5}}$$

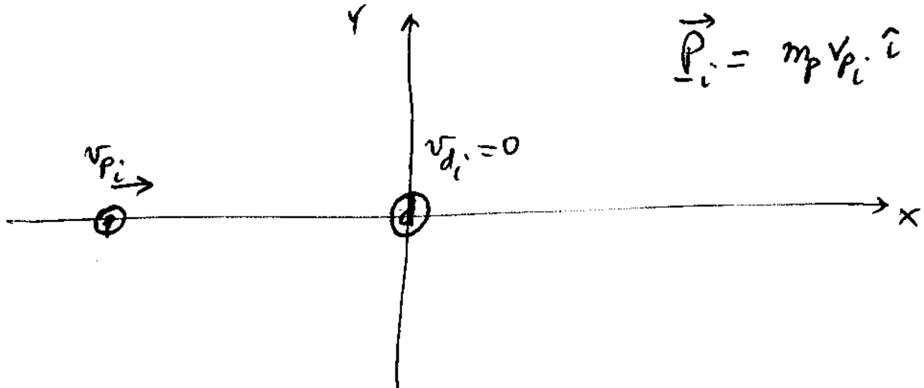
$$= \boxed{0.989\,m}$$

b) Speed of two cars when they rebound from spring:
gain back same KE from EPE_{spring} $\rightarrow$ $\vec{v} = 3.92\,\hat{i}\ \frac{m}{s}$.

$$\begin{cases} \cdot\ \vec{P}_i = \vec{P}_f \\ \cdot\ KE_i = KE_f : \ KE_{p_i} = KE_{pf} + KE_{df} \rightarrow KE_{df} = KE_{p_i} - KE_{pf} \end{cases}$$

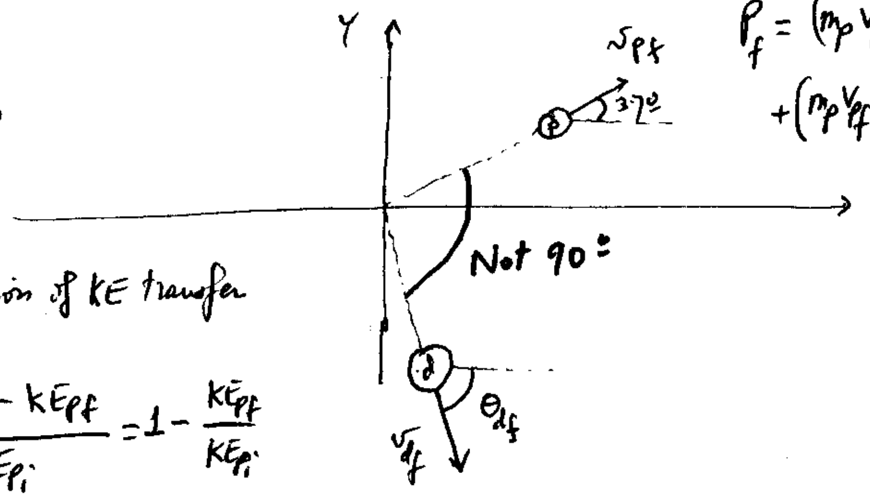
9.65 <u>2D Elastic collision</u> : proton (1u) on a stationary deuteron (2u)

Before Collision



$\vec{P}_i = m_p v_{p_i} \hat{i}$

After Collision



$\vec{P}_f = (m_p v_{pf} \cos 37° + m_d v_{df} \cos\theta_{df})\hat{i} + (m_p v_{pf} \sin 37° - m_d v_{df} \sin\theta_{df})\hat{j}$

<u>Question</u> : fraction of KE transfer from p to d:

$$\frac{KE_{df}}{KE_{p_i}} = \frac{KE_{p_i} - KE_{pf}}{KE_{p_i}} = 1 - \frac{KE_{pf}}{KE_{p_i}}$$

