Ch 8: Gravitation

$$F = G \cdot \frac{m_1 m_2}{r^2}$$

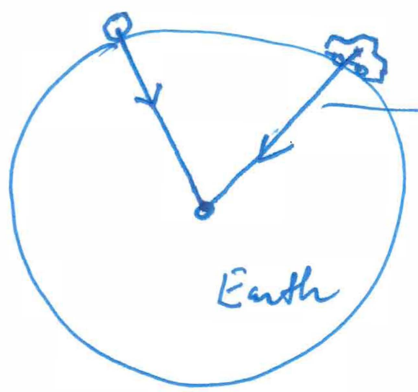
Universal law of gravitation
↓
it applies to marbles as well as to planets

↓

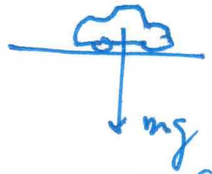
Force of gravitational attraction b/w two objects of mass m_1 & m_2, separated by distance r

→ G: "universal gravitational constant" $= 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

→ Direction of force is center to center, toward the the more massive object



→ So far we say the gravitational attraction is vertical & downward (since for small distances the Earth's curvature is not noticeable)



What about g? What does the Universal Law of Gravitation with the acceleration of gravity $g = 9.81\ m/s^2$?

$M_{Earth} = 5.97 \times 10^{24}$ kg

$R_E = 6.37 \times 10^6$ m

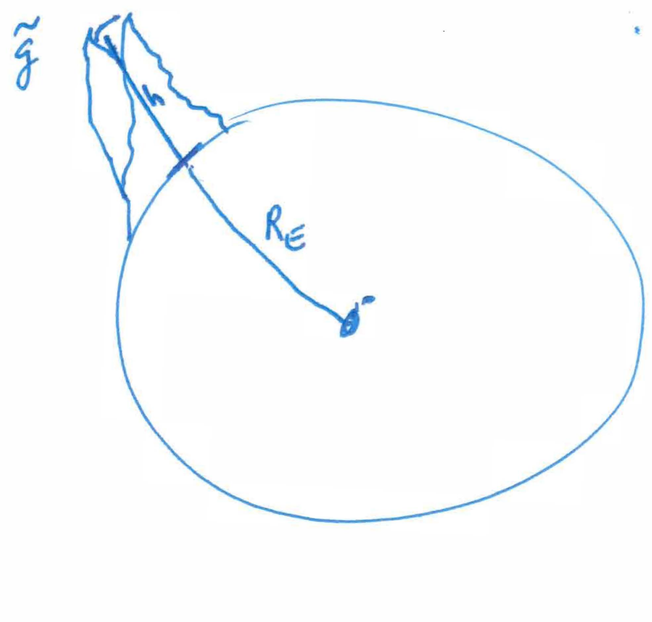
For an object of mass $m_2 = m$ on the surface of the Earth $r = R_E \rightarrow F = G \frac{M_E \cdot m}{R_E^2}$

$$= \underbrace{6.67 \times 10^{-11} \times \frac{5.97 \times 10^{24}}{(6.37 \times 10^6)^2}}_{} \, m$$

$$= \quad 9.81 \frac{m}{s^2} \quad m$$

$$\boxed{F = mg \quad \text{where } g = G \frac{M_E}{R_E^2}}$$

↳ g actually depends on r — Mt Everest: $\tilde{g} < g$; Bottom ~~sea~~: $\tilde{\tilde{g}} > g$



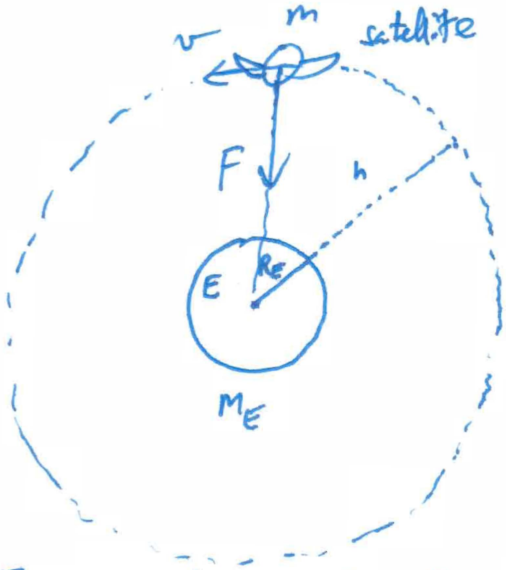
$$\tilde{g} = G \frac{M_E}{(R_E + h)^2} < g$$

if $h \sim 10$ m $\rightarrow \tilde{g} \simeq g$

$h \sim 5000 \rightarrow \tilde{g} < g$

Orbital motion:

Circular orbital motion: satellite under uniform circular motion (UCM) at constant speed v



$R = R_E + h$

Force that provide the change of direction or radial acceleration in UCM is that of gravitational attraction:

$$F = G\frac{M_E m}{R^2} = m\frac{v^2}{R} \rightarrow \text{orbital speed: } v = \sqrt{\frac{GM_E}{R}}$$

F: Net force on satellite.

$G\frac{M_E m}{R^2}$: Universal Law of Gravitation.

$m\frac{v^2}{R}$: 2nd Newton's Law for satellite under radial acceleration.

Orbital period: time to complete one circular orbit:

$$\text{UCM} \Rightarrow T = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{\frac{GM_E}{R}}} = \underbrace{\frac{2\pi}{\sqrt{GM_E}}}_{\text{constant}} R^{3/2}$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{GM_E}} R^{3/2} \Rightarrow T^2 = \left(\frac{4\pi^2}{GM_E}\right) R^3$$

Kepler's 3rd Law: (generalized into elliptical orbits)

"the period squared is proportional to the semimajor axis cubed"

(the semimajor axis is the equivalent of the radius in circular orbits)

Earth follows elliptical orbits around the Sun!

