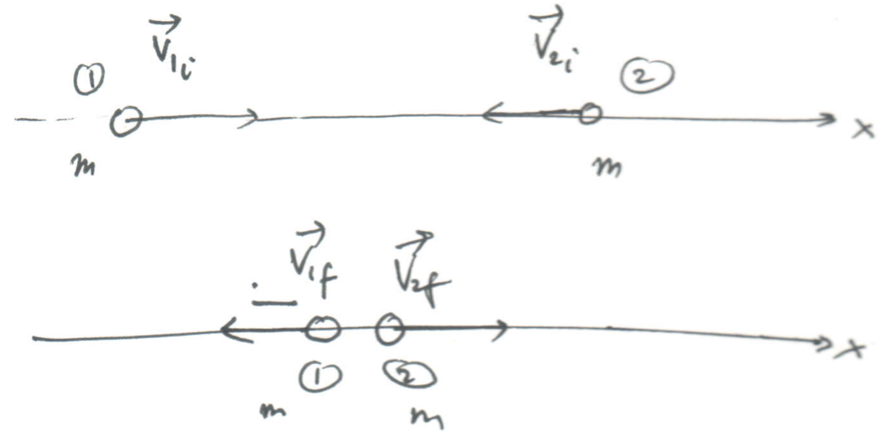
1D Elastic Collisions

$$\left\{ \begin{array}{l} \vec{P}_i = \vec{P}_f \\ KE_i = KE_f \end{array} \right.$$

(Two particles in head-on collision)

Initial

Final:



Conserv. of linear momentum : (1) $\cancel{m} v_{1i} + \cancel{m} v_{2i} = \cancel{m} v_{1f} + \cancel{m} v_{2f}$ (1D)

Conserv. of kin. energy : (2) $v_{1i}^2 + v_{2i}^2 = v_{1f}^2 + v_{2f}^2$

(equal masses, and $\frac{1}{2}$ in all terms have been eliminated)

2 equations: assume initial velocities are known → 2 unknowns: v_{1f} & v_{2f}

(1) → $v_{2f} = v_{1i} + v_{2i} - v_{1f}$

(2) $\underbrace{v_{1i}^2 + v_{2i}^2}_{\text{number}} = v_{1f}^2 + \underbrace{(v_{1i} + v_{2i} - v_{1f})^2}_{\underbrace{(v_{1i}+v_{2i})^2}_{\text{number}} + v_{1f}^2 - 2 v_{1f} \underbrace{(v_{1i}+v_{2i})}_{\text{number.}}}$

$$2 v_{1f}^2 - 2 v_{1f} \underbrace{(v_{1i} + v_{2i})}\ \ \ - (v_{1i}^2 + v_{2i}^2) = 0$$

$$\underbrace{(v_{1i} + v_{2i})^2}_{(v_{1i}^2 + v_{2i}^2) + 2 v_{1i} v_{2i}}$$

$$\rightarrow \cancel{2} v_{1f}^2 - \cancel{2} v_{1f} \underbrace{(v_{1i} + v_{2i})}_{\text{number}} + \cancel{2} \underbrace{v_{1i} v_{2i}}_{\text{number}} = 0$$

Quadratic equation in v_{1f}:

$$ax^2+bx+c=0 \rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$v_{1f}=\frac{(v_{1i}+v_{2i})\pm\sqrt{(v_{1i}+v_{2i})^2-4v_{1i}v_{2i}}}{2} \qquad \begin{cases} b\equiv -(v_{1i}+v_{2i}) \\ a\equiv 1 \\ c\equiv v_{1i}v_{2i} \\ x\equiv v_{1f}\end{cases}$$

$$=\frac{(v_{1i}+v_{2i})\pm\sqrt{v_{1i}^2+v_{2i}^2-2v_{1i}v_{2i}}}{2}$$

$$=\frac{(v_{1i}+v_{2i})\pm\sqrt{(v_{1i}-v_{2i})^2}}{2}$$

$$=\frac{(v_{1i}+v_{2i})\pm(v_{1i}-v_{2i})}{2}=\begin{cases} v_{1i} & (a) \\ v_{2i} & (b)\end{cases}$$

(a) $v_{1f}=v_{1i} \Rightarrow v_{2f}=v_{1i}+v_{2i}-v_{1f}$
$= \cancel{v_{1i}}+v_{2i}-\cancel{v_{1i}}=v_{2i}$

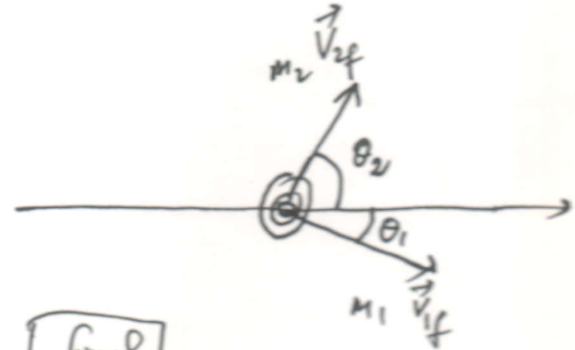
(both particles would continue as if no collision has happened!) $\rightarrow$ does not make sense.!

(b) $v_{1f}=v_{2i} \Longrightarrow v_{2f}=v_{1i}+v_{2i}-v_{1f}$
$=v_{1i}+\cancel{v_{2i}}-\cancel{v_{2i}}$

$$\boxed{\begin{array}{l} v_{1f}=v_{2i} \\ v_{2f}=v_{1i}\end{array}}$$ $\rightarrow$ both particles (equal masses) have exchanged velocities.

2D Elastic Collision:

$$\left\{\begin{array}{l} \vec{P}_i = \vec{P}_f \quad (x \,\&\, y \to 2 \text{ eqs.}) \\ KE_i = KE_f \quad (1 \text{ eq.}) \end{array}\right\} \text{ can solve for 3 unknowns}$$



$\boxed{\text{initial:}}$

v_{1i} & v_{2i} are known

m_1 & m_2 are known

$\boxed{\text{final}}$

θ_1 is known.