$\rightarrow v_{pf}$ in term of v_{p_i}

<u>Note</u> : a) since $\vec{P}_i$ is in $+x$ direction, and proton heads 37° (in 1st quad.), the deuteron will head in the 4th quadrant ($\vec{P}_f$ should have no y-component)

$$\vec{P}_i = \vec{P}_f \begin{cases} m_p v_{p_i} = m_p v_{pf} \cos 37° + m_d v_{df} \cos\theta_{df} \\ 0 = m_p v_{pf} \sin 37° - m_d v_{df} \sin\theta_{df} \end{cases}$$

b) $m_d = 2 m_p$ ($37° + \theta_{df} \neq 90°$, this only happens if $m_2 = m_1$)

$$\vec{P}_i = \vec{P}_f \begin{cases} v_{p_i} = v_{pf} \cos 37° + 2 v_{df} \cos\theta_{df} \quad (1) \\ 0 = v_{pf} \sin 37° - 2 v_{df} \sin\theta_{df} \quad (2) \end{cases}$$

Looking for v_{pf} in term of v_{p_i} $\rightarrow$ Need to solve for $v_{pf}, v_{df}, \theta_{df}$: 3 unknowns $\rightarrow$ write down next $KE_i = KE_f$.

c) $KE_i = KE_f$ $\left\{ \begin{array}{l} \frac{1}{2} m_p v_{pi}^2 = \frac{1}{2} m_d v_{df}^2 + \frac{1}{2} m_p v_{pf}^2 \\ \boxed{m_d = 2m_p} \\ \hookrightarrow v_{pi}^2 = 2 v_{df}^2 + v_{pf}^2 \quad (3) \end{array} \right.$

From now on use algebraic manipulations to solve for the 3 unknowns: $v_{pf}, v_{df}, \theta_{df}$ in term of v_{pi}.

i) Eliminate θ_{df} : use $\cos^2\theta_{df} + \sin^2\theta_{df} = 1$

$$\cos\theta_{df} = \frac{(v_{pi} - v_{pf}\cos 37°)}{2 v_{df}} \qquad (1)$$

$$\sin\theta_{df} = \frac{v_{pf}\sin 37°}{2\ v_{df}} \qquad (2)$$

$$1 = \cos^2\theta_{df} + \sin^2\theta_{df} = \frac{1}{4v_{df}^2}\Big[v_{pi}^2 - 2v_{pi}v_{pf}\cos 37° + \underbrace{v_{pf}^2\cos^2 37° + v_{pf}^2\sin^2 37°}_{v_{pf}^2}\Big]$$

$$1 = \frac{1}{4v_{df}^2}\Big[v_{pi}^2 + v_{pf}^2 - 2v_{pi}v_{pf}\cos 37°\Big]$$

2) Eliminate v_{df} : from eq (3) $v_{df}^2 = \dfrac{v_{pi}^2 - v_{pf}^2}{2}$

$$2v_{pi}^2 - 2v_{pf}^2 = v_{pi}^2 + v_{pf}^2 - 2v_{pi}v_{pf}\cos 37°$$

$$3v_{pf}^2 - 2v_{pi}\cos 37°\, v_{pf} - v_{pi}^2 = 0 \qquad (ax^2 + bx + c = 0)$$

$$\rightarrow v_{pf} = \frac{2v_{pi}\cos 37° \oplus \sqrt{4v_{pi}^2\cos^2 37° + 12 v_{pi}^2}}{6}$$

(Negative sign give a negative magnitude)

$$\boxed{v_{pf}} = v_{pi}\left[\frac{2\cos 37° + \sqrt{4\cos^2 37° + 12}}{6}\right] = \boxed{v_{pi}\ 0.902}$$

Note $\dfrac{V_{pf}}{V_{pi}}$ has to be less than one! (some of the proton's energy went into the deuteron after the collision!)

<u>Question</u>: KE transferred from proton to deuteron:

$$\frac{KE_{df}}{KE_{pi}} = 1 - \frac{KE_{pf}}{KE_{pi}} = 1 - \frac{\frac{1}{2}\cancel{m_p} V_{pf}^2}{\frac{1}{2}\cancel{m_p} V_{pi}^2} = 1 - \left(\frac{V_{pf}}{V_{pi}}\right)^2 = 1 - (0.902)^2$$

$$= 0.186 \quad \text{or} \quad 18.6\,\%$$

For curiosity: we can also solve for V_{df} & θ_{df}:

