Dimensional Analysis

- Dimension of speed: \([\nu] = \frac{[\Delta s]}{[\Delta t]} = \frac{L}{T}\) (length)

  "\(\Delta\)" delta, "change of" or "increment of"

- Dimension of acceleration: \([\dot{\nu}] = \frac{[\Delta \nu]}{[\Delta t]} = \frac{\frac{L}{T}}{T} = \frac{L}{T^2}\)

- Dimension of energy \([E] = [\frac{1}{2} m \nu^2] = [m][\nu^2]\)

  Numeric constants have dimension of 1

  \[E = M \left(\frac{L}{T}\right)^2 = \frac{M L}{T^2}\]

Application: check a formula:

\[\nu = \frac{1}{2} g h^2 \Rightarrow [\frac{1}{2} g h^2] = [g][h]^2 = \frac{L}{T^2} \cdot L^2 = \frac{L^2}{T^2}\]

\[\nu = \sqrt{gh} \Rightarrow [\sqrt{gh}] = \sqrt{[g][h]} = \sqrt{\frac{L}{T} \cdot L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}\]

- \(g\): acceleration of gravity
- \(h\): height (length)
Units:

<table>
<thead>
<tr>
<th>S.I. (system of international units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L : ( m ) (meter)</td>
</tr>
<tr>
<td>T : ( s ) (second)</td>
</tr>
<tr>
<td>M : ( \text{kg} ) (kilogram ( \Rightarrow ) kilo: ( \times 10^3 ))</td>
</tr>
<tr>
<td>Area : ( m^2 )</td>
</tr>
<tr>
<td>Volume : ( m^3 )</td>
</tr>
<tr>
<td>Energy : ( \frac{kg \cdot m^2}{s^2} = J ) (Joules)</td>
</tr>
</tbody>
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Unit conversion:

\[
\begin{array}{l|cccccccc}
\text{in} & 1\text{in} & 1\text{cm} & 1\text{mm} & 1\text{km} & 1\text{light-year} & 1\text{mi} & 1\text{ft} & 1\text{in.} \\
\text{m} & 10^{-1} & 10^{-2} & 10^{-3} & 10^{-3} & 9.46 \times 10^{15} & 1609 \text{m} & 0.3048 & 2.54
\end{array}
\]

\[
1 \text{lb} = 0.454 \text{kg} \\
1 \text{h} = 3600 \text{s} ; \ 1 \text{day} = 86400 \text{s} ; \ etc. \\
1 \text{km}^2 = 10^6 \text{m}^2 ; \ 1 \text{cm}^2 = 10^{-4} \text{m}^2 ; \ 1 \text{mm}^2 = 10^{-6} \text{m}^2 \\
1 \text{km}^3 = 10^9 \text{m}^3 ; \ 1 \text{cm}^3 = 10^{-6} \text{m}^3 ; \ 1 \text{mm}^3 = 10^{-9} \text{m}^3
Accuracy & Significant Figures:

- Scientific notation: \( \Delta s = 3.105 \times 10^4 \text{ m} \)
  - Easy for multiplication & division

\[
\Delta t = 3000 \text{ s} = 3 \times 10^3 \text{ s}
\]

\[
\rightarrow \text{ speed } = v = \frac{3.105 \times 10^6}{3 \times 10^3} \text{ m/s} = \frac{3.105}{3} \times 10^{6-3} \text{ m/s}
\]

\[
= 1.035 \times 10^3 \text{ m/s}
\]

Accuracy:

- \( \pi = 3.1416 \) is more accurate than 3.14

Addition & Subtraction:

- \( 3.1416 - 1.14 = 2.0016 \)
  - 2.00
  - Keep same accuracy as least accurate term: LHS

Significant Figures:

- 6370 000
  - 3
  - (Zeros at end)

- 6370 001
  - 7
  - (Zeros in middle)

Multiplication & Division:

- Keep smallest number of significant figures (except for numeric constants)

Earth's circumference:

\[
2\pi R_E = 2 \times 3.1416 \times 6.37 \times 10^6 \text{ m/s}^2
\]

\[
= 4.012398 \times 10^7 = 4.00 \times 10^7
\]
1. Estimate volume of water flowing over Niagara Falls in 1s