Use conservation laws to determine V_{1f}, V_{2f}, θ_2

Particular scenario: $V_{2i} = 0$

① $\frac{1}{2} m_1 V_{1i}^2 = \frac{1}{2} m_1 V_{1f}^2 + \frac{1}{2} m_2 V_{2f}^2$

C.L.M in x ② $m_1 V_{1i} = m_1 V_{1f} \cos\theta_1 + m_2 V_{2f} \cos\theta_2$

C.L.M in y ③ $0 = m_1 V_{1f} \sin\theta_1 + m_2 V_{2f} \sin\theta_2$

②^2: $V_{1i}^2 = \left(V_{1f}\cos\theta_1 + \frac{m_2}{m_1} V_{2f} \cos\theta_2\right)^2$

$= V_{1f}^2 \cos^2\theta_1 + \frac{m_2^2}{m_1^2} V_{2f}^2 \cos^2\theta_2$

$2 \frac{m_2}{m_1} V_{1f} V_{2f} \cos\theta_1 \cos\theta_2$

③^2: $0 = m_1^2 \left(V_{1f} \sin\theta_1 + \frac{m_2}{m_1} V_{2f} \sin\theta_2 \right)^2$

$0 = V_{1f}^2 \sin^2\theta_1 + \frac{m_2^2}{m_1^2} V_{2f}^2 \sin^2\theta_2$

$+ 2 \frac{m_2}{m_1} V_{1f} V_{2f} \sin\theta_1 \sin\theta_2$

$$② ^2 + ③ ^2 \Rightarrow V_{1i}^2 = V_{1f}^2 \underbrace{\left(\cos^2\theta_1 + \sin^2\theta_1\right)}_{1} + \frac{m_2^2}{m_1^2} V_{2f}^2 \underbrace{\left(\cos^2\theta_2 + \sin^2\theta_2\right)}_{1}$$

$$+ 2\frac{m_2}{m_1} V_{1f} V_{2f} \underbrace{\left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right)}_{\substack{\cos(\theta_1 - \theta_2) \\ \text{or } \cos(\theta_2 - \theta_1)}}$$

Result: ⓐ $$\boxed{V_{1i}^2 = V_{1f}^2 + \frac{m_2^2}{m_1^2} V_{2f}^2 + 2\frac{m_2}{m_1} V_{1f} V_{2f} \cos(\theta_2 - \theta_1)}$$

Observation:

① Divide both sides by $\frac{m_1}{2}$:

ⓑ $$\boxed{V_{1i}^2 = V_{1f}^2 + \frac{m_2}{m_1} V_{2f}^2}$$

ⓐ-ⓑ : $$0 = \frac{m_2}{m_1}\left(\frac{m_2}{m_1} - 1\right) V_{2f}^2 + 2\frac{m_2}{m_1} V_{1f} V_{2f} \cos(\theta_2 - \theta_1)$$

Divide both sides by $\frac{m_2}{m_1}$
$$0 = \left(\frac{m_2}{m_1} - 1\right) V_{2f}^2 + 2 V_{1f} V_{2f} \cos(\theta_2 - \theta_1)$$

Divide both sides by V_{2f}
$$\boxed{0 = \left(\frac{m_2}{m_1} - 1\right) V_{2f} + 2 V_{1f} \cos(\theta_2 - \theta_1)}$$

If $m_1 = m_2 \rightarrow \left(\frac{m_2}{m_1} - 1\right) = 0 \Rightarrow 0 = 0 + \underbrace{2V_{1f}\cos(\theta_2 - \theta_1)}_{0}$

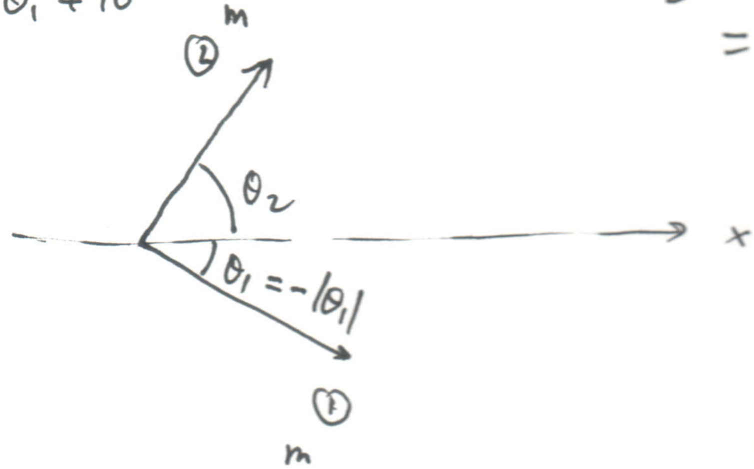
$\Rightarrow \boxed{\cos(\theta_2 - \theta_1) = 0}$

(From the conservation equations!)

$\theta_2 - \theta_1 = 90° \rightarrow \boxed{\theta_2 = \theta_1 + 90°}$

$\theta_2 = \theta_1 + 90^\circ \longrightarrow \theta_2 = -|\theta_1| + 90$

$= 90 - |\theta_1|$ } They form a 90° angle



$\boxed{\text{The two particles of equal masses will leave at } 90^\circ \text{ angle after a 2D elastic collision.}}$

For example: Atomic & sub-atomic head on collisions and billiard ball collisions. !

Ch10 Rotational Motion:

So far we have considered only linear motion with translation. Rotational motion is a different type of motion, e.g. a rotating ball is not changing position. However an object can undergo both linear and rotational motion → Rolling motion is an example.

Linear Motion
(change of position)

$$\bar{v} = \frac{v_0 + v}{2}$$

$$v = v_0 + at \quad (1)$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \quad (2)$$

$$\frac{v^2 - v_0^2}{x - x_0} = 2a \quad (3)$$

$$F_{net} = ma$$

Rotational Motion
(change of angle)

$$\bar{\omega} = \frac{\omega_0 + \omega}{2}$$

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\frac{\omega^2 - \omega_0^2}{\theta - \theta_0} = 2\alpha$$

$$\tau_{net} = I.\alpha$$

ω: angular velocity (omega)
α = angular acceleration (alpha)
θ = angle (theta)
τ = torque (tau)
I = moment of intertia

Angular velocity ω

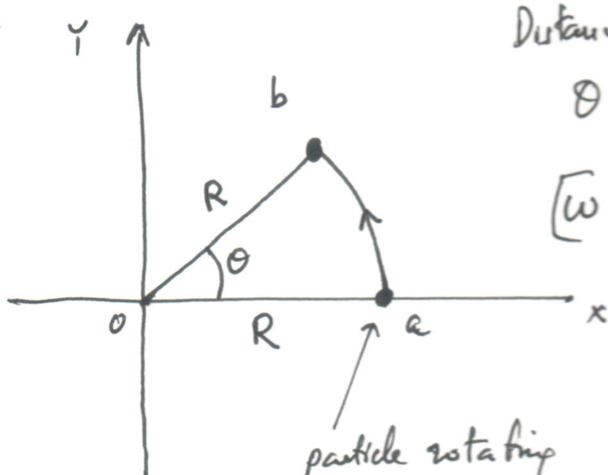
$\bar{\omega} = \dfrac{\Delta\theta}{\Delta t}$ (average angular velocity, unit: $\dfrac{rad}{s}$ or $\dfrac{1}{s}$ or s^{-1})