Circular Orbital Motion: satellites:

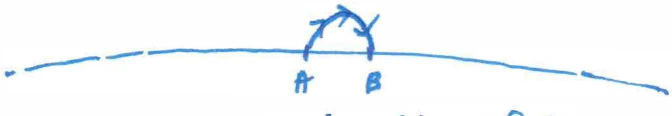
Orbital period for a satellite: @ $h = 250\,km$ above surface

$$T = \frac{2\pi}{\sqrt{GM_E}}(R_E + h)^{3/2} = \frac{2\pi}{\sqrt{6.67\times10^{-11} \times 5.97\times10^{24}}}\left[(6370+250)10^3\right]^{3/2}$$

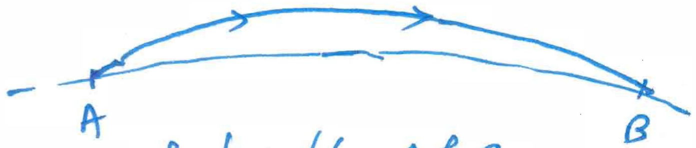
$$= 5400\,s = 1.5\,h.$$

Projectile Motion: a motion that is $\begin{cases} \text{in } y: & \text{due to gravitation} \\ \text{in } x: & \text{uniform motion.} \end{cases}$

↳ so far (Ch. 3): trajectory was an inverted parabola
→ Actually is only good for relatively small range where the surface can be considered as flat



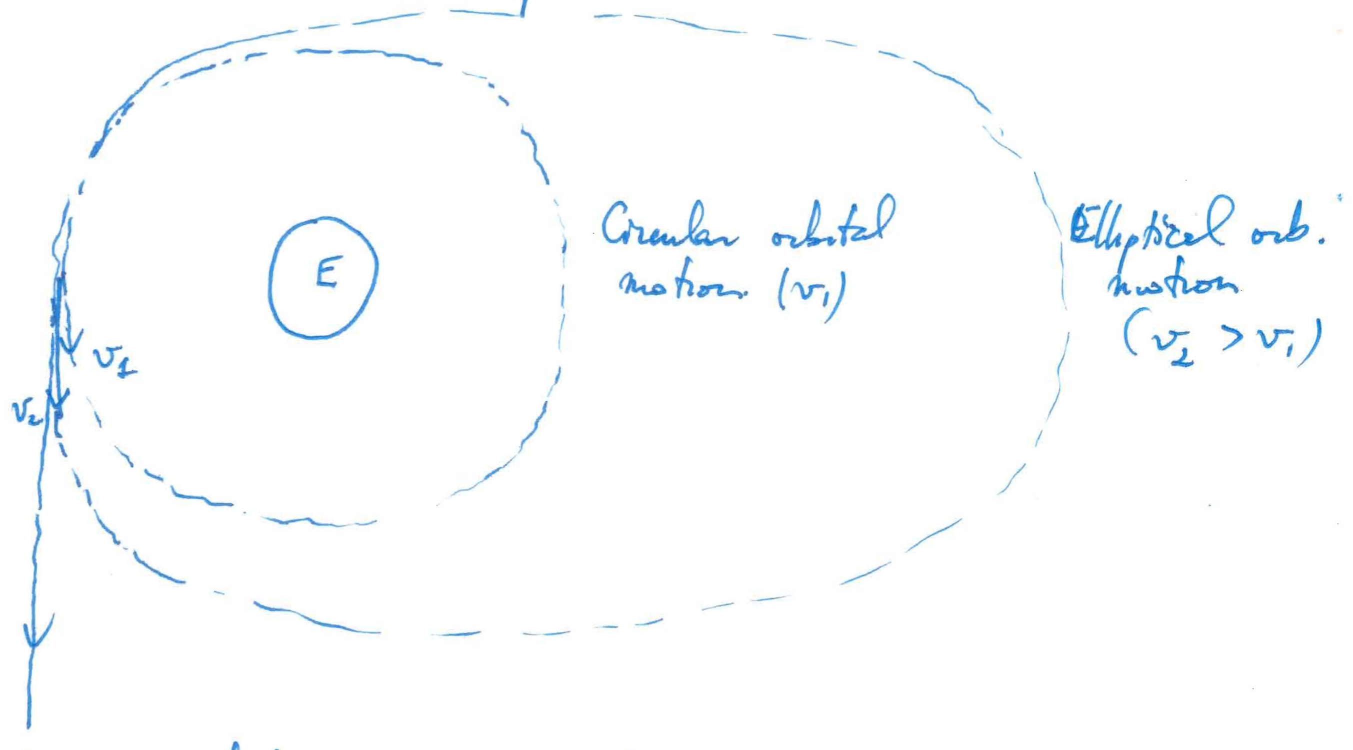
surface b/w A & B can be considered as flat



surface b/w A & B no longer flat!
(long-range ballistic motion)
→ trajectory is part of an elliptical orbit!

$\boxed{\text{Escape speed}}$: speed @ which an object can escape gravitational attraction → it can then follow an "open" orbit.

(under gravitational attraction an object normally follows a closed orbit; satellite around Earth, moon around Earth, etc...). For a rocket to visit other planets, it has to reach escape speed



v_3 ; open orbit = escape speed.

Value on surface? → using
- → Conservation of energy
- → Universal gravitation.

Mechanical energy = K.E + P.E

An object under gravitational attraction KE + PE < 0

Can escape when ~~at~~ KE + PE ≥ 0 → at least KE + PE = 0

$$KE + PE = 0$$

$$\frac{1}{2}\cancel{m} v_{esc}^2 \boxed{- G\frac{M_E \cancel{m}}{r}} = 0 \quad \Rightarrow \quad v_{esc} = \sqrt{\frac{2GM_E}{r}}$$

Escape speed on surface: $v_{esc} = \sqrt{\frac{2 \times 6.67\times10^{-11} \times 5.97\times10^{24}}{6.37\times10^{6}}}$

$$= 11.2\ \frac{km}{s} = 40320\ \frac{km}{h}$$

Why $PE = -G\frac{M_E m}{r}$?

$\Downarrow$ Work (Ch. 6) Grav. force (Ch 8)

$$\Delta U = -\int_A^B \vec{F} \cdot d\vec{r} = -\int_A^B G\frac{M_E m}{r^2}\, dr$$

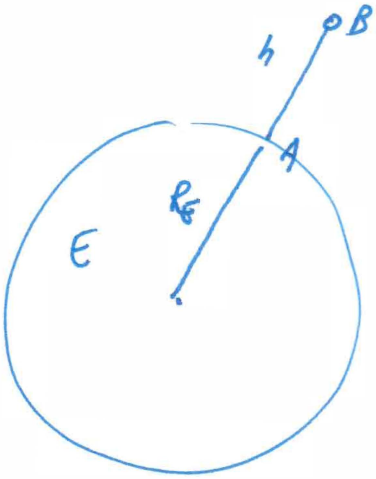
$$= -GM_E m \int_A^B \frac{dr}{r^2} = GM_E m \left(\frac{1}{r}\right)_A^B$$

$\Delta U \propto \frac{1}{r} \rightarrow$ zero PE @ $B = \infty$ $\rightarrow$ Ref. ~~pt~~ point is ∞

$$\Delta U = GM_E m \left(\frac{1}{r}\right)_A^\infty = GM_E m \left(\frac{1}{\infty} - \frac{1}{r}\right)$$

$$\boxed{\Delta U = -\frac{GM_E m}{r}}$$

What is the connection b/w $PE = -G\frac{M_E m}{r}$ & $PE = mgh$?