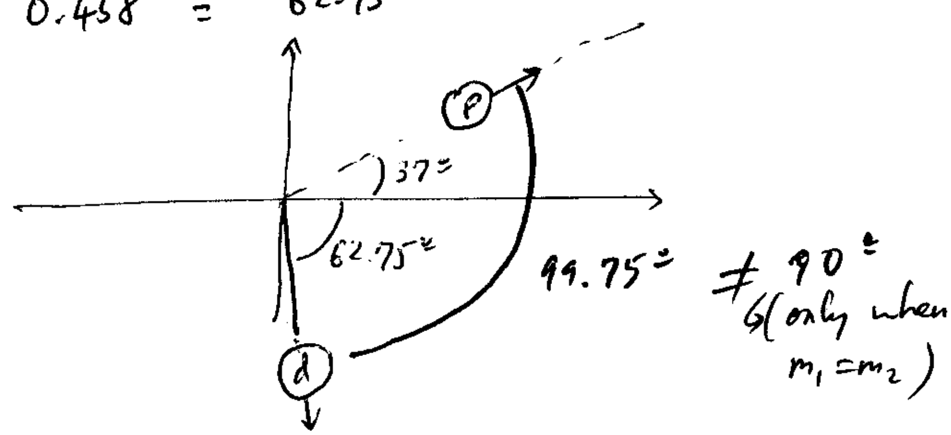
3) $$V_{df}^2 = \frac{V_{pi}^2 - V_{pf}^2}{2} = \frac{V_{pi}^2}{2}\left(1 - \frac{V_{pf}^2}{V_{pi}^2}\right) = \frac{V_{pi}^2}{2}\underbrace{\left(1 - 0.902^2\right)}_{0.186}$$

$$\boxed{V_{df} = V_{pi}\sqrt{\frac{0.186}{2}} = 0.305\, V_{pi}}$$

1) $$\cancel{V_{pi}} = \cancel{V_{pi}}\, 0.902 \cos 37° + 2 \cdot 0.305 \cancel{V_{pi}} \cos\theta_{df}$$

$$\cos\theta_{df} = \frac{1 - 0.902 \cos 37°}{2 \cdot 0.305} = 0.458$$

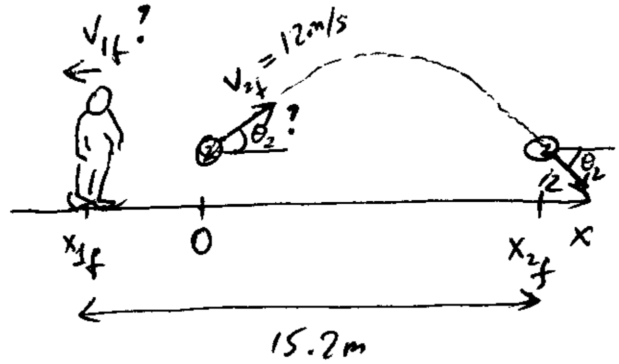
$$\theta_{df} = \cos^{-1} 0.458 = 62.75°$$



9.52

Initial

Final



$m_1 = 65\,kg$

$m_2 = 4.5\,kg$

Interpretation: $\vec{P}_i = \vec{P}_f$

$0 = \vec{P}_f \rightarrow$ as rock moves in $+x$ $\rightarrow$ person moves in $-x$!

Quantitative analysis:

$\vec{P}_f = 0 = m_1 \vec{V}_{1f} + m_2 \vec{V}_{2f}$ (Note: rock followed a 2D ~~not~~ projectile motion!)

Since we are only asked for V_{1fx} : we'll use the x-component of the conservation of linear momentum equation:

$$0 = m_1 V_{1fx} + m_2 \underbrace{V_{2fx}}_{12\cos\theta_2} \rightarrow V_{1fx} = -\frac{m_2 \cdot 12\cos\theta_2}{m_1}$$

$$V_{1fx} = -\frac{4.5 \cdot 12}{65} \cos\theta_2$$

$$\boxed{V_{1fx} = -0.83\cos\theta_2}$$

Next use $\boxed{x_{2f} - x_{1f} = 15.2\,m}$; via kinematic equations:

Solve for x_{2f} & x_{1f} in terms of speeds & angle (or velocity)