↓ angle is considered dimensionless ($\theta = \dfrac{arc}{radius} = \dfrac{L}{L}$)

$\left(\bar{v} = \dfrac{\Delta x}{\Delta t} \right)$

$\omega = \dfrac{d\theta}{dt}$ (instantaneous angular velocity)

$\left(v = \dfrac{dx}{dt} \right)$



Distance b/w a & b = arc s

$\theta = \dfrac{arc\ s}{R} = \dfrac{s}{R}$

$\boxed{\omega = \dfrac{d\theta}{dt} = \dfrac{1}{R}\dfrac{ds}{dt} = \dfrac{v}{R}}$ → Units: $\dfrac{m/s}{m} = s^{-1}$

↓ R constant.

$\dfrac{ds}{dt} = v$ = linear speed along circular trajectory

particle rotating wrt origin of coordinates. going from a ($\theta = 0$) to b ($\theta = \theta$)

Alternative units : rpm (revolution per minute)

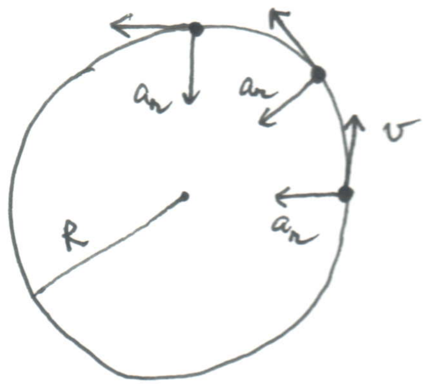
Angular Acceleration : α

$\bar{\alpha} = \dfrac{\Delta\omega}{\Delta t}$ average angular acceleration

$\alpha = \dfrac{d\omega}{dt}$ instantaneous angular acceleration

Units: SI : $\dfrac{1}{s^2}$ or s^{-2} or $\dfrac{rad}{s^2}$.

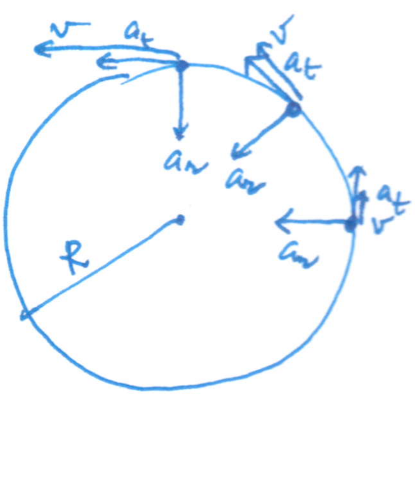
UCM (Uniform circular motion) only radial acceleration a_r



$$\underset{\substack{\downarrow \\ \text{UCM}}}{a_r} = \frac{v^2}{R} \underset{\substack{\downarrow \\ \omega = \frac{v}{R}}}{=} \frac{\omega^2 R^2}{R} = R\omega^2$$

Non-UCM (Non-uniform circular motion): in addition to the radial acceleration a_r (to change direction) we also have tangential acceleration a_t

$$\begin{cases} a_r = R\omega^2 \\ a_t = \dfrac{dv}{dt} \underset{\omega = \frac{v}{R}}{=} \dfrac{d(\omega R)}{dt} \underset{\substack{\downarrow \\ R = \text{constant}}}{=} R\dfrac{d\omega}{dt} = R\alpha \end{cases}$$



Torque $\vec{\tau}$: $\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta \, \hat{\tau}$

$\vec{r}$: position vector from pivot (or center rotation) to force application point

$\vec{F}$ = force applied

(if $\vec{F} = 0 \rightarrow \vec{\tau} = 0$

if $\vec{r} = 0 \rightarrow \vec{\tau} = 0$)

↳ if you apply a force on the axis of rotation you are not affecting the rotational motion !)

$\times$: "cross product" : a product b/w two vectors that gives another vector! (different than the "scalar product" in work & energy)

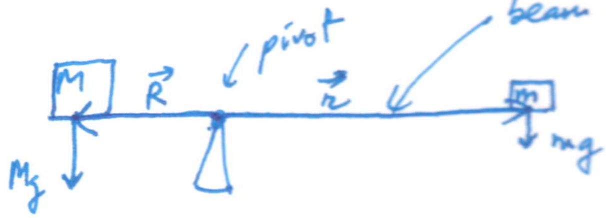
θ : angle b/w the two vectors $\vec{r}$ & $\vec{F}$

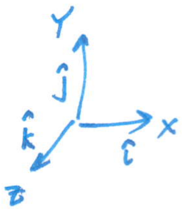
$\hat{\tau}$: unit vector that is perpendicular to the plane formed by $\vec{r}$ & $\vec{F}$

Torque is a vector whose magnitude is the product of the magnitudes of the two vectors involved time sin of the angle between them, whose direction is perpendicular to both of the two vectors involved : given by the $\boxed{\text{Right Hand Rule}}$

Unit : SI : Nm (will keep Nm to distinguish from energy/work $\rightarrow$ J)

Example : find total or ~~net~~ torque : $\vec{\tau}_{net}$



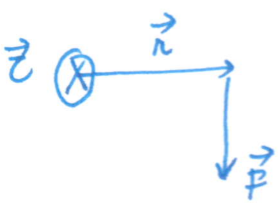


$\hat{k}$: out of the page
$\hat{i}$ & $\hat{j}$ form the plane of the page

$$\vec{\tau}_{net} = \vec{\tau}_M + \vec{\tau}_m$$

Cross-product Right Hand Rule:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

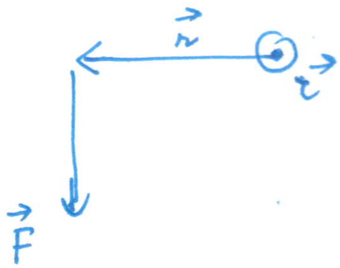


use Right Hand: point fingers along 1st vector ($\vec{r}$), turn these fingers towards 2nd vector ($\vec{F}$), your thumb will indicate direction of the torque!

