Rotational Motion

12.1 Angular Speed and Acceleration

Angular speed: the rate at which a body rotates.

— a quantity that describes how rapidly the angular position of any point on the body changes.

Average angular speed:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

$\Delta\theta$: change in angle occurring in the time Δt.

If the angular speed is not a constant, then we can define instantaneous angular speed,

$$\omega = \frac{d\theta}{dt}$$

It is similar to speed except that x is replaced with θ.

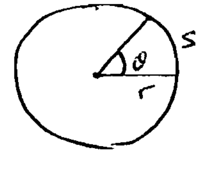
Unit :

$[\theta]$ = radian

$[\omega]$ = radians per second.

Knowing the ω, we can find its linear speed, v.

$$\theta = \frac{s}{r}$$



s: arc length, r, radius.

$$\frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt}$$

or $$\omega = \frac{v}{r}, \qquad \underline{\underline{v = \omega r}}$$

Angular Acceleration

$$\alpha = \frac{d\omega}{dt}$$

$$[\alpha] = \frac{\text{rad}}{\text{sec}^2}$$

Since $\omega = v/r$

$$\alpha = \frac{d\omega}{dt} = \frac{1}{r}\frac{dv}{dt} = \frac{1}{r}a_t$$

$$a_t = \underline{\underline{r\alpha}} \longleftarrow \text{tangential acceleration}$$

For the radial acceleration a_r (centripetal)

$$a_r = \frac{v^2}{r} = \underline{\underline{\omega^2 r}}$$

Table 12-1 in the text indicates that the analysis of rotational and linear motions are mathematically analogous to each other. If angular acceleration is constant, then all our constant-acceleration formulas apply when we substitute θ for x, ω for v, and α for a.

$$\bar{v} = \frac{1}{2}(v_0 + v) \qquad \bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$$

$$v = v_0 + at \qquad \omega = \omega_0 + \alpha t$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

12.2 Torque

Can we formulate an analogous law like $\vec{F} = m\vec{a}$ to deal with rotational quantities? To do so we need rotational analogs of force, mass, and acceleration.

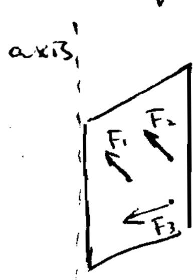
Analog of force: torque

Analog of mass: rotational inertia (moment of inertia)

Analog of acceleration: angular acceleration.

The effectiveness of a force in bringing about changes in rotational motion depends not only on the magnitude of the force, but also on how far from the rotation axis it is applied as well as its direction.

Examples: Opening a door

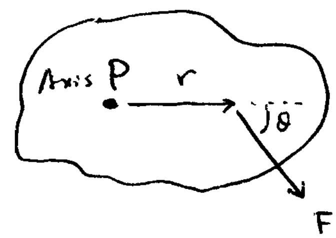


F_2 is more effective than F_1

F_3 points to the axis, does not open door.

We measure the effectiveness of a given force in producing rotational motion by the product of the distance r from the rotation axis with the component of force perpendicular to that axis.

$$\tau = rF\sin\theta \quad \text{— torque}$$



Torque has direction. In fact, its direction can be given by the combination of $\vec{r}$ and $\vec{F}$, using cross-product $\vec{\tau} = \vec{r} \times \vec{F}$.

To keep it simple here we only use clockwise or counterclockwise.

12.3 Rotational Inertia and the Analog of Newton's Law

We still need the rotational analog of mass to develop a rotational analog of Newton's law.

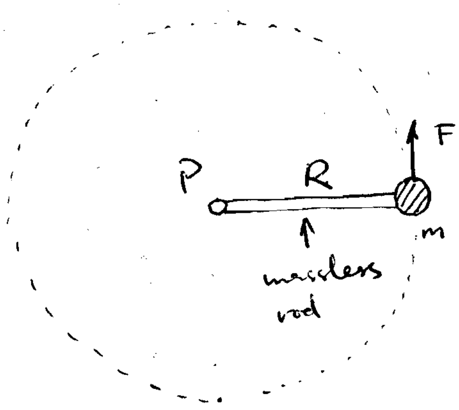
Tangential direction:

$$F = ma_t = mR\alpha$$

$$\tau = RF$$

$$= mR^2\alpha$$

$$= I\alpha$$



$I = mR^2$ — rotational inertia (moment of inertia)

$[I] = kg \cdot m^2$

$I = mR^2$ is the rotational inertia for a single, localized mass. For extended objects, it should be calculated as the sum of the rotational inertias of the individual mass elements making up the object.