- Rotate Falls:
- Slab of water:

\[ \text{Vol} = h w d \]

Flow rate (volume of water per second) = \( \frac{\text{Vol}}{\text{time}} = \frac{h w d}{t} \)

1) \( \frac{h w d}{t} \)

2) \( h \left( \frac{w}{t} d \right) \)

3) \( h w d \) [water speed along depth direction]

\[ h d \left( \frac{w}{t} \right) = 5000 \text{ m}^3/\text{s} \] (educated guess)

- Water speed over falls:

\[ 1 \text{m} \quad 1000 \text{m} \quad 5 \text{ m/s} \]

\[ \frac{100 \text{ km}}{3600 \text{ s}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} = \frac{100}{36} \approx 28 \text{ m/s} \]

b) Lake Erie Surface = \( A_{\text{Erie}} \)

\[ \text{How long?} = \frac{V_{\text{Erie}}}{5000 \text{ m}^3/\text{s}} \]

\[ = \frac{1 \text{ m} A_{\text{Erie}}}{5000 \text{ m}^3/\text{s}} \]
\[ A_{\text{true}} = 75 \text{km} \times 375 \text{km} = 75 \times 375 \text{ km}^2 \]

\[ \text{How long} = \frac{75 \times 375 \times 10^6 \times 1 \text{ m}^3}{5000 \times \frac{1}{1000}} = 9 \times 10^6 \frac{1 \text{ day}}{84600 \text{ s}} = 100 \text{ day} \]
Average motion:

\[ \text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{velocity} = \frac{\text{displacement}}{\text{time}} \]

\[ \begin{array}{c}
\text{speed} = \frac{12 \text{ mi}}{\text{hr}} \\
\text{velocity} = \frac{8 \text{ mi}}{\text{hr}}
\end{array} \]

(takes into consideration the direction of motion)

Average velocity:

\[ \bar{v} = \frac{\Delta x}{\Delta t} \] (\(\Delta\) delta = displacement in or change of)

Instantaneous velocity:

\[ v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \] ("derivative of x with time")

\[ x = \text{position} \]

\[ \rightarrow \text{example:} \quad x = at^3 \rightarrow v = \frac{dx}{dt} = \frac{d(at^3)}{dt} = 3at^2 \]

\[ \left\{ \begin{array}{c}
\frac{dt^n}{dt} = nt^{n-1} \\
x = at^n \rightarrow v = na t^{n-1}
\end{array} \right. \]

\[ \text{units:} \quad \frac{m}{s} ; \quad \frac{\text{km}}{\text{hr}} ; \quad \frac{\text{mi}}{\text{hr}} ; \quad \text{etc...} \]
Acceleration: change of velocity over time.

Average acceleration: \( \bar{a} = \frac{\Delta v}{\Delta t} \) (change of velocity over time)

Instantaneous acceleration: \( a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \)

\( x = at^3 \) → \( v = \frac{dx}{dt} = 3at^2 \) → \( a = \frac{dv}{dt} = 3at \frac{d(t^2)}{dt} = 6at \)

units: \( [a] = \frac{L}{T^2} \) → \( \frac{m}{s^2}; \frac{km}{s^2}; \text{etc} \)

Speed: velocity increasing linearly with time.

Since: \( \frac{dv}{dt} \) is also the slope of the curve \( v(t) \)

Motion along a straight line (1D motion)

Constant acceleration: \( \bar{a} = \bar{a} = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - 0} \)

\( v - v_0 = at \) → \( v = v_0 + at \) [1st kinematic equation]
Average velocity, \( \bar{v} \), can also be calculated as:

\[
\bar{v} = \frac{1}{t-0} \int_{0}^{t} v \, dt
\]

\[
(1) \quad \frac{1}{t} \int_{0}^{t} (v_0 + at) \, dt
\]

\[
= \frac{1}{t} \left[ v_0 t + \frac{at^2}{2} \right]_{0}^{t} = \frac{1}{t} \left[ v_0 t + \frac{1}{2}at^2 \right]
\]

\[
= v_0 + \frac{1}{2}at
\]

\[
= \frac{1}{2}v_0 + \frac{1}{2}a \cdot 0 + \frac{1}{2}v_0 + \frac{1}{2}a \cdot t
\]

\[
= \frac{1}{2} \left[ (v_0 + a \cdot 0) + (v_0 + a \cdot t) \right]
\]

\[
\Rightarrow \quad \bar{v} = \frac{1}{2} (v_0 + v) \quad (B)
\]