In this case the thumb points into the page! $\otimes \vec{\tau}$ or $\vec{\tau} = \tau(-\hat{k})$

$$= rF\sin\theta\,(-\hat{k})$$
$$= rF\,(-\hat{k})$$

$\theta = 90°$



RHR : here thumb points out of page

$\odot$ or $\vec{\tau} = \tau\,(+\hat{k})$

$= rF\sin\theta\,(+\hat{k})$

$\theta = 90^\circ$: $= rF\,\hat{k}$

Balancing beam example : $\vec{\tau}_{net} = RMg\,\hat{k} - rmg\,\hat{k}$

$= (RM - rm)\,g\,\hat{k}$

equilibrium $\rightarrow$ $RM - rm = 0$

$$\boxed{\frac{R}{r} = \frac{m}{M}}$$

Analog of 2nd Newton's Law for rotational motion :

$F_{net} = m.a$

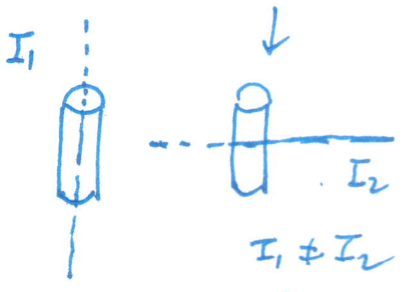
$\tau_{net} = I\alpha$

net torque, moment of inertia, angular acceleration

Moment of inertia : $I = \sum_i m_i r_i^2$

(sum over all components. m_i = mass of component i; r_i = position of component i w.r.t. axis of rotation or pivot point)

↳ w.r.t. an axis of rotation!



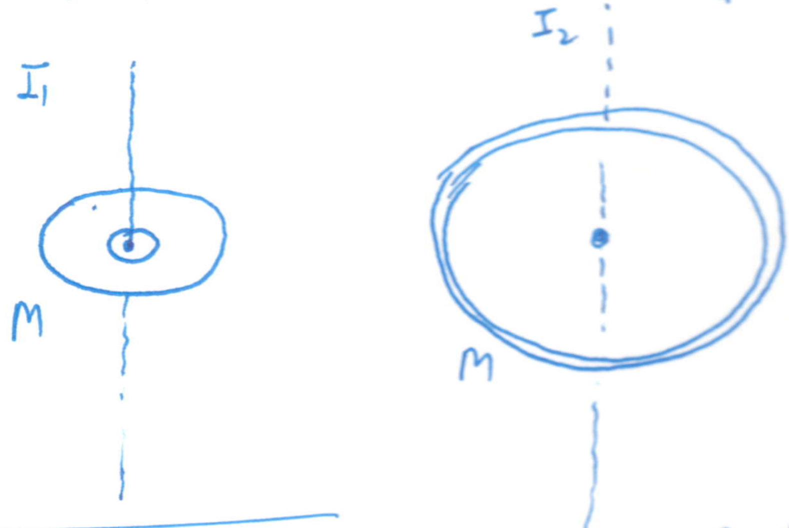
$I = \int dm\, r^2$

$I_1 \neq I_2$

$I_? > I_?$

Difference b/w linear & rotational motion:

Linear	Rotational
1) Any force will change linear motion	A force applied on the axis of rotation does not change rotational motion
2) Same mass, same inertia	Same mass, different axis of rotation, different inertia !
3) Same mass, any distribution, same inertia	Same mass, different distribution, different inertia.



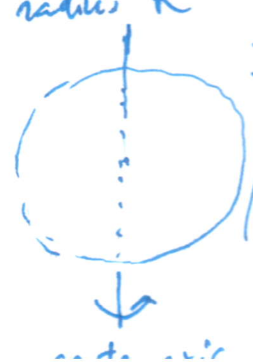
$I_2 > I_1$

Parallel Axis Theorem:

if the axis is moved a distance R, still parallel, moment of inertia is increased by MR^2

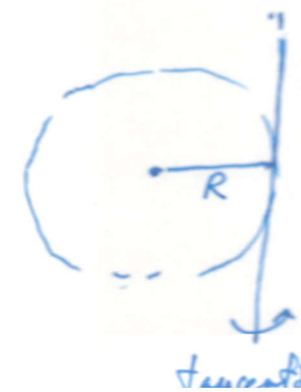
solid sphere of mass M, radius R

$I_1 = \frac{2}{5}MR^2$ ↓ $\boxed{\text{Table in Ch.10}}$



same sphere

$I_2 = I_1 + MR^2$
$= \frac{7}{5}MR^2$



Round & Symmetrical objects: (disks, rods, spheres ...)

$$I = \alpha M R^2$$

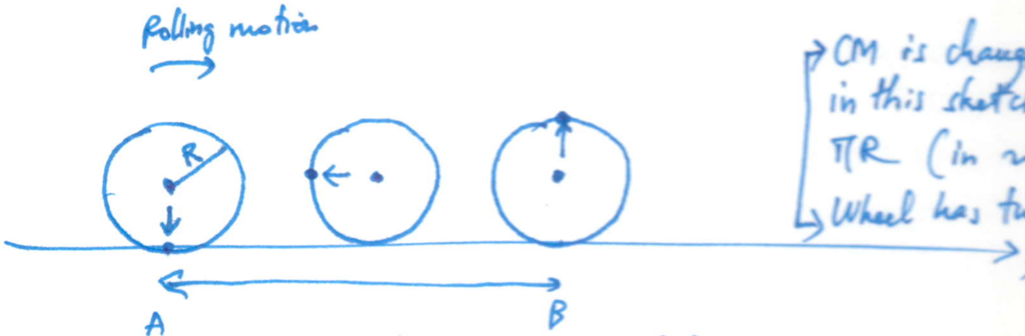
M mass of object

R radius or length.

- sphere w.r.t. center axis: $\alpha = \frac{2}{5}$
- cylinder w.r.t. center axis: $\alpha = \frac{1}{2}$
- thin rod of length L: $\alpha = \frac{1}{12} \rightarrow I = \frac{1}{12} M L^2$

Rolling Motion:

→ non skidding: under normal conditions (with friction!), car wheels undergo rolling motion!