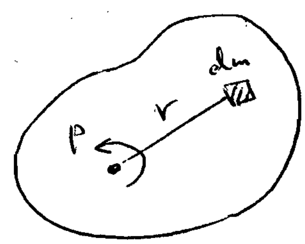
$$I = \sum m_i r_i^2$$

m_i : mass of the ith mass point

r_i : distance from the rotation axis.

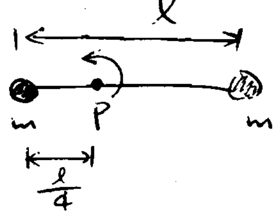
When an object consists of a continuous distribution of matter, in the limit of an arbitrarily large number of very small mass elements, the sum becomes an integral,

$$I = \int r^2 dm$$



over the entire object.

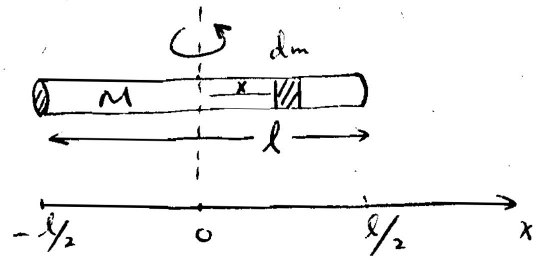
Example



$$I = m\left(\frac{3}{4}\ell\right)^2 + m\left(\frac{1}{4}\ell\right)^2$$

$$= \frac{5}{8} m\ell^2$$

Example: Uniform rod.



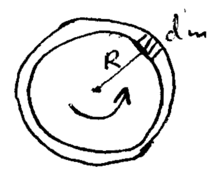
$$I = \int_{-l/2}^{l/2} x^2 \, dm$$

$$\frac{dm}{dx} = \frac{M}{l} \Rightarrow dm = \frac{M}{l} dx$$

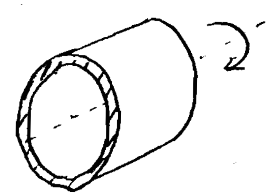
$$I = \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx = \frac{M}{l} \frac{x^3}{3} \Big|_{-l/2}^{l/2} = \frac{1}{12} M l^2$$

Example: thin ring of radius R or thin pipe of radius R.

$$I = \int R^2 dm = R^2 \int dm = MR^2$$

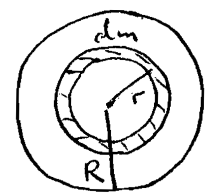


Same for a thin, long pipe.



Example A uniform disk of radius R with a mass of M.

$$I = \int_{r=0}^{r=R} r^2 dm$$

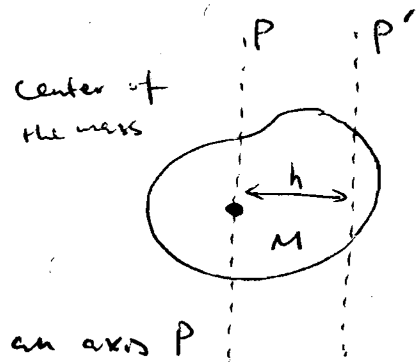


$$\frac{dm}{2\pi r \, dr} = \frac{M}{\pi R^2}, \quad dm = \frac{2M}{R^2} r \, dr$$

$$I = \int_0^R \frac{2M}{R^2} r^3 dr = \frac{1}{2} M R^2$$

Table 2-2 lists rotational inertias for some common shapes. Obviously the rotational inertia depends on the position of rotation axis. For the same object, if the axis moves, the rotational inertia also changes.