(B) into (A)

\[
x = x_0 + \frac{1}{2}(v_0 + v) t
\]

\[
= x_0 + \frac{1}{2}(v_0 + v_0 + a t) t
\]

\[
x = x_0 + v_0 t + \frac{1}{2}at^2 \quad (2)
\]

Second kinematic equation.
Constant acceleration motion in 1D:

\[ v = v_0 + at \quad (1) \]
\[ x = x_0 + v_0 t + \frac{1}{2} at^2 \quad (2) \]

Can derive from (1) & (2):

\[ \frac{v^2 - v_0^2}{x - x_0} = 2a \quad (3) \]

(m time)

\( x_0 \): initial position \quad \( x \): final position \quad \( v_0 \): initial velocity \quad \( v \): final velocity

Applications:

2.35

Car with \( v_0 = 50 \text{ mi/h} \) constant acceleration (negative acceleration)

\( x - x_0 = 100 \text{ ft} \) → \( a \)?

Sketch

\[ v_0 = 50 \text{ mi/h} \quad a ? \quad v = 0 \]

\( x - x_0 = 100 \text{ ft} \) stop light

No time information → try eq. (3): \( 2a = \frac{0 - 22.35}{30.48} \)

Make sure units are in S.I.

\[ v_0 = 50 \text{ mi/h} = \frac{160 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ mi}}{3600 \text{ s}} = 22.35 \text{ m/s} \]

\( x - x_0 = 100 \text{ ft} \) \( \cdot \frac{0.3048 \text{ m}}{1 \text{ ft}} = 30.48 \text{ m} \)

\[ a = -8.192 \text{ m/s}^2 \]

deceleration, sign is provided by the equation!
2.61

i) 
\[ x - x_0 = 140 \text{m} \quad t = 3.6 \text{s} \quad v = 53 \text{m/s} \]

(1) \[ v_0 = v - at \quad \text{(now need } a \text{ in exp (2))} \]

(2) \[ x - x_0 = v_0 t + \frac{1}{2} at^2 \]

One alternative: Find \( a \) : eliminate \( v_0 \) use (1) in (2)

\[ x - x_0 = (v - at) t + \frac{1}{2} at^2 = vt - \frac{1}{2} at^2 \]

\[ a = -2 \left( \frac{x - x_0 - vt}{t^2} \right) = -2 \left( \frac{140 - 53 \times 3.6}{3.6^2} \right) = 7.83 \frac{\text{m}}{\text{s}^2} \]

Back in (1): \( v_0 = v - at = 53 - 7.83 \times 3.6 = 24.8 \frac{\text{m}}{\text{s}} \)

Second alternative: Eliminate \( a \):

(1) \( a = \frac{v - v_0}{t} \)

(2) \[ v_0 t = x - x_0 - \frac{1}{2} at^2 = x - x_0 - \frac{1}{2} (v - v_0) t \]

\[ = x - x_0 - \frac{1}{2} v t + \frac{1}{2} v_0 t \]

\[ \frac{1}{2} v_0 t = x - x_0 - \frac{1}{2} v t \]

\[ v_0 = 2 \left( \frac{x - x_0}{t} \right) - v = 2 \frac{140}{3.6} - 53 \]

\[ = 24.8 \frac{\text{m}}{\text{s}} \]

b) 

\[ v_0 = 0 \text{ m/s} \quad 24.8 \text{ m/s} \quad 53 \text{ m/s} \]

Distance??
Alternative:

1) Find $AB$, then $AC = AB + BC = 140\text{m}$.