→ CM is changing position along x: in this sketch, CM has gone πR (in rolling motion)

→ Wheel has turned an angle of π b/w A & B

half of wheel has laid or touched the road b/w A & B → πR

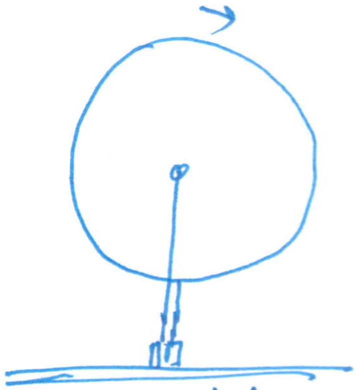
$$v_{CM} = \frac{\Delta x}{\Delta t} = \frac{\pi R}{\Delta t}$$

$$\omega = \frac{\Delta \theta}{\Delta t} = \frac{\pi}{\Delta t}$$

$\boxed{v_{CM} = \omega R}$ in rolling motion.

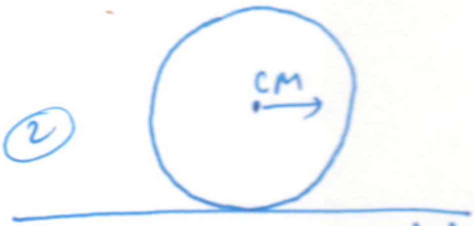
→ linear speed is angular speed times radius of the wheel.

1



only rotation
no translation of CM

2



no friction

skidding w/ no rotation.
translation of CM w/o rotation.

3

w/ friction.

Rolling motion: $\left\{ \begin{array}{l} \text{Translation of CM} \\ \quad \& \\ \text{Rotation wrt CM !} \end{array} \right\}$ $\boxed{v_{CM} = \omega R}$

Kinetic Energy

Linear Motion (2)

$\frac{1}{2} m v^2$

Rolling Motion ($v = \omega R$)

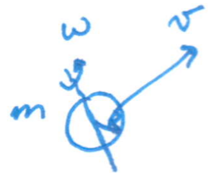
$$\underbrace{\frac{1}{2} m v^2}_{\text{Translational K.E}} + \overbrace{\frac{1}{2} I \omega^2}^{}_{\text{Rotational K.E.}}$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} I \left(\frac{v^2}{R^2} \right)$$

$$= \frac{1}{2} \left(m + \underbrace{\frac{I}{R^2}}_{} \right) v^2$$

compared to skidding motion 2, in a rolling motion inertia is increased by $\left(\frac{I}{R^2} \right)$!

10.37



Baseball $\begin{cases} \text{linear motion} \rightarrow \frac{1}{2}mv^2 \\ \text{rotational motion} \rightarrow \frac{1}{2}I\omega^2 \end{cases}$

→ I: sphere w.r.t. center axis $\frac{2}{5}MR^2$

$v = 33\ \frac{m}{s}$; $\omega = 42\ \frac{rad}{s}$

$m = 0.15\ kg$; $R = 0.037m$

Fraction of rotational energy $= \dfrac{\cancel{\frac{1}{2}}I\omega^2}{\cancel{\frac{1}{2}}mv^2 + \cancel{\frac{1}{2}}I\omega^2}$

$$= \frac{\frac{2}{5}\cancel{m}R^2\omega^2}{\cancel{m}v^2 + \frac{2}{5}\cancel{m}R^2\omega^2}$$

$$= \frac{\frac{2}{5}\, 0.037^2 \times 42^2}{33^2 + \frac{2}{5}\, 0.037^2 \times 42^2}$$

$$= 8.86 \times 10^{-4} = 0.0886\ \%$$

(Very small!)

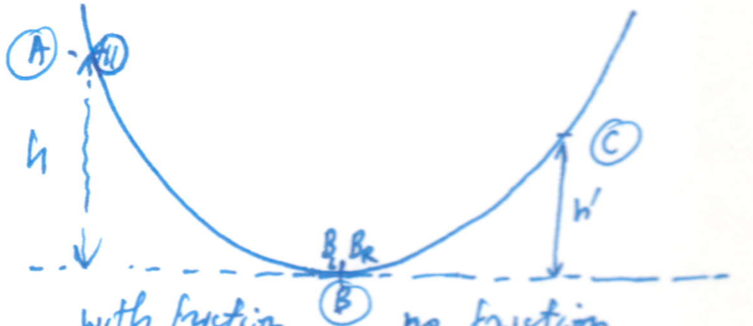
<u>Note:</u> linear speed for a point on surface of baseball is

$$\omega R = 42 \times 0.037m$$
$$= 1.55\ \frac{m}{s}$$

(This is much smaller than $v_{cm} = 33\ m/s$!)

10.64

M, R, starting @ rest



with friction → $\boxed{\text{Rolling motion}}$ $v_{CM} = \omega R$

no friction $\boxed{\text{Only motion of CM}}$ (Center of Mass)
slipping → no rotation.
→ will rise to a different height h'?

$$\underbrace{Mgh}_{\text{No KE b/c ball was at rest}} = \underbrace{\tfrac{1}{2} M v_{CM}^2}_{\substack{\text{translational}\\ \text{KE}\\ \text{(linear motion}\\ \text{of CM)}}} + \underbrace{\tfrac{1}{2} I \omega^2}_{\substack{\text{rotational}\\ \text{KE}}}$$

(A) (B_L)

→ $I = \frac{2}{5} M R^2$ (sphere w.r.t. center axis)

$$= \frac{1}{2} M v_{CM}^2 + \frac{1}{2}\frac{2}{5} M \cancel{R^2} \frac{v_{CM}^2}{\cancel{R^2}}$$

$$= \frac{1}{2} M \left(1 + \frac{2}{5}\right) v_{CM}^2$$

B/w A & B_L (a little bit to the left of B, where there is rolling motion)

$$\cancel{M} g h = \frac{1}{2} \cancel{M} \left(\frac{7}{5}\right) v_{CM}^2 \rightarrow \boxed{v_{CM}^2 = \frac{10}{7} g h}$$

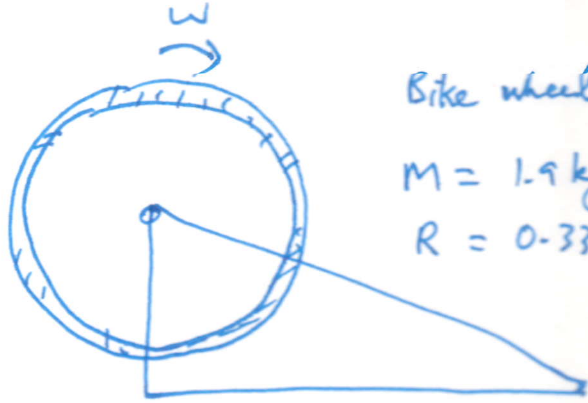
A little bit to the right of B ← B_R

$$\frac{1}{2} \cancel{M} v_{CM}^2 = \overset{C}{\cancel{M} g h'}$$

(ball turns back → no KE)

$$\boxed{h' = \frac{\frac{1}{2} v_{CM}^2}{g} = \frac{\frac{1}{2}\frac{10}{7}\cancel{g} h}{\cancel{g}} = \frac{5}{7} h}$$