Projectile motion: $x_{2f} = \underbrace{v_{2f}\cos\theta_2}_{v_{2fx}} \cdot 2t_{up}$ (x-motion is uniform!)

t_{up}: time for rock to go to its max. altitude point:

$$v_{2fy} = v_{2fy0} - g \cdot t$$

$$0 = \underbrace{v_{2f}}_{12}\sin\theta_2 - 9.81 \cdot t_{up}$$

$$\boxed{t_{up} = \frac{12\sin\theta_2}{9.81}}$$

$$x_{2f} = 12\cos\theta_2 \cdot 2 \cdot \frac{12\sin\theta_2}{9.81}$$

$$= \frac{12^2\; 2\cos\theta_2 \sin\theta_2}{9.81} \qquad \left(x_{Range} = \frac{v_0^2 \sin 2\theta_2}{g}\right)$$

$$\boxed{x_{2f} = \frac{12^2 \sin 2\theta_2}{9.81}}$$

Person's final position: $x_{1f} = v_{1fx} \cdot 2t_{up} = -0.83\cos\theta_2 \cdot 2\,\frac{12\sin\theta_2}{9.81}$

no friction (on ice) → uniform motion

$$= -\frac{0.83 \cdot 2 \cdot 12}{9.81}\cos\theta_2 \sin\theta_2$$

$$\boxed{x_{1f} = -\frac{0.83 \cdot 12}{9.81}\sin 2\theta_2}$$

person moves in $-x$ direction!

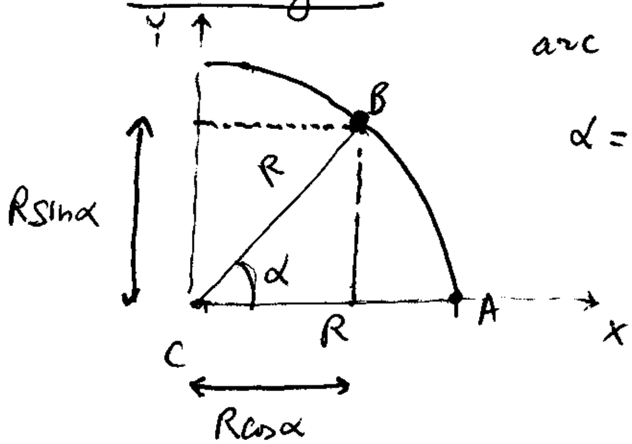
$$15.2\,m = x_{2f} - x_{1f} = \frac{12^2\sin 2\theta_2}{9.81} + \frac{0.83 \cdot 12 \sin 2\theta_2}{9.81} = \frac{12\sin 2\theta_2}{9.81}(12 + 0.83)$$

$$\boxed{\theta_2 = \frac{1}{2}\sin^{-1}\left[\frac{15.2 \cdot 9.81}{12(12.83)}\right] = 37.78^{\circ}}$$

$$V_{1fx} = -0.83 \cdot \cos\theta_2 = -0.83 \cdot \cos 37.78^\circ = -0.658 \,\frac{m}{s}$$

recoil in $-x$-direction!

About angles:



arc AB

$$\alpha = \frac{AB}{R}$$

Position B $\begin{cases} (R, \alpha) \to \text{Polar coords.} \\ (R\cos\alpha, R\sin\alpha) \to \text{Cartesian coords.} \end{cases}$

Trigonometric identities: $\begin{cases} \cos^2\alpha + \sin^2\alpha = 1 \quad (\text{for any } \alpha) \\ 2\sin\alpha\cos\alpha = \sin 2\alpha \end{cases}$

9.57 , See pg. 108

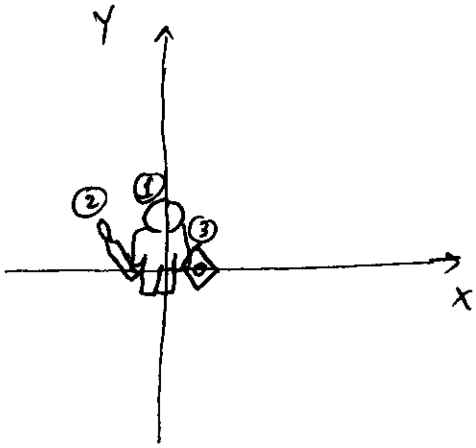
9.48

Recoils at angle 200° → 2D "collision"