2) Look at motion b/w $A$ & $C$: constant acceleration.

\[ v_0 = 0, \quad v = 53\text{m/s} \]

\[ a = \frac{7.83\text{m/s}^2}{5} \]

\[ \Rightarrow \text{Eq}(3): \quad (x - x_0)_{AC} = \frac{v^2 - v_0^2}{2a} = \frac{53^2 - 0}{2 \times 7.83} \]

\[ = 179.37\text{m} \]

$(AB \rightarrow 39.37\text{m})$
\[
\left( 6.4 \times 10^{19} \right)^{\frac{1}{3}} = \sqrt[3]{6.4 \times 10^{19}} \\
= (64 \times 10^{18})^{\frac{1}{3}} \\
= (4^3 \times 10^{18})^{\frac{1}{3}} = 4 \times 10^6
\]

\[
R = 3.4 \text{ km (radius)} \\
S = 2.1 \text{ km (arc)} \\
\alpha = \frac{\text{arc}}{\text{radius}} = \frac{2.1 \text{ km}}{3.4 \text{ km}} = 0.62 \text{ rad} \\
\text{"unit"} \text{ for angles.}
\]

1.39 \(\) Consumption 1.5 kW \(\) \(\) (energy rate = energy use per unit time) \(\)
\(\) Collection (solar): 300 W per \(m^2\) \(\) (energy rate density) \(\)
\(\) Total area with solar cells to get needed energy rate:

\[
\text{energy rate} = \text{energy rate density} \times \text{area}
\]

\[
\text{area} = \frac{\text{energy rate}}{\text{energy rate density}} = \frac{1.5 \times 10^3 \text{ W}}{300 \text{ W/m}^2}
\]

Solar cell efficiency = 20% \(\)
\(\) \(5 \text{ m}^2 \) (per person) \(\)
\(\)
\(\text{Total area} = 5 \text{ m}^2 \times 250 \times 10^6 = 1.25 \times 10^8 \text{ m}^2\)
\(\)
\(\text{Total surface of US} = \frac{5000 \text{ km}}{5000 \text{ km}} = 10^7 \text{ km}^2\)
\(\)
\(\text{Fraction: } 1.25 \times 10^{-4} = 0.0001\)
Chapter 3: Motion in 2 & 3 Dimensions

2D: Position is determined by 2 quantities:
\[
\begin{align*}
(x, y) &\rightarrow \text{ Cartesian coordinates} \\
(r, \theta) &\rightarrow \text{ Polar coordinates}
\end{align*}
\]

- \( r \) is the radius (theta), and \( \theta \) is the angle.

Position vector: \( \vec{r} = (x, y) = (r, \theta) \)

Direction is important → use a vector to describe position. Even if \( r_1 = r_2 \), we still have different positions because \( \theta_1 \neq \theta_2 \).

2D
\[
\begin{align*}
\text{Position vector: } \vec{r} &= (x, y) = (r, \theta) \\
\text{Velocity vector: } \vec{v} &= (v_x, v_y) = (v_1, \theta_1) \\
\text{Acceleration vector: } \vec{a} &= (a_x, a_y) = (a_1, \theta_1)
\end{align*}
\]
Changing from Cartesian cords to Polar cords:

\[ \vec{r} = (x, y) \rightarrow (r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \frac{y}{x}) \]

Reversing \( \vec{r} = (r, \theta) \rightarrow (x = r \cos \theta, y = r \sin \theta) \)

Unit vectors: vectors with magnitude 1 (normalised or length of vector)

- along x axis: \( \hat{i} = (1, 0) = (1, 0^\circ) \) (length is 1)
- along y axis: \( \hat{j} = (0, 1) = (1, 90^\circ) \) (length is 1)

\[ \vec{r} = (x, y) = x \hat{i} + y \hat{j} = x(1, 0) + y(0, 1) \]

\[ \vec{v} = (v_x, v_y) = v_x \hat{i} + v_y \hat{j} \]

Preparing for 2D (and 3D) equations of motion (or kinematic equations) – Look at addition & subtraction of vectors!
Addition & Subtraction of Vectors:

\[ \vec{A} + \vec{B} \text{ graphically: draw a copy of } \vec{B} \text{ from tip of } \vec{A} \text{. Addition vector } \vec{A} + \vec{B} \text{ from origin of } \vec{A} \text{ to tip of copy of } \vec{B} \]

\[ \vec{A} - \vec{B} \text{ repeat same but to reverse direction of copy of } \vec{B} \]