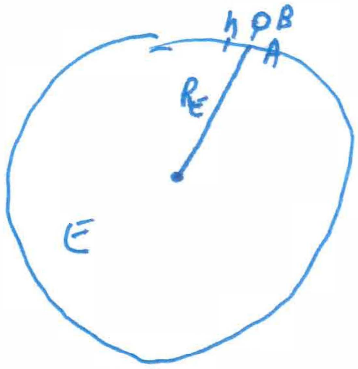
$$\Delta U_{AB} = \Delta PE_{AB} = U_B - U_A$$

$$= -\frac{GM_E m}{\underbrace{(R_E+h)}_{r_B}} + \frac{GM_E m}{\underbrace{R_E}_{r_A}}$$

$$= GM_E m\left(-\frac{1}{R_E+h} + \frac{1}{R_E}\right)$$

$$= GM_E m\left(\frac{-\cancel{R_E} + \cancel{R_E} + h}{(R_E+h)R_E}\right)$$

$$\Delta U_{AB} = GM_E m\frac{h}{(R_E+h)R_E}$$



if $h \ll R_E$
(object very close to surface: what we can see around us)

$$R_E + h \simeq R_E$$

$$(6370000\,m + 300\,m \simeq R_E)$$

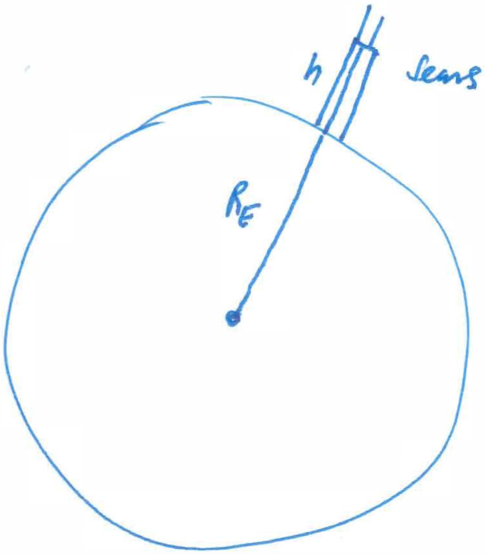
$$\Delta U_{AB} = \underbrace{\left(\frac{GM_E}{R_E^2}\right)}_{g} m h$$

$$\Delta U_{AB} = mgh$$

Conclusion:

$$\underbrace{PE = -\frac{GM_E m}{r}}_{\text{any distance from surface (satellites)}} \qquad \underbrace{PE = mgh}_{\text{smaller distances from surface. (top Sears tower)}}$$

8.18



$$F = G\frac{M_E m}{r^2} \longrightarrow g = \begin{cases} \text{@ surface } g = \dfrac{GM_E}{R_E^2} \\ \\ \text{@ top of Sears tower } \tilde{g} = \dfrac{GM_E}{(R_E+h)^2} \end{cases}$$

$$\Delta g = g - \tilde{g} = GM_E\left[\frac{1}{R_E^2} - \frac{1}{(R_E+h)^2}\right]$$

$\shortparallel$

$0.00136 \frac{m}{s^2}$

$$= GM_E \frac{(R_E+h)^2 - R_E^2}{R_E^2 (R_E+h)^2}$$

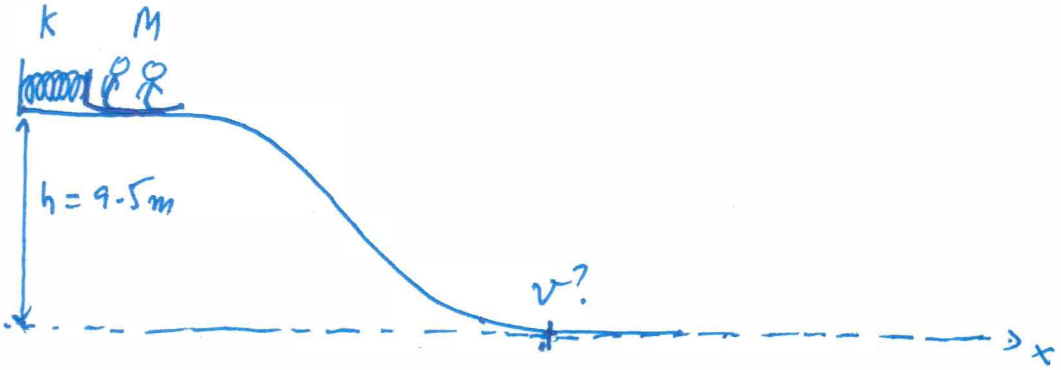
$$= GM_E \frac{2R_E h + h^2}{R_E^2 (R_E+h)^2} = \underbrace{\frac{GM_E}{R_E^2}}_{\text{no } h} \underbrace{\frac{h(2R_E+h)}{(R_E+h)^2}}_{\text{some } h}$$

Now apply an approximation : $h \simeq 300\,m \ll 6370000\,m = R_E$

$$\hookrightarrow \begin{cases} R_E + h \simeq R_E \\ 2R_E + h \simeq 2R_E \end{cases} \Big\rangle\ \Delta g = \underbrace{\frac{GM_E}{R_E^2}}_{g} \frac{h\, 2\cancel{R_E}}{R_E^{\cancel{2}}}$$

$$\Delta g = g\frac{2h}{R_E} \rightarrow \boxed{\frac{\Delta g}{g}\frac{R_E}{2} = h}$$

7.43



$k = 890\ \frac{N}{m}$

$\Delta x = -2.6\,m$

$M = 80\ kg$

(total mass: kids + toboggan)

No friction

Speed @ bottom of hill?