Parallel - axis theorem:



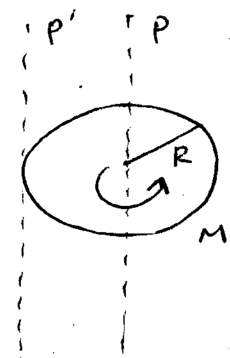
$$I = I_{cm} + M h^2$$

I_{cm}: rotational inertia about an axis P through the center of mass

I: rotational inertia about an axis P' in parallel with P, separated by h.

This theorem allows us to calculate the rotational inertia through any parallel axis.

Example: Find I about P' for a disk



$$I_{cm} = \frac{1}{2} M R^2$$

$$I = I_{cm} + M R^2$$

$$= \frac{3}{2} M R^2$$

Rotational Dynamics

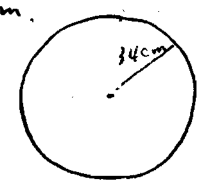
Now we have the analogs of force, mass, and acceleration, we can use the rotational analog of Newton's second law to determine its behavior.

$$\tau_{net,\,ext} = I\alpha$$

$\tau_{net,ext}$: net external torque. — Sum of all external torques acting on the object.

Example:

A bicycle wheel is spinning freely at 210 rpm. What frictional force must the brakes apply to the wheel rim if they are to bring the wheel to a stop in 0.92 s?



bicycle wheel
$m = 740\,g$

Solution:

$$\Delta\omega = 210\text{ rpm} = \frac{210\,(2\pi)}{60\text{ sec}} = 7\pi/s$$

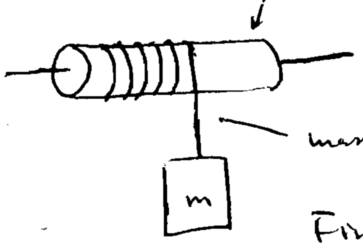
$$\Delta t = 0.92\text{ sec.}$$

$$\alpha = \frac{\Delta\omega}{\Delta t} = 23.9/s^2$$

$$I = mr^2 = 0.0855\ kg\cdot m^2$$

$$F = \frac{\tau}{r} = \frac{I\alpha}{r} = mr\alpha = 6.01\ N.$$

Example: Solid cylinder of M and R

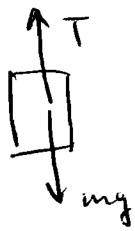


axis

massless rope wrapped around the cylinder

Find the acceleration of the bucket m?

Free body diagram for the bucket:



$$mg - T = ma$$

Rope exerts a torque $\tau = RT$ on the cylinder, giving it an angular acceleration

$$\alpha = \frac{RT}{I} = \frac{RT}{\frac{1}{2}MR^2} = \frac{2T}{MR}$$

As the rope unwinds, the tangential acceleration of the cylinder edge must be equal to the bucket's linear acceleration,

$$a = \alpha R = \frac{2T}{M}$$

or $$T = \frac{1}{2}Ma$$

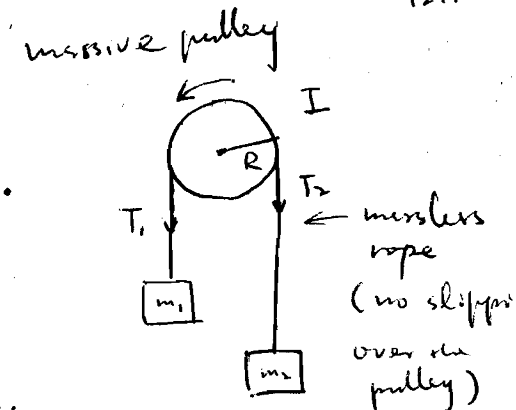
Then $$mg - \frac{1}{2}Ma = ma$$

Finally $$a = \frac{m}{m+\frac{1}{2}M}g$$

Example:

$m_1 > m_2$

Find the downward acceleration of m_1.