Mathematically, using Cartesian coordinates, best for addition & subtraction, polar would be best for multiplication & division.

\[ \vec{A} = A_x \hat{i} + A_y \hat{j} \quad (A_x = 3; A_y = 2 \text{ in the example above}) \]

\[ \vec{B} = B_x \hat{i} + B_y \hat{j} \quad (B_x = -1; B_y = 1) \]

\[ \vec{A} + \vec{B} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} \]
\[ = (3 - 1) \hat{i} + (2 + 1) \hat{j} = 2 \hat{i} + 3 \hat{j} \]

\[ \vec{A} - \vec{B} = (A_x - B_x) \hat{i} + (A_y - B_y) \hat{j} \]
\[ = 4 \hat{i} + 1 \hat{j} \]
Relative Motion:

1D:

1. \[ \mathbf{v}' \]

2. \[ \mathbf{v}' \]

Both 1 and 2 walk at constant velocity \( \mathbf{v}' \), then 1 chooses to use walkway while 2 stays out. 1 gets to terminal faster. Why?

Relative motion:

- Person 1 (wrt. walkway):
  \[ \mathbf{v}' \]

- Person 2 (wrt. floor):
  \[ \mathbf{v}' \]

Since \( \mathbf{V} \) is in the same direction as \( \mathbf{v}' \), \( \mathbf{v} + \mathbf{V} > \mathbf{v}' \).

wrt. floor:

\[ \mathbf{v}' = \mathbf{v}' + \mathbf{V} \]

\[ \mathbf{v} = \mathbf{v}' \]

2D:

\( B \) (fixed point on river bank)

\( A \) (fixed point on river bank)

Rowing boat going at velocity \( \mathbf{v}' \) (wrt. water). To get to B where should \( \mathbf{v}' \) point to:

1) To B  2) Left of B  3) Right of B
Velocity of boat in water: \( \vec{v} \)  
"..." = velocity water + velocity river bank (ground) = \( \vec{v} + \vec{V} \)

1) \( \vec{v} \) pointing A to B:

\[
\begin{array}{c}
\vec{V} \\
\downarrow \\
\vec{v} + \vec{V}
\end{array}
\]

since \( \vec{v} + \vec{V} \) points to right of B → boat will get to the right of B

2) \( \vec{v} \) pointing right to left of B

\[
\begin{array}{c}
\vec{V} \\
\downarrow \\
\vec{v} + \vec{V}
\end{array}
\]

boat can get from A to B

3) \( \vec{v} \) points A to right of B

\[
\begin{array}{c}
\vec{V} \\
\downarrow \\
\vec{v} + \vec{V}
\end{array}
\]

boat goes further away from B than situation 1)
Equations of motion in 2D:

- Motions along perpendicular directions are independent

Give car a push, release the spring by pressing button, ball is launched vertically upward, car keeps going forward.

1) Ball falls back down behind car
2) Ball falls back in front of car
3) Ball falls into car (funnel) ✅

Car motion is along x (4D)
Ball motion is composed of
- Vertical launch by spring
- Horizontal motion of car by spring
- Motion of car

Motions along perpendicular directions combine, however they are independent of each other.
With this observation, what are the kinematic equations in 2D? (constant acceleration)

\[ \begin{align*}
1D: & \quad \vec{v} = \vec{v}_0 + \vec{a} \times t \\
& \quad \vec{x} = \vec{x}_0 + \vec{v}_0 \times t + \frac{1}{2} \vec{a} \times t^2
\end{align*} \]

\[ \Rightarrow \quad 2D \begin{align*}
\begin{align*}
\vec{v}_x &= \vec{v}_{0x} + \vec{a}_x \times t \\
\vec{v}_y &= \vec{v}_{0y} + \vec{a}_y \times t \\
\vec{x} &= \vec{x}_0 + \vec{v}_0 \times t + \frac{1}{2} \vec{a} \times t^2
\end{align*}
\end{align*} \]