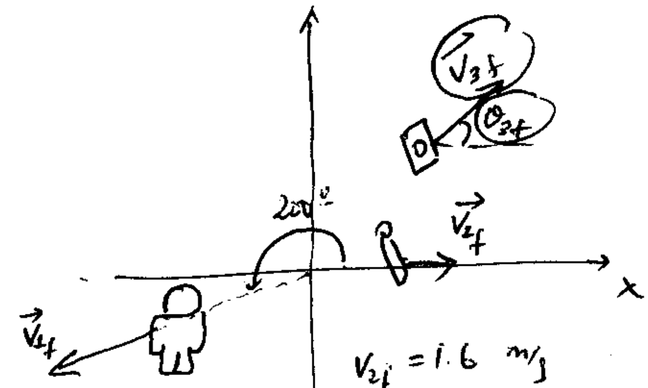
Astronaut - oxigen tank - camera 3 component system

$m_1 = 60kg$ $m_2 = 14kg$ $m_3 = 5.8kg$

initial



final



$V_{2f} = 1.6$ m/s
$V_{1f} = 0.85$ m/s

Note: camera will go off in the 1st quadrant (its x-component along with the O_2-tank will cancel the x-component of astronaut)

$$\vec{F}_{net\ external} = 0 \rightarrow \vec{P}_i = \vec{P}_f$$

$$0 = m_1\vec{V}_{1f} + m_2\vec{V}_{2f} + m_2\vec{V}_{3f}$$

$$\rightarrow 0 = (m_1 V_{1f}\cos 200 + m_2 V_{2f} + m_3 V_{3f}\cos\theta_{3f})\hat{i} + (m_1 V_{1f}\sin 200 + m_3 V_{3f}\sin\theta_{3f})\hat{j}$$

$$\rightarrow \begin{cases} 60\cdot 0.85\cdot\cos 200° + 14\cdot 1.6 + 5.8\cdot \overset{V_{3fx}}{\boxed{V_{3f}\cos\theta_{3f}}} = 0 \rightarrow V_{3fx} = +4.4 \text{ m/s} \\ 60\cdot 0.85\cdot\sin 200° + 5.8\cdot\underset{V_{3fy}}{\boxed{V_{3f}\sin\theta_{3f}}} = 0 \rightarrow V_{3fy} = +3 \text{ m/s} \end{cases}$$

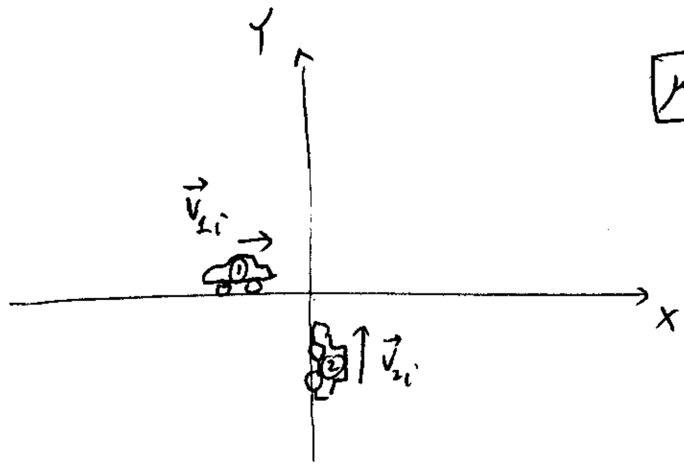
2 eqs. & 2 unknowns →

$$\vec{V}_{3f} = 4.4\hat{i} + 3\hat{j}\ \frac{m}{s} \rightarrow \begin{cases} V_{3f} = \sqrt{4.4^2 + 3^2} \\ \theta = \tan^{-1}\frac{3}{4.4} \end{cases}$$