<u>Note:</u> we worked out the speed @ bottom hill before, however there the slope is curved & angle is unknown → can't use kinematic equations & 2nd Newton's Law.

a) <u>Conservation of mechanical energy</u>:
- initial: kids & toboggan at top with spring compressed and @ rest.
- final: @ bottom of hill

initial:
- No KE (@ rest)
- Yes grav. pot. energy.
- Yes elastic potential energy

final:
- No elastic pot energy
- "No" gravitational pot energy.
- Yes KE.

$$ME_{initial} = ME_{final}$$

$$Mgh + \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}Mv^2$$ (v: speed @ bottom of hill)

$$\rightarrow v = \sqrt{2gh + \frac{k}{M}(\Delta x)^2} = \sqrt{2 \times 9.81 \times 9.5 + \frac{890}{80}(2.6)^2}$$

$$\boxed{v \simeq 16.2\,m/s}$$

b) initial elastic potential energy is only a fraction of the final KE → $\frac{\frac{1}{2}k(\Delta x)^2}{\frac{1}{2}Mv^2} = \frac{890 \times (2.6)^2}{80 \times 16.2^2} = 0.28 \rightarrow 28\%$

(Note: remaining 72% is from grav. pot. energy)

Is there any change to 2nd Newton's Law when components of a system are considered? No,

Ch9: System of Particles:

Until now: we have reduced an object to a point (the center of mass or CM). Example: the free-body diagrams (FBD)

Center of Mass: the average position of all components of a system weighted by their masses

↳ $\vec{R}$

weighted average:

(equal masses)

(unequal masses)

Discrete system → $$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M}$$

m_i : mass of component i
$\vec{r}_i$: position of component i
M : total mass of the system. $M = \sum_i m_i$

Continuous system → $$\vec{R} = \frac{1}{M}\int \vec{r}\, dm$$

dm : infinitesimal mass
$\vec{r}$: position of this mass
M : total mass of system.

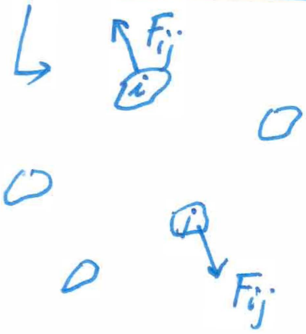
Is there any change to 2nd Newton's Law when components of a system are considered? No, only a subtlety:

2nd Newton's law for a system of particles is:

$$\vec{F}_{net} = M \frac{d^2\vec{R}}{dt^2}$$

Net force on the system is still equal to the total mass times the acceleration of the CM.

<u>Subtlety</u>: is $\vec{F}_{net}$ = only comes from external forces <u>not internal forces</u> between components.



internal forces are pairs of equal & opposite actions & reactions (3rd Newton's laws).
→ they cancel out.

Linear Momentum of an object (system)

$$\vec{P} = M\vec{V} = M\frac{d\vec{R}}{dt} = M\frac{d}{dt}\frac{\sum_i m_i \vec{r}_i}{M} = \sum_i m_i \cdot \frac{d\vec{r}_i}{dt}$$

M ↓ total mass; $\vec{V}$ ↓ velocity of the CM; $=$ ↓ def. of CM; $\frac{d\vec{r}_i}{dt}$: velocity of component i ↓ $\vec{v}_i$

$$\vec{P} = \sum_i \underbrace{m_i \cdot \vec{v}_i}_{\vec{p}_i} = \sum_i \vec{p}_i$$

$\vec{p}_i$ lower case $\vec{p}_i$: linear momentum of component i

2nd Newton's Law:

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

Net external force on a system is equal to the change of it's total momentum over time!

Consequence: $\vec{F}_{net} = 0$ (no net external force on system)

$$\rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} = \text{is conserved.}$$

Conservation of linear momentum

Two different conservation laws
- Conservation of Mechanical energy (only conservative forces are applying; not when there is friction!)
- Conservation of linear momentum (no net external force on system).

Collisions

Inelastic collisions: two components will stick together after collision ($\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f$). Total KE is not conserved (some has been used to change internal structure of one or more components). Eg: throwing a sticky ball on a running kid.

~~Elastic collisions~~

Only momentum is conserved. ($\vec{F}_{net\ external} = 0$)

$$\boxed{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2)\vec{v}_f}$$

Elastic collisions: total KE of the colliding components is also conserved!

$$\boxed{\begin{aligned} m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} &= m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \\ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 &= \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \end{aligned}}$$

→ In 1D: 2 equations { 1 Con. Lin. Mom.; 1 Con. Kin. Energy }

can solve any problem w/ 2 unknowns (eg. v_{1f} & v_{2f} knowing the rest: m_1, m_2, v_{1i}, v_{2i})

→ 1D elastic collision:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

also: $v_{1i} + v_{1f} = v_{2i} + v_{2f}$

(these equations were derived from the 2 conservation laws for 1D elastic collision)

→ 2D: (2D elastic collision):

3 equations $\left\{ \begin{array}{l} 2 \text{ (from conserv. of lin. momentum } (x \,\&\, y)) \\ 1 \text{ (from conserv. of kin. energy).} \end{array} \right.$

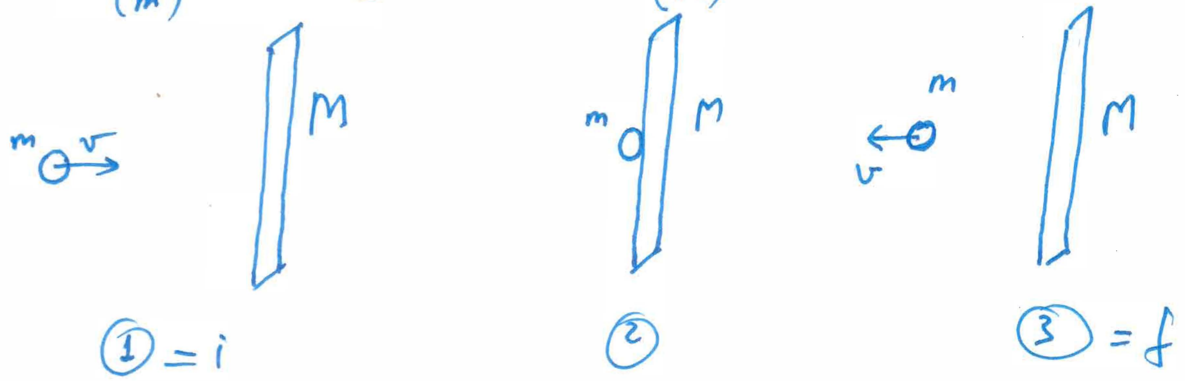
↓

can only solve for 3 unknowns!

$\vec{v}_{1f}$ & $\vec{v}_{2f}$ can't be solved (these are 4 unknowns) You may be asked to find v_{1f}, v_{2f}, and the angle b/w them. (3 unknowns).

<u>A simple but useful example of 1D elastic collision:</u>

A ball (m) colliding with a wall (M): $m \ll M$



- Assume wall is very heavy compared to m ($m \ll M$)
  ↳ $\vec{V}_{wall} = 0$
→ System with two components: ball & wall
→ Net external force on this system (v high enough & ball close enough to wall so that effect of gravity can be ignored)

$\rightarrow \vec{F}_{net_{ext}} = 0 \rightarrow \vec{P}$ is conserved!