10.58

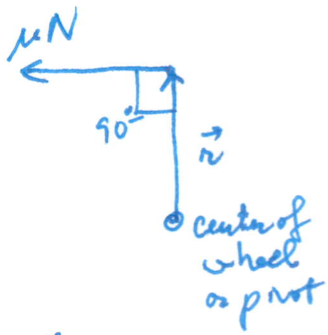


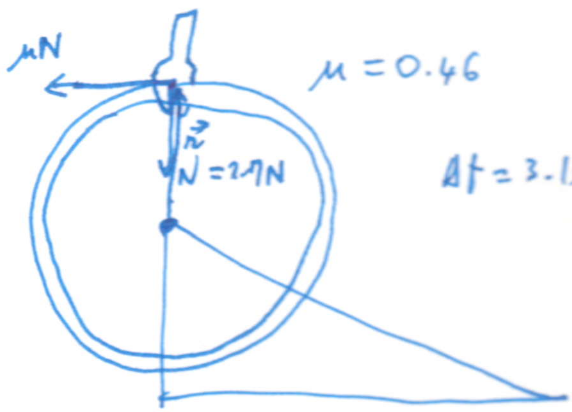
Bike wheel spinning @ $\omega = 230\,rpm$

$M = 1.9\,kg$ concentrated @ rim

$R = 0.33\,m \rightarrow I = MR^2$

(ring w.r.t. center axis)

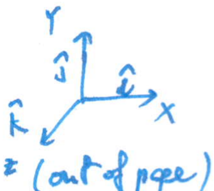


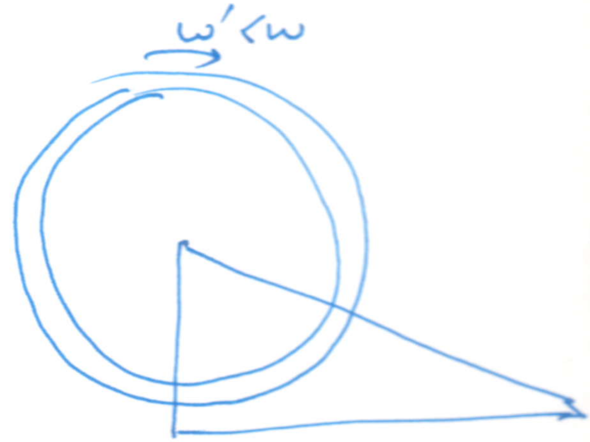


$\mu = 0.46$

$\Delta t = 3.1\,s$

wheel is then slowed down using friction b/w tire and a wrench during 3.1 s.





ω' ?

→ There is only rotational motion! (Bike is not moving!)

→ Rotational motion is being changed ⟹ Analog of 2nd Newton law

for rotational motion: $\tau_{net} = I\alpha$

$\vec{\tau}_{net} = \underset{\downarrow\ \text{pivot to force application point}}{\vec{r}} \times \vec{F} = \underbrace{R}_{\text{magnitude of } \vec{r}}\underbrace{\mu N}_{\text{magnitude of } \vec{F}}\underbrace{\sin 90°}_{1}\ \underset{\downarrow\ \text{used RHR}}{\hat{k}}$

ang. acceleration

$I\alpha = MR^2\frac{\Delta\omega}{\Delta t} = MR^2\frac{\omega-\omega'}{\Delta t}$

$\cancel{R}\mu N = MR^{\cancel{2}}\frac{\omega-\omega'}{\Delta t}$

$\omega-\omega' = \frac{\mu N\Delta t}{MR}$

$\omega' = \underset{\downarrow}{\omega} - \underbrace{\frac{\mu N\Delta t}{MR}}_{rad/s}$

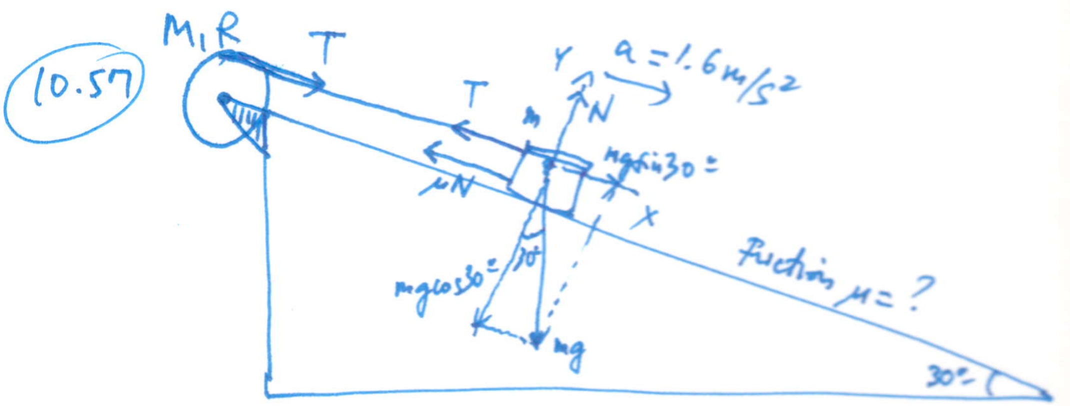
$$\Delta\omega = \frac{\mu N \Delta t}{MR} = \frac{0.46 \times 2.7 \times 3.1}{1.9 \times 0.33} = 6.14 \ \frac{rad}{s}$$

$$= 6.14 \ \frac{rad}{s} \cdot \frac{60 s}{1 min} \ \frac{1 rev}{2\pi \ rad}$$

$$= 58.6 \ rpm$$

$$\rightarrow \omega' = \omega - \Delta\omega = 230 rpm - 58.6 rpm = 171 rpm.$$

10.57



Block of m connected to a drum (disk) of mass M, radius R, via a rope (massless → same tension throughout the rope)

→ Multiple objects $\begin{cases} \text{Block: } m = 2.4 kg \rightarrow \text{linear motion} \\ \text{Drum} \begin{cases} M = 0.85 kg \\ R = 0.05 m \end{cases} \rightarrow \text{rotational motion.} \end{cases}$

Block: 2nd Newton's Law: $F_{net} = ma$ $\begin{cases} x \rightarrow mg\sin 30° - T - \mu N = ma \\ y \rightarrow N - mg\cos 30° = 0 \end{cases}$

↳ linear motion.

$$\mu = \frac{mg\sin 30° - T - ma}{mg\cos 30°}$$

(Need T!)