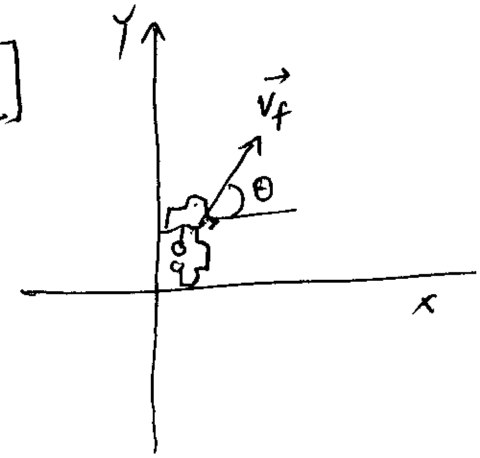
$V_{3f} = 5.33$ m/s

$\theta_{3f} = 34.3°$ (CCW from axis)

9.57



$\boxed{\mu_k = 0.91}$



① Toyota $m_1 = 1200\,kg$

② Buick $m_2 = 2200\,kg$

They lock together after collision → Inelastic Collision → $\vec{V}_{2f} = \vec{V}_{1f} = \vec{V}_f$

and $\boxed{\text{skid together } 22\,m.}$

$$\vec{P}_i = \vec{P}_f$$

1) $\boxed{m_1 \vec{V}_{1i} + m_2 \vec{V}_{2i} = (m_1 + m_2)\vec{V}_f}$

2) Skidding will stop when all KE after collision is used up in friction (work done by friction to stop the cars)

$$\boxed{\frac{1}{2}(m_1+m_2)V_f^2 = \mu_k (m_1+m_2) g \cdot x}$$

1) $\begin{cases} m_1 V_{1i} = (m_1+m_2) V_f \cos\theta & \rightarrow \quad 1200\, V_{1i} = 3400\, V_f \cos\theta \\ m_2 V_{2i} = (m_1+m_2) V_f \sin\theta & \rightarrow \quad 2200\, V_{2i} = 3400\, V_f \sin\theta \end{cases}$

2) $\frac{1}{2} \cancel{3400}\, V_f^2 = 0.91 (\cancel{3400}) \cdot 9.81 \cdot 22$

$(\cos^2\theta + \sin^2\theta = 1)$

$$1200^2 V_{1i}^2 + 2200^2 V_{2i}^2 = 3400^2 V_f^2$$

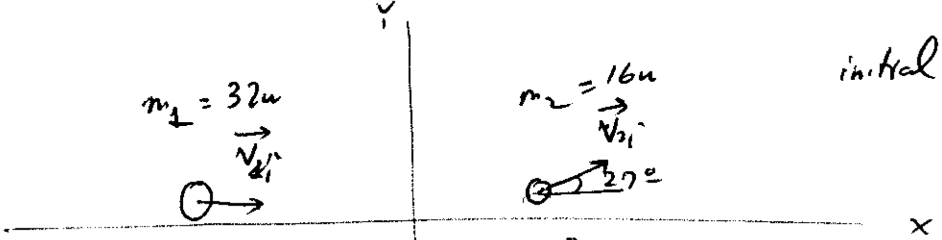
$$\underbrace{1200^2}_{1.44\times10^6} V_{1i}^2 + \underbrace{2200^2}_{4.84\times10^6} V_{2i}^2 = \underbrace{3400^2 \cdot 2 \cdot 0.91 \cdot 9.81 \cdot 22}_{4.54\times10^9}$$

$$V_{1i}^2 + 3.36\, V_{2i}^2 = 3.15\times10^3 \longrightarrow$$

$25\frac{km}{h}$ is $6.94\frac{m}{s}$ Clearly this equation is not satisfied if both V_{1i} and V_{2i} are 6.94 or less!