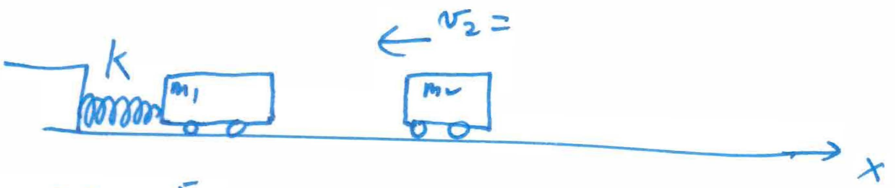
$$\vec{P}_i = \vec{P}_f$$

$$mv = m(-v) + \underbrace{2mv}_{}$$

Momentum transferred to the wall!

$$\underset{\substack{\downarrow \\ \text{mass of wall}}}{2mv = M}\underset{\substack{\downarrow \\ \text{speed of wall}}}{V} \rightarrow V = \frac{2mv}{M} \simeq 0$$

9.41



$k = 3.2 \times 10^5 \frac{N}{m}$

$m_1 = 11000\,kg$; $m_2 = 9400\,kg$

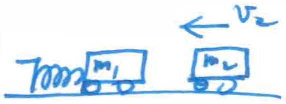
$v_2 = -8.5\,\hat{i}$ m/s

The two cars then couple together $\rightarrow$ inelastic collision (conserv. of linear momentum)

a) Max. compression of spring?

$\longrightarrow \vec{F}_{net_{ext}} = 0 \rightarrow \vec{P}_i = \vec{P}_f$

initial :



$v_{1i} = 0$; $\vec{v}_{2i} = -8.5\hat{i}$ m/s

final :

two cars moving together @ $\vec{v}_f$ compressing the spring

$$\vec{P}_i = \vec{P}_f$$

$$m_2 \vec{v}_{2i} = (m_1+m_2)\vec{v}_f \rightarrow \vec{v}_f = \frac{m_2}{m_1+m_2}\vec{v}_{2i}$$

$$= \frac{9400}{9400+11000}(-8.5\hat{i})$$

$$\boxed{\vec{v}_f = -3.92\ \hat{i}\ m/s}$$

The two cars are moving together @ $\vec{v}_f$ after collision.

↳ spring will slow them down to a final zero speed at max. compression (then will push the cars back in the +x direction)

↳ $$\frac{1}{2}k(\Delta x_{max})^2 = \frac{1}{2}(m_1+m_2)v_f^2 \rightarrow \Delta x_{max} = v_f\sqrt{\frac{m_1+m_2}{k}}$$

↓ max. compression when cars have transfered all of their KE after their collision into the spring!

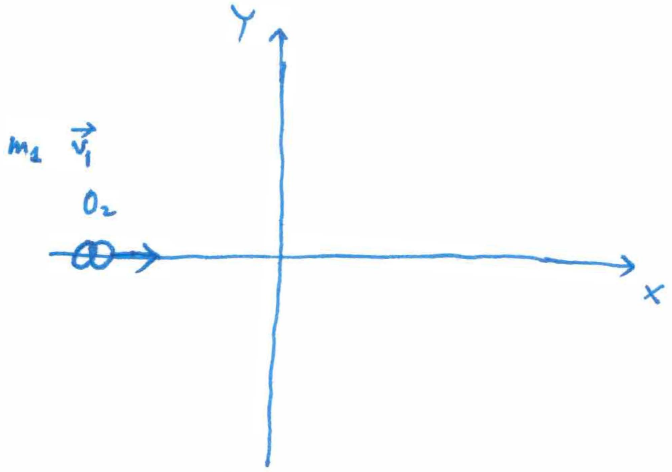
$$\Delta x_{max} = 3.92\sqrt{\frac{9400+11000}{3.2\times10^5}} = 0.989\ m.$$

b) Speed of two cars when they rebound from the spring?

$$\vec{v} = +3.92\ \hat{i}\ m/s$$

when all elastic pot. energy in spring has been returned to the 2 cars in form kinetic energy!

9.67



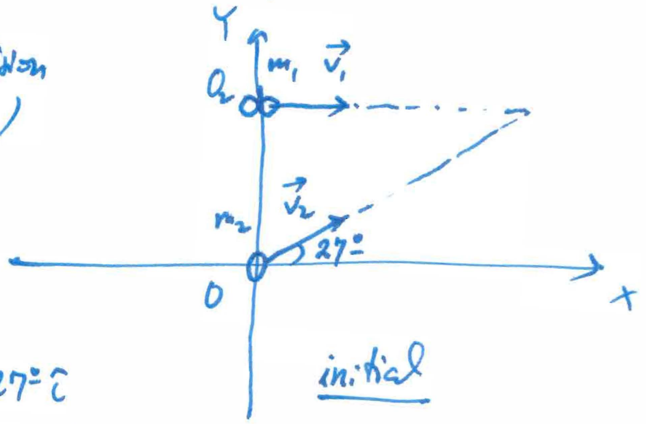
$m_1 = 32u$

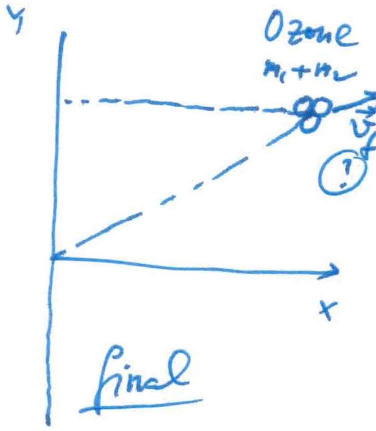
$\vec{v}_1 = 580\hat{i}\ \frac{m}{s}$

2D inelastic collision

$(\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f)$

$\hookrightarrow \boxed{\vec{p}_i = \vec{p}_f}$





$m_2 = 16u$

$\vec{v}_2 = 870\cos 27°\hat{i} + 870\sin 27°\hat{j}\ \frac{m}{s}$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1}{m_1+m_2}\vec{v}_1 + \frac{m_2}{m_1+m_2}\vec{v}_2$$

$$= \frac{32}{32+16}580\hat{i} + \frac{16}{32+16}(870\cos 27°\hat{i} + 870\sin 27°\hat{j})$$

conversion factors from u to kg will cancel out

$$= \left(\frac{2}{3}580 + \frac{1}{3}870\cos 27°\right)\hat{i} + \frac{1}{3}870\sin 27°\hat{j}$$