Drum: Analog of 2nd Newton's Law: $\tau_{net} = I\alpha$

↳ rotational motion

$\vec{\tau}_{net} = \vec{r} \times \vec{F} = RT\sin 90° (-\hat{k})$

$\tau_{net} = RT$

$I = \frac{1}{2}MR^2$ (disk wrt

Also: $\alpha = \frac{a}{R}$

$RT = \frac{1}{2}M\cancel{R^2}\frac{a}{\cancel{R}} \rightarrow \boxed{T = \frac{Ma}{2}}$

$$\rightarrow \mu = \frac{mg\sin 30^\circ - \frac{M}{2}a - ma}{mg\cos 30^\circ} = \frac{2.4 \times 9.81 \times \sin 30^\circ - \frac{0.85}{2} \times 1.6 - 2.4 \times 1.6}{2.4 \times 9.81 \times \cos 30^\circ}$$

$$\boxed{\mu = 0.36}$$

Ch 11: Rotational Vectors & Angular Momentum:

Linear motion

$\vec{F}_{net} = m\vec{a}$

More general 2nd Newton's law:

$\vec{F}_{net} = \frac{d\vec{p}}{dt}$ (when m may change)

$\vec{p}$: linear momentum

$\vec{p} = m\vec{v}$

Rotational motion

$\vec{\tau}_{net} = I\vec{\alpha}$

$\vec{\tau}_{net} = \frac{d\,(?)}{dt}$

$(?) \rightarrow \vec{L}$: angular momentum.

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{r}$ → position vector

$\times$ → cross product

$\vec{p}$ → linear momentum

$\vec{L}$ is perpendicular to both $\vec{r}$ & $\vec{p}$.

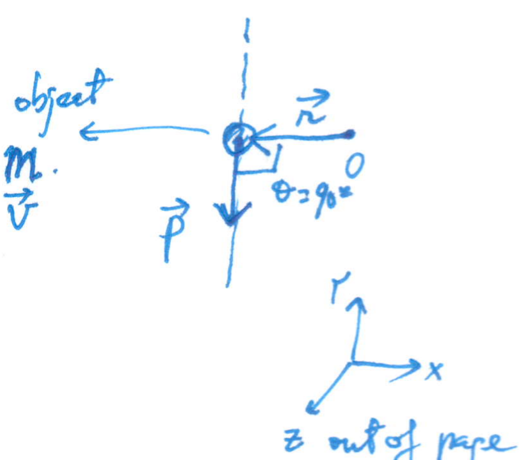
Angular momentum $\vec{L}$ of an object w.r.t. center of rotation O

$\vec{L} = \vec{r} \times \vec{p} \qquad = rp\sin\theta\ \hat{k}$

$\vec{r}$ → center of rotation to object

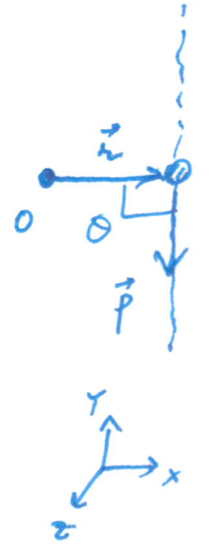
$\vec{p}$ → $m\vec{v}$

$\hat{k}$ → from using RHR (out of the page!)



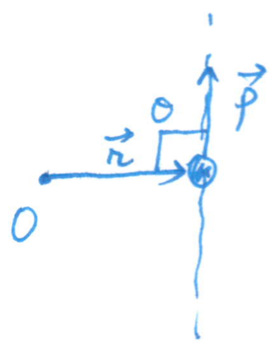
RHR: right hand fingers along 1st vector of cross product ($\vec{r}$), as we turn the fingers towards the 2nd vector ($\vec{p}$), direction of cross product ($\vec{L}$) is indicated by thumb.

With respect to a different center of rotation:



$$\vec{L} = \vec{r} \times \vec{p} = rp\sin\theta \, (-\hat{k})$$

Another situation:



$$\vec{L} = \vec{r} \times \vec{p} = rp\sin\theta \, (\hat{k})$$

(Reflecting the center of rotation wrt line of motion or reversing the direction of motion will change the sign of the angular momentum)

$$\vec{L} = \sum_{i} \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

i → sum over components of a system.

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} \right)$$

$$= \sum_i \Big(\underbrace{\underbrace{\vec{v}_i \times \vec{v}_i m_i}_{\sin\theta = 0}}_{0} + \underbrace{\vec{r}_i \times \vec{F}_i}_{\vec{\tau}_i} \Big)$$

(torque on component i)

$$\rightarrow \frac{d\vec{L}}{dt} = \underbrace{\sum_i \vec{\tau}_i}_{\vec{\tau}_{net}} \rightarrow \boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

So it's not just analog of 2nd Newton's Law. It is a consequence of

1) Def. of $\vec{L}$ ($\vec{L}_i = \vec{r}_i \times \vec{p}_i$)
2) 2nd Newton's Law ($\frac{d\vec{p}_i}{dt} = \vec{F}_i$)
3) Def of $\vec{\tau}$ ($\vec{\tau}_i = \vec{r}_i \times \vec{F}_i$)
4) Property of cross product ($\vec{v}_i \times \vec{v}_i = 0$)

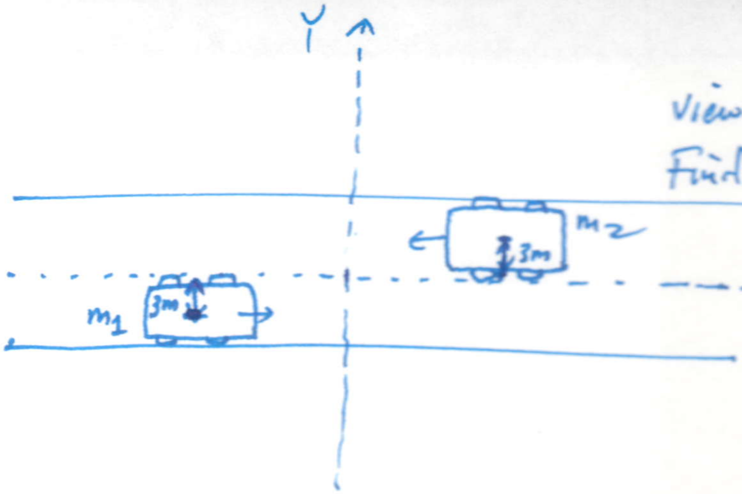
Consequence: $\vec{\tau}_{net} = 0 \rightarrow \frac{d\vec{L}}{dt} = 0 \rightarrow \boxed{\vec{L}_i = \vec{L}_f}$