9.67

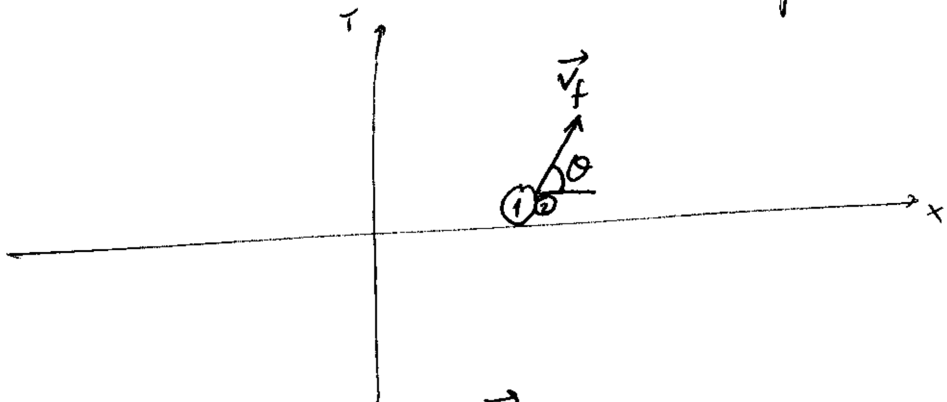
initial



$\vec{V}_{1i} = 580 m/s\ \hat{i}$

$\vec{v}_{2i} = 870 \frac{m}{s}$ @ $27°$
$= 870 \cos 27 \hat{i} + 870 \sin 27 \hat{j}$

final



Inelastic collision: $\vec{P}_i = \vec{P}_f$

$$m_1 \vec{V}_{1i} + m_2 \vec{V}_{2i} = (m_1 + m_2) \vec{v}_f$$

$m_1 = 2m_2$

$$2 \cdot 580 \hat{i} + 870 \cos 27 \hat{i} + 870 \sin 27 \hat{j} = 3(V_{fx} \hat{i} + V_{fy} \hat{j})$$

$$\rightarrow \begin{cases} 2 \cdot 580 + 870 \cos 27 = 3 V_{fx} \rightarrow V_{fx} = 645.06 \text{ m/s} \\ 870 \sin 27 = 3 V_{fy} \rightarrow V_{fy} = 131.66 \text{ m/s} \end{cases}$$

$$\rightarrow V_f = \sqrt{645^2 + 131.66^2} = 658.4 \text{ m/s}$$

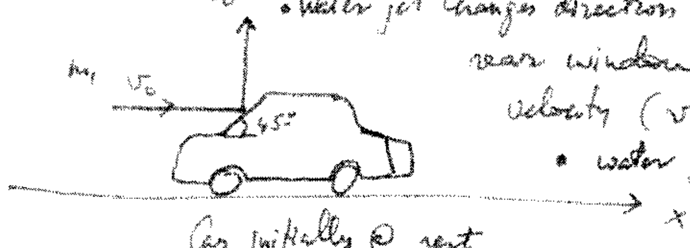
$$\theta = \tan^{-1} \frac{131.66}{645.06} = 11.54°$$

$$\frac{1}{2}\cancel{m} v_{min}^2 = \cancel{m} g\, 17\left(1-\cos\left(\sin^{-1}\frac{10}{17}\right)\right)$$

$$v_{min} = \sqrt{2\times 9.81\times 17\,(1-\cos 36^\circ)} = \boxed{7.98\ m/s}$$

9.43

• No friction



• water jet changes direction (+x to +y) after hitting rear window, not magnitude of velocity (v_0)

• water jet mass rate is $\frac{dm_1}{dt}$

Car initially @ rest

M_2

a) a_x for the car?

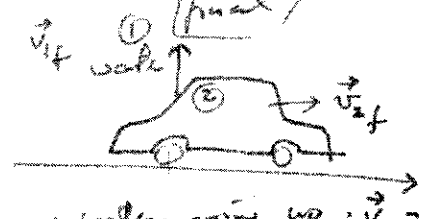
→ The collision b/w jet of water & car @ rest transfers some of its momentum into the car allowing it to go from zero speed to non-zero speed : acquiring an acceleration in the horizontal direction → a_x.

→ $\vec{F}_{net\,ext} = 0$ all forces are b/w components (water jet & car) of the system. (No friction).