$$\vec{v}_f = (645\hat{i} + 132\hat{j})\ \frac{m}{s}$$

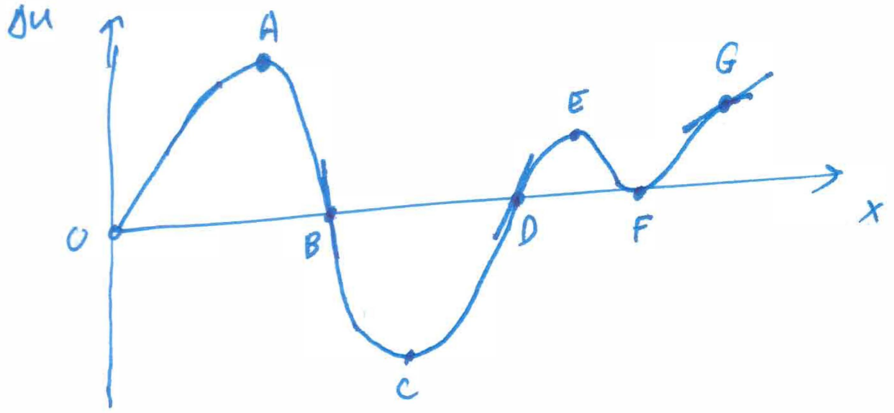
or $648\ \frac{m}{s}$ @ $\theta_v = 11.5°$.

$(1u = 1.661\times10^{-27}\ kg.)$

7.10

$$\Delta U = -\int_1^2 F \cdot dx \quad \overset{\frac{\partial}{\partial x}}{\longleftrightarrow} \quad \underbrace{\frac{\partial \Delta U}{\partial x}}_{\text{slope of potential energy curve}} = -F$$

ΔU → potential energy; F → force; dx → displacement



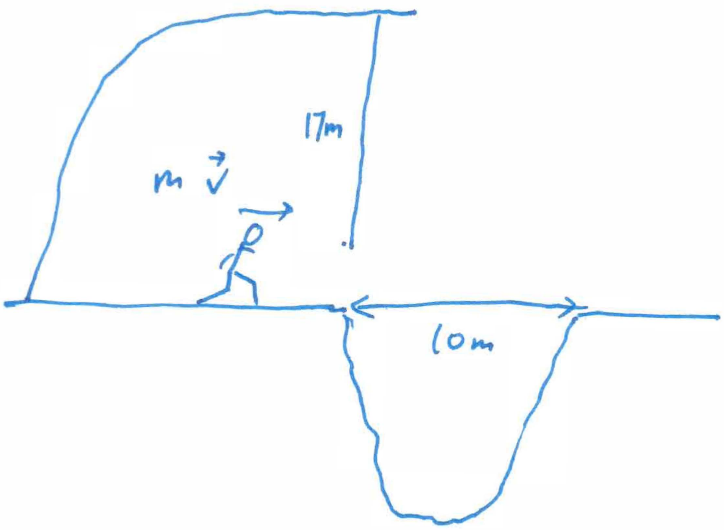
a) Greatest magnitude for F? or steepest slopes in the potential energy curve → B & D

b) F pointing in $-x$ direction ⇒ positive slope: or when the potential is increasing → D & G

c) $F = 0$ → potential energy w/ zero slope: max or min
→ A, C, E, F

d) $F = 0$ & $\Delta U = 0$ ⇒ point F.

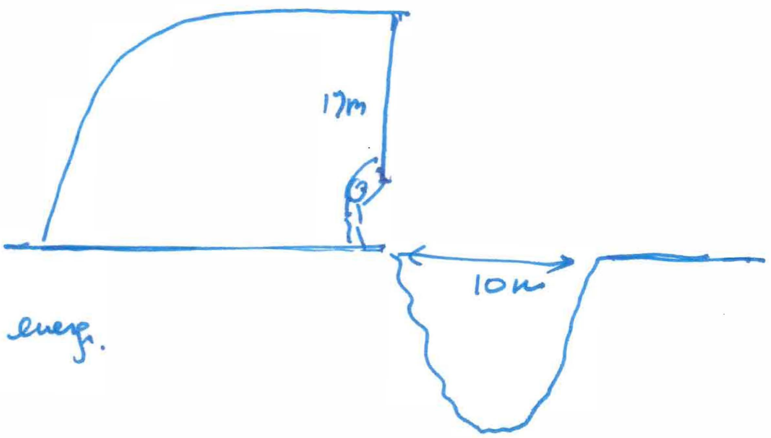
7.64

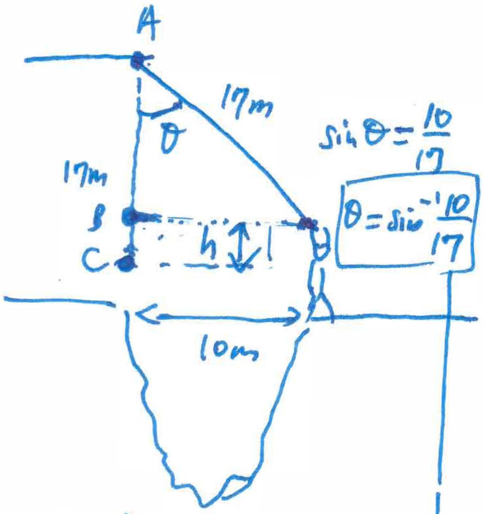


Tarzán's minimum speed : where he barely makes the 10 m gorge.

Conservation of energy:

Ref level for grav. pot. energy.





$\sin\theta = \frac{10}{17}$

$\boxed{\theta = \sin^{-1}\frac{10}{17}}$

initial:

Tarzan grabbing on vine

energy = K.E.

$\frac{1}{2}mv^2$

$h = AC - AB$

$= 17 - 17\cos\theta$

$= 17(1-\cos\theta)$

final:

Tarzan used up all his KE to land on the other side (@10m) (min initial speed).

Gained some grav. pot. energy by going up $\boxed{h = 17(1-\cos\theta)}$

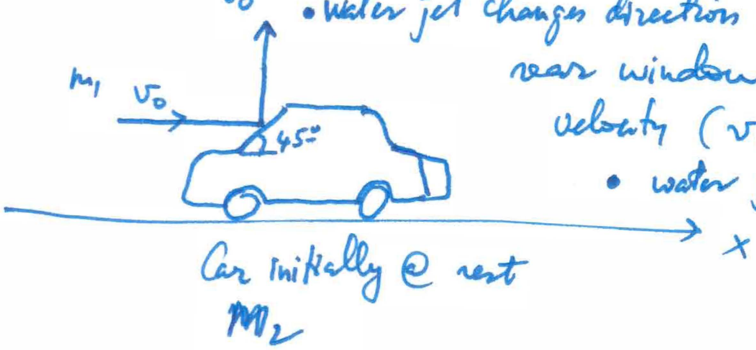
$$\frac{1}{2}mv^2 = mgh$$

$$\frac{1}{2}\cancel{m}v_{min}^2 = \cancel{m}g\,17\left(1-\cos\left(\sin^{-1}\frac{10}{17}\right)\right)$$

$$v_{min} = \sqrt{2\times 9.81\times 17\,(1-\cos 36^\circ)} = \boxed{7.98\ m/s}$$

9.43

• No friction!