Conservation of angular momentum

Similar to the conservation of linear momentum:

$$\vec{F}_{net} = 0 \rightarrow \frac{d\vec{p}}{dt} = 0 \rightarrow \boxed{\vec{p}_i = \vec{p}_f}$$

11-37



view from above!
Find total $\vec{L} = \vec{L}_1 + \vec{L}_2$
$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$

$m_1 = m_2 = 1800\ kg$

$90 \frac{km}{h} = 25 \frac{m}{s}$

$\vec{v}_1 = 25\ \hat{i}\ m/s \quad ; \quad \vec{r}_1 = x_1 \hat{i} - 3\hat{j}\ (m)$

$\vec{v}_2 = 25(-\hat{i})\ m/s \quad ; \quad \vec{r}_2 = x_2 \hat{i} + 3\hat{j}\ (m)$

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= (x_1\hat{i} - 3\hat{j}) \times \underbrace{25\hat{i} \cdot 1800}_{45000\hat{i}} + (x_2\hat{i} + 3\hat{j}) \times \underbrace{25(-\hat{i})1800}_{-45000\hat{i}}$$

$$= 135000\ \hat{k} + 135000\ \hat{k} = 270000\ \hat{k}$$

$\hat{i} \times \hat{i} = 0$

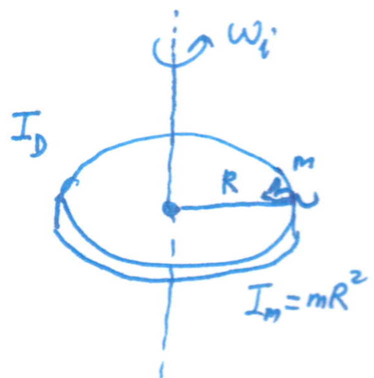
$\hat{j} \times \hat{i} = -\hat{k}$

$kg \frac{m^2}{s}$

$= J.s$

$\hat{j} \times \hat{i}$
into page $\rightarrow (-\hat{k})$

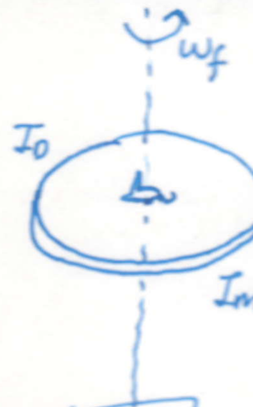
11.40



initial

System:
disk + mouse
No external force & torque on this system. $\vec{\tau}_{net} = 0$

$$\boxed{\vec{L}_i = \vec{L}_f}$$



final

Turn table is a disk

$R = 0.25\,m$

$I_D = 0.0154\ kg\,m^2$

$\omega_i = 22\ rpm$

Mouse at edge (at R)

$m = 0.0195\ kg$

While $\vec{L}$ is conserved $I_T = I_D + I_m$ has changed from initial to final situation, as a consequence ω_i has changed to a different value ω_f !

Why?

How does ω relate to L ?

$$\underbrace{\tau = I\alpha \qquad \tau = \frac{dL}{dt}}$$

$$\tau = I\frac{d\omega}{dt} = \frac{d}{dt}(I\omega) = \frac{d}{dt}(L)$$

I is not depending on time

$$\boxed{L = I\omega}$$

initial

Final

a) $I_D\omega_i + I_m\omega_i = I_D\omega_f + 0\cdot\omega_f$

mouse standing on disk → same angular speed as the disk

$$\rightarrow \boxed{\omega_f = \frac{I_D + I_m}{I_D}\omega_i}$$

Equations:

$$L_i = L_f$$

$$I_1 \omega_i + |\vec{r}_2 \times \vec{p}_2| = (I_1 + I_2)\,\omega_f$$

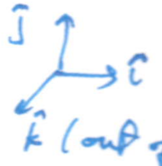
$$I_2 \to m_2 r_2^2 \to ?$$

Clay

$$\vec{r}_2 \times \vec{p}_2 = (x_2 \hat{i} + y_2 \hat{j}) \times v_2 \hat{j}\, m_2$$

$$= x_2 v_2 m_2 \underbrace{\hat{i} \times \hat{j}}_{\hat{k}} = \underbrace{0.15 \times 1.3}_{0.195} \times m_2 \hat{k}$$

$$\downarrow \hat{j} \times \hat{j}$$



$$\boxed{\begin{aligned} \vec{L}_{Table} &= I_1 \omega_i (-\hat{k}) \\ \vec{L}_{clay} &= 0.195 m_2 (+\hat{k}) \end{aligned}}$$ } opposite signs

→ clay was opposing rotation!

Turn table

$I_1 \vec{\omega}_i = I_1 \omega_i (-\hat{k})$ (Direction by RHR: fingers turning as $\vec{\omega}_i$, thumb is direction of $\vec{\omega}_i$)

$$-0.021 \times 0.29 + 0.195 m_2 = -(0.021 + m_2\, 0.15^2)\, 0.085$$

$$+0.021 \times 0.29 - 0.195 m_2 = (0.021 + 0.0225)\,0.085 \quad 0.0225$$

$$+\frac{0.021 \times 0.29}{0.085} - 0.021 = m_2 \left(0.0225 + \frac{0.195}{0.085}\right)$$

$$m_2 = \frac{\frac{0.021 \times 0.29}{0.085} - 0.021}{0.0225 + \frac{0.195}{0.085}}$$

$$\boxed{m_2 = 0.0218\, kg = 21.8 g.}$$

→ Pay attention on the direction of angular momentum!

10.65

$$m = \frac{\pi \left(\frac{R}{4}\right)^2}{\pi R^2} M = \frac{M}{16}$$

$$- \overbrace{\left(\frac{1}{2} m \left(\frac{R}{4}\right)^2 + m \left(\frac{R}{4}\right)^2 \right)}^{\frac{3}{32} mR^2} = 0.496$$