↳ $\boxed{\vec{P}_i = \vec{P}_f}$

initial

final



• water jet $\frac{dm_1}{dt}$ @ $\vec{v}_{1i} = v_0\hat{i}$

• car @ rest, mass M_2

• water going up : $\vec{v}_{1f} = v_0\hat{j}$ (vertical)

• car going +x @ $\vec{v}_{2f}\,\hat{i}$ (horizontal)

$$m_1\vec{v}_{1i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

Note: water & car are not sticking together after collision!

observation : $\vec{a} = \frac{d\vec{v}}{dt}$ → car $\vec{a}_2 = \frac{d\vec{v}_{2f}}{dt}$

$$\Rightarrow \quad \vec{v}_{2f} = m_1 \frac{1}{m_2}\left(\vec{v}_{1i} - \vec{v}_{1f}\right) = m_1 \frac{1}{m_2}\left(v_0 \hat{i} - v_0 \hat{j}\right)$$

$$\rightarrow \quad \vec{a}_2 = \frac{d\vec{v}_{2f}}{dt} = \frac{dm_1}{dt} \frac{v_0}{m_2}\left(\hat{i} - \hat{j}\right)$$

only m_1 is changing

Observation : final acceleration of car has 2 components.

$$\boxed{a_x = \frac{dm_1}{dt} \frac{v_0}{m_2}}$$ ← forward in + x.

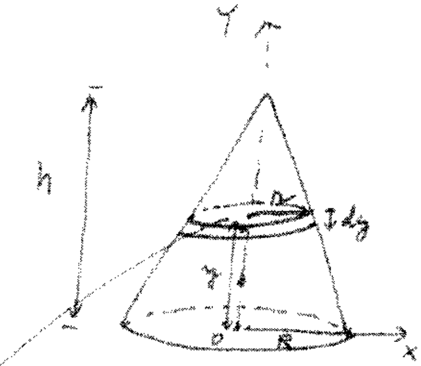
$$a_y = - \frac{dm_1}{dt} \frac{v_0}{m_2}$$ (since initially there was no momentum in the y direction, and since finally the water goes up → car gets pushed down)

b) Max. speed reached by the car?

Why max? or can we accelerate the car to ∞ speed with this water jet? No b/c when the car reaches v_0 (same speed as the water jet) no more pushing or momentum transfer is possible.

→ max speed for car is v_0!

9.39



$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm \begin{cases} x_{cm} = \frac{1}{M}\int x\, dm \\ y_{cm} = \frac{1}{M}\int y\, dm \end{cases}$$

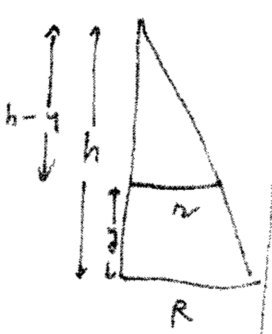
M → total mass.

$\vec{R}$ → position vector of center of mass.

b/c symmetry $x_{cm} = 0$ in the coord. syst. shown. (Y axis coincide with the axis of symmetry of the cone).

infinitesimal disk of thickness dy & mass dm, radius r, located @ distance y above the base.

$$\rightarrow\ y_{cm} = \frac{1}{M}\int y\, dm = \frac{1}{M}\int_0^h y\, \rho \pi R^2 \left(1-\frac{y}{h}\right)^2 dy =$$



$$\frac{r}{h-y} = \frac{R}{h} \rightarrow r = R\,\frac{(h-y)}{h}$$

$$\boxed{r = R\left(1-\frac{y}{h}\right)}$$

Infinitesimal disk volume is

$$dV = \pi r^2 dy$$

Since $\rho = \frac{dm}{dV} \rightarrow dm = \rho\, dV$

$$dm = \rho\, \pi r^2\, dy = \rho \pi R^2 \left(1-\frac{y}{h}\right)^2 dy$$

$$y_{cm} = \frac{\rho \pi R^2}{M}\int_0^h y\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy$$

$$= \frac{\rho \pi R^2}{M}\int_0^h \left(y - \frac{2}{h}y^2 + \frac{1}{h^2}y^3\right) dy$$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}}$$

$$\boxed{\rho = \frac{M}{\text{Vol of cone}} = \frac{M}{\pi R^2 \frac{h}{3}}}$$

$$= \frac{3}{h}\left[\frac{y^2}{2} - \frac{2}{h}\frac{y^3}{3} + \frac{1}{h^2}\frac{y^4}{4}\right]_0^h$$

$$= \frac{3}{h}\left[\frac{h^2}{2} - \frac{2}{3}h^2 + \frac{1}{4}h^2\right] = 3h\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= 3h\left[\frac{6-8+3}{12}\right] = \frac{3h}{12} = \boxed{\frac{h}{4}}$$