• water jet changes direction (+x to +y) after hitting rear window, not magnitude of velocity (v_0)

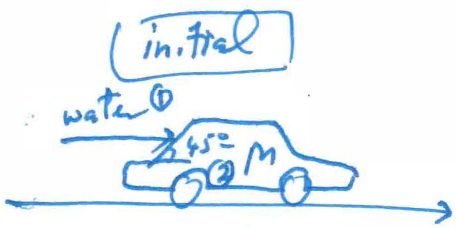
• water jet mass rate is $\frac{dm_1}{dt}$

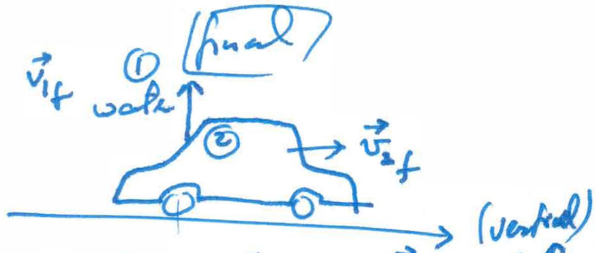
a) a_x for the car?

→ The collision b/w jet of water & car @ rest transfers some of its momentum into the car allowing it to go from zero speed to non-zero speed: acquiring an acceleration in the horizontal direction → a_x.

→ $\vec{F}_{net\,ext} = 0$ all forces are b/w components (water jet & car) of the system. (No friction).

↳ $\boxed{\vec{P}_i = \vec{P}_f}$





• water jet $\frac{dm_1}{dt}$ @ $\vec{v}_{1i} = v_0\,\hat{i}$

• car @ rest, mass M_2

• water going up: $\vec{v}_{1f} = v_0\,\hat{j}$ } (vertical)

• car going $+x$ @ $\vec{v}_{2f}$ } (horizontal)

$$m_1\vec{v}_{1i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

Note: water & car are not sticking together after collision!

observation : $\vec{a} = \frac{d\vec{v}}{dt}$ → car $\vec{a}_2 = \frac{d\vec{v}_{2f}}{dt}$

$$\rightarrow \vec{V}_{2f} = m_1 \frac{1}{m_2}\left(\vec{V}_{1i} - \vec{V}_{1f}\right) = m_1 \frac{1}{m_2}\left(v_0 \hat{\imath} - v_0 \hat{\jmath}\right)$$

$$\rightarrow \vec{a}_2 = \frac{d\vec{v}_{2f}}{dt} = \frac{dm_1}{dt} \frac{v_0}{m_2}\left(\hat{\imath} - \hat{\jmath}\right)$$

only m_1 is changing

→ Observation : final acceleration of car has 2 components

$$\begin{cases} \boxed{a_x = \frac{dm_1}{dt} \frac{v_0}{m_2}} & \leftarrow \text{forward in } +x. \\ a_y = -\frac{dm_1}{dt} \frac{v_0}{m_2} & \end{cases}$$

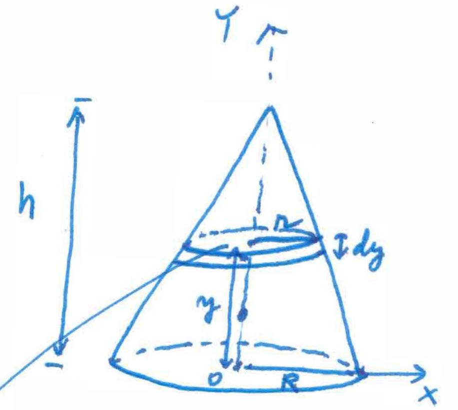
(since initially there was no momentum in the y direction, and since finally the water goes up → car gets pushed down)

b) Max. speed reached by the car?

Why max? or can we accelerate the car to ∞ speed with this water jet? No b/c when the car reaches v_0 (same speed as the water jet) no more pushing or momentum transfer is possible.

→ max speed for car is v_0 !

9.39



$$\vec{R} = \frac{1}{M}\int \vec{r}\,dm \begin{cases} x_{cm} = \frac{1}{M}\int x\,dm \\ y_{cm} = \frac{1}{M}\int y\,dm \end{cases}$$

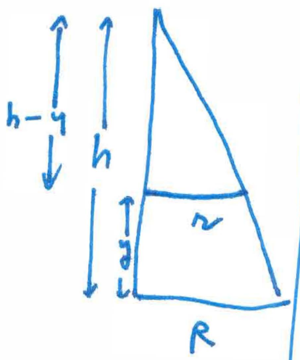
M → total mass.

$\vec{R}$ → position vector of center of mass.

b/c symmetry $x_{cm} = 0$ in the coord. syst. shown. (Y axis coincide with the axis of symmetry of the cone).

infinitesimal disk of thickness dy & mass dm, radius r, located @ distance y above the base.

$$\to y_{cm} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^h y\,\rho\pi R^2\left(1-\frac{y}{h}\right)^2 dy =$$



$$\frac{r}{h-y} = \frac{R}{h} \to r = R\frac{(h-y)}{h}$$

$$\boxed{r = R\left(1-\frac{y}{h}\right)}$$

Infinitesimal disk volume is $dV = \pi r^2 dy$

Since $\rho = \frac{dm}{dV} \to dm = \rho\,dV$

$$dm = \rho\,\pi r^2\,dy = \rho\pi R^2\left(1-\frac{y}{h}\right)^2 dy$$

$$y_{cm} = \frac{\rho\pi R^2}{M}\int_0^h y\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)dy$$

$$= \frac{\rho\pi R^2}{M}\int_0^h \left(y - \frac{2}{h}y^2 + \frac{1}{h^2}y^3\right)dy$$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}}$$

$$\boxed{\rho = \frac{M}{\text{Vol of cone}} = \frac{M}{\pi R^2 \frac{h}{3}}}$$

$$= \frac{3}{h}\left[\frac{y^2}{2} - \frac{2}{h}\frac{y^3}{3} + \frac{1}{h^2}\frac{y^4}{4}\right]_0^h$$

$$= \frac{3}{h}\left[\frac{h^2}{2} - \frac{2}{3}h^2 + \frac{1}{4}h^2\right] = 3h\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$$

$$= 3h\left[\frac{6-8+3}{12}\right] = \frac{3h}{12} = \boxed{\frac{h}{4}}$$

9.48

3 component system
① astronaut $m_1 = 60\,kg$
② oxygen tank $m_2 = 14\,kg$
③ camera $m_3 = 5.8\,kg$