Solution:

Previously when we consider similar problems, the pulley is treated with zero mass. As a result, the tensions in the rope on both sides of the pulley are equal. $T_1 = T_2$.

Now with a massive pulley, $T_1 \neq T_2$ causes the pulley to rotate.

m_1: $\quad m_1 g - T_1 = m_1 a \qquad (1)$

m_2: $\quad T_2 - m_2 g = m_2 a \qquad (2)$

pulley. $\quad (T_1 - T_2) R = I \cdot \alpha \qquad (3)$

Constraint: $\quad a = \alpha R \qquad (4)$

(1) + (2) $\quad T_2 - T_1 = (m_2 - m_1) g + (m_1 + m_2) a$

From (3) $\quad T_2 - T_1 = -\frac{I\alpha}{R} = -\frac{I}{R^2} a$

$$(m_2 - m_1) g + (m_1 + m_2) a = -\frac{I}{R^2} a$$

$$a\left[m_1 + m_2 + \frac{I}{R^2}\right] = (m_1 - m_2) g$$

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}} g$$

As $I \to 0$, massless pulley,

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Same result we have obtained before, if we take

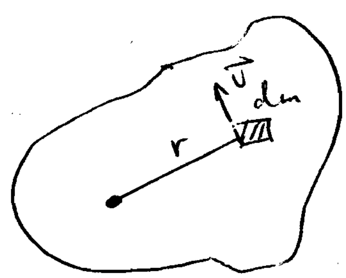
$T_1 = T_2$.

12.4 Rotational Energy.

When an object rotates, it clearly has kinetic energy. The rotational kinetic energy is defined as the sum of the kinetic energies of all its mass elements, taken with respect to the rotation axis.

$$dK = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(\omega r)^2$$

$$= \frac{1}{2}(dm) r^2 \omega^2$$



The total energy

$$K_{rot.} = \int \frac{1}{2}(dm) r^2 \omega^2$$

$$= \frac{1}{2}\omega^2 \int r^2\, dm$$

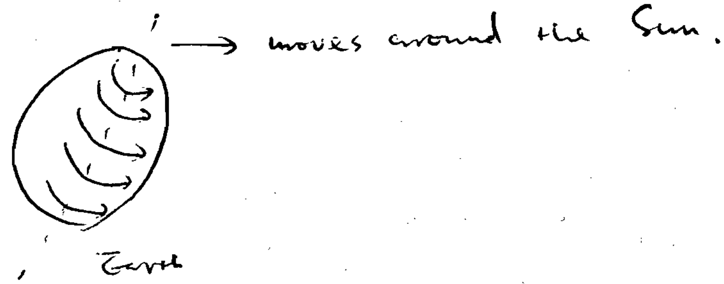
$$= \frac{1}{2} I \omega^2$$

Similar to the expression for the kinetic energy of translational motion, $\frac{1}{2} M v^2$.

· If the object is also doing translational motion

$$K_{total} = K_{rot} + K_{trans}.$$

for example, the position of the axis is also moving in a fix direction like the motion of Earth.

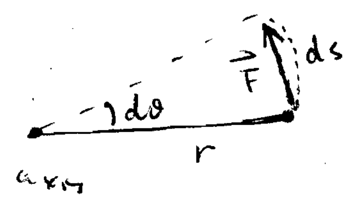


Energy and Work in Rotational Motion

A torque does work on an object to change its kinetic energy

The arc length:

$$ds = r\,d\theta$$



Work done by F: $dW = F \cdot ds = F \cdot r\,d\theta = \tau\,d\theta$

If the torque is constant over $\Delta\theta$.

$$W = \tau\,\Delta\theta$$

otherwise $$W = \int \tau\,d\theta$$

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta = \int_{\theta_i}^{\theta_f} I \alpha \, d\theta = \int_{\theta_i}^{\theta_f} I \frac{d\omega}{dt} \, d\theta$$

$$= \int_{\theta_i}^{\theta_f} I \omega \, d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

— rotational analog of the work-energy theorem.