Using vector notation:

\[ \vec{z} = x \hat{i} + y \hat{j} \]

\[ \vec{v} = \vec{v}_x \hat{i} + \vec{v}_y \hat{j} = \frac{d\vec{x}}{dt} \hat{i} + \frac{d\vec{y}}{dt} \hat{j} \]

\[ \vec{a} = \vec{a}_x \hat{i} + \vec{a}_y \hat{j} = \frac{d\vec{v}_x}{dt} \hat{i} + \frac{d\vec{v}_y}{dt} \hat{j} = \frac{d^2\vec{x}}{dt^2} \hat{i} + \frac{d^2\vec{y}}{dt^2} \hat{j} \]

In 2D, add a third component \( z \), kinematic equations in vector notation look exactly the same:

\[ \vec{z} = x \hat{i} + y \hat{j} + \vec{a}_z \hat{k} \]

So, unit vector along \( z \)-direction:

\[ \vec{v} = \vec{v}_x \hat{i} + \vec{v}_y \hat{j} + \vec{v}_z \hat{k} \]

\[ \vec{a} = \vec{a}_x \hat{i} + \vec{a}_y \hat{j} + \vec{a}_z \hat{k} \]

\[ 3D \begin{align*}
\vec{v} &= \vec{v}_0 + \vec{a} \times t \\
\vec{x} &= \vec{x}_0 + \vec{v}_0 \times t + \frac{1}{2} \vec{a} \times t^2
\end{align*} \]
Example:

Airplane flies eastward at \( \vec{v}_i = 2100 \text{ km/h} \) and turns southward to \( \vec{v}_f = -1800 \text{ km/h} \).

Average acceleration?

\[
\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{-500 \hat{j} - 583.3 \hat{i}}{150 \text{s}}
\]

\[
|\vec{v}_i| = 2100 \text{ km/h} = \frac{2100 \times 1000 \text{ m}}{3600 \text{ s}} = 583.3 \frac{\text{m}}{\text{s}} \Rightarrow \vec{v}_i = 583.3 \hat{i}
\]

\[
|\vec{v}_f| = 1800 \text{ km/h} = \frac{1800 \times 1000 \text{ m}}{3600 \text{ s}} = 500 \frac{\text{m}}{\text{s}} \Rightarrow \vec{v}_f = 500 \hat{j}
\]

\[
\vec{a} = -3.3 \hat{j} - 3.9 \hat{i} \frac{\text{m}}{\text{s}^2} \quad \text{(Cartesian)} \Rightarrow \left\{ \begin{array}{l}
|\vec{a}| = \sqrt{(-3.3)^2 + (-3.9)^2} = 5.1 \text{ m/s}^2 \\
\theta_a = \tan^{-1} \left( -\frac{3.3}{3.9} \right) = 40.5^\circ \\
\end{array} \right.
\]

\[
\Theta_a = 40.5^\circ + 180^\circ = 220.5^\circ \\
\]

Not correct!
Diver # 2: 
\( (2) \Rightarrow \frac{v^2 - v_0^2}{x - x_0} = 2a \Rightarrow v^2 = \frac{2g(x - x_0)}{x - x_0} \) 
\( v_2 = \sqrt{2 \times 9.81 \times 3} = 7.67 \text{ m/s} \)

Diver # 1: 
\( v_0 = 10 \text{ m/s} \)

\( v_x = v_0 - gt = 0 \Rightarrow t = \frac{v_0}{g} \)
\( x - x_0 = v_0 t - \frac{1}{2} gt^2 = \frac{1}{2} \frac{v_0^2}{g} = \frac{1.8}{9.81} = 0.187 \text{ m} \)
\( v_x = \sqrt{2 \times 9.81 \times (3 + 0.18)} = 7.88 \text{ m/s} \)

\( v_z = 2g(x - x_0) + v_0 \)

Diver # 1 will hit water first.

\( t_2 = \frac{v_z}{g} = \frac{7.67 - 0}{9.81} = 0.782 \text{ s} \)
\( t_1 = \frac{v_z - v_0}{g} = \frac{7.88 - 1.8}{9.81} = 0.618 \text{ s} \)