$\vec{F}_{net_{ext}} = 0 \longrightarrow \boxed{\vec{P}_i = \vec{P}_f}$

(in outer space!)
All forces are internal b/w astronaut & objects

initial   final

$\vec{v}_{1i} = \vec{v}_{2i} = \vec{v}_{3i} = 0$

$\vec{v}_{2f} = 1.6\,\hat{i}\ m/s$

$-0.8\hat{i} - 0.3\hat{j} = \vec{v}_{1f} = 0.85 \cos 200°\,\hat{i} + 0.85 \sin 200°\,\hat{j}$

$\vec{v}_{3f} = ?$

$$0 = 60(-0.8\hat{i} - 0.3\hat{j}) + 14 \times 1.6\,\hat{i} + 5.8\,\vec{v}_{3f}$$

$$\vec{v}_{3f} = \frac{60}{5.8}(0.8\hat{i} + 0.3\hat{j}) - \frac{14 \times 1.6}{5.8}\hat{i}$$

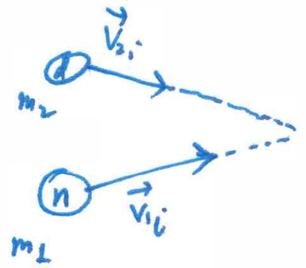
$$\vec{v}_{3f} = \left(\frac{60 \times 0.8}{5.8} - \frac{14 \times 1.6}{5.8}\right)\hat{i} + \frac{60 \times 0.3}{5.8}\hat{j}\ \frac{m}{s}$$

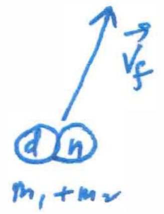
$$= \frac{32 \times 0.8}{5.8}\hat{i} + \frac{60 \times 0.3}{5.8}\hat{j}\ \frac{m}{s}$$

$$\vec{v}_{3f} = 4.4\,\hat{i} + 3.1\,\hat{j}\ \frac{m}{s}$$

or $= 5.33\ m/s$ @ $\theta_v = 34.3°$ ccw from x axis.

9.28





2D inelastic collisions

initial

final.

$$\vec{F}_{net\,ext} = 0 \rightarrow \vec{P}_i = \vec{P}_f$$

$$\begin{cases} \vec{v}_{1i} = (28\hat{i} + 17\hat{j})10^6 \ \frac{m}{s} \\ m_1 = 1u \\ \vec{v}_f = (12\hat{i} + 20\hat{j})10^6 \ \frac{m}{s} \\ (m_1 + m_2) = 3u \end{cases}$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\Rightarrow \vec{v}_{2i} = \frac{m_1 + m_2}{m_2}\vec{v}_f - \frac{m_1}{m_2}\vec{v}_{1i}$$

$$= \frac{3}{2}(12\hat{i} + 20\hat{j})10^6 - \frac{1}{2}(28\hat{i} + 17\hat{j})10^6$$

~~conversion~~
conversion
from u to kg
not necessary

$$\boxed{\vec{v}_{2i} = (4\hat{i} + 21.5\hat{j})10^6 \ \frac{m}{s}}$$

9.52 Person tossing a rock on frictionless ice $\rightarrow \vec{F}_{net\,ext} = 0$

$\rightarrow \vec{P}_i = \vec{P}_f$

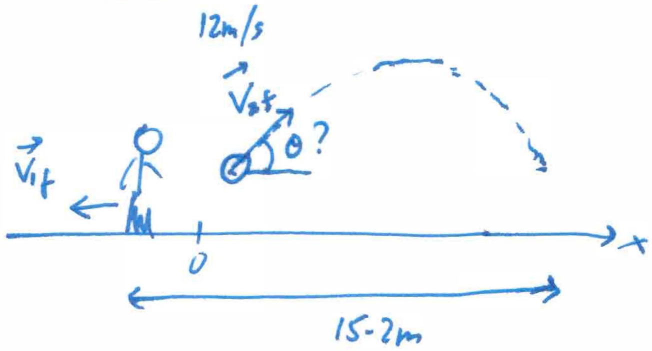
a) initial

$\vec{v}_{1i} = \vec{v}_{2i} = 0$

$m_1 = 65\,kg$

$m_2 = 4.5\,kg$

Final



$$0 = m_1 \vec{V}_{1f} + m_2 \vec{V}_{2f} \quad \begin{cases} x = \checkmark \text{ we'll focus on this eq.} \\ y : \text{since } x - x' = 15.2\text{m} \end{cases}$$

$$\Rightarrow -m_1 V_{1fx} = m_2 V_{2fx} = m_2\, 12 \cos\theta$$

$$V_{1fx} = -\frac{4.5}{65}\, 12 \cos\theta$$ (person recoils in $-x$ direction)

When rock lands it took $2t_{up}$ where t_{up} is time it took to go up to max. altitude point:

$\boxed{\text{Vertical motion for } m_2 \text{ (rock)}}$

$$V_{2y} = V_{2y0} - gt$$

$$0 = V_{2y0} - 9.81\, t_{up}$$

$$V_{2fy} = 12 \sin\theta$$

$$t_{up} = \frac{12 \sin\theta}{9.81}$$

$\Rightarrow$ During $2t_{up}$, person went back at constant speed (no gravity affecting horizontal motion!)

$$x' = V_{1fx} \cdot 2t_{up} = -\frac{4.5}{65}\, 12\cos\theta \times \frac{24 \sin\theta}{9.81}$$

$\boxed{\text{Horizontal motion of } m_2 \text{ (rock)}}$: is also uniform:

the rock ~~which~~ travelled a distance $X = V_{2fx} \cdot 2t_{up}$

$$x = 12\cos\theta \times \frac{24 \sin\theta}{9.81}$$

$\Rightarrow$ Rock-person separation: $$x - x' = \frac{12 \times 24 \cos\theta \sin\theta}{9.81}\left(1 + \frac{4.5}{65}\right)$$

$\boxed{\cos\theta \sin\theta = \frac{\sin 2\theta}{2}}$

$$15.2 = \frac{12^2 \sin 2\theta}{9.81}\left(1 + \frac{4.5}{65}\right)$$

(100)

$$2\theta = \sin^{-1}\left[\underbrace{\frac{15.2 \times 9.81}{144\left(1+\frac{4.5}{65}\right)}}_{0.968}\right] \Rightarrow \boxed{\theta = 37.78^{\circ}}$$

b) $V_{1fx} = -\frac{4.5}{65} 12 \cos(37.78^{\circ}) = -0.658 \frac{m}{s}$