Static Equilibrium

As we know the magnitude of the torque

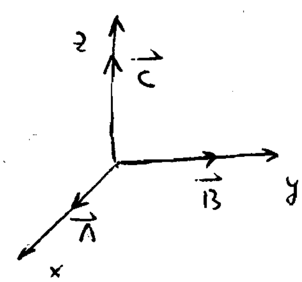
$$\tau = rF\sin\theta$$

r: distance from the rotation axis to the force application point.

F: magnitude of the force.

θ: angle between the corresponding vector $\vec{r}$ and force $\vec{F}$.

Now we would to sign torque a direction. Mathematically, this can be done with the right-hand rule.



$$\vec{C} = \vec{A} \times \vec{B}$$

the magnitude $C = AB\sin\theta$

The direction can be obtained by curling your right fingers in a direction that would rotate $\vec{A}$ onto $\vec{B}$, then your right thumb points in the direction of $\vec{C}$.

Based on the cross-product definition,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

However, if you have trouble understanding the cross product and right-hand rule, you can still treat torque with directions of clockwise and counterclockwise.

14.1 Conditions for Equilibrium.

An object is in equilibrium when the net external force and torque on the object are both zero. It may still be moving or rotating uniformly. In the special case where the object is at rest and not rotating, then it is in static equilibrium.

$$\sum \vec{F}_i = 0$$

$$\sum \vec{\tau}_i = \sum (\vec{r}_i \times \vec{F}_i) = 0 \qquad \text{or}$$

$$\sum \tau_i = 0 \quad (\text{algebraic sum})$$

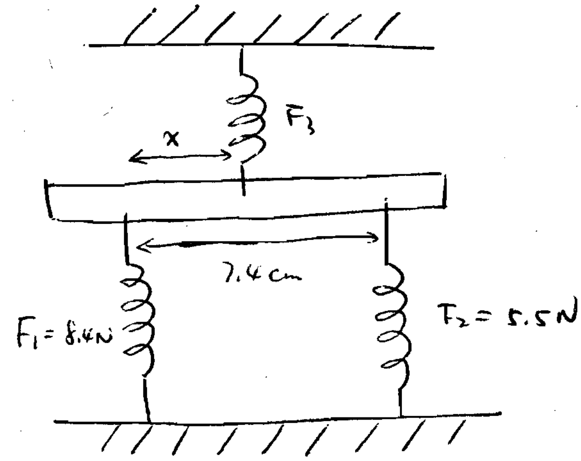
i labels the different forces that act on the object.

The condition for the zero torque was not specified with respect to any point. But a body in equilibrium can't rotate about any point, so it does not matter what point we choose.

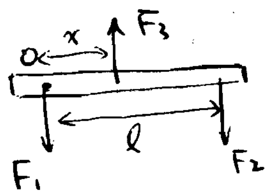
So in solving equilibrium problems, we can choose any convenient point about which to evaluate the torques. An appropriate point is usually the application point of one force so the torque due to that force is zero.

Example:

A metal rod is secured in the horizontal position by three vertical springs. Neglect the mass of the rod, find the tension force F_3 and the point of attachment x for the upper spring.



$$\Sigma F_i = 0$$



$$F_3 = F_1 + F_2 = 13.9 \text{ N}$$

Pick point O where F_1 is applied,

$$\Sigma \tau_i = 0$$

$$l F_2 - x F_3 = 0$$

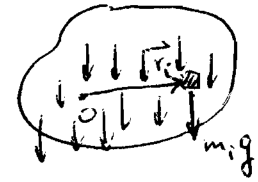
$$x = \frac{F_2}{F_3} l = 2.9 \text{ cm}$$

14.2 Center of Gravity

One force often present is gravity, which acts on all parts of a body. The vector sum of these gravitational forces are mg.

But how do we calculate the torque due to the gravity?

If we sum these torques from different parts of the object



$$\tau = \sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times m_i \vec{g}$$

$$= \left(\sum m_i \vec{r}_i\right) \times \vec{g}$$

$\vec{r}_i$: vector from some pivot point O to that mass element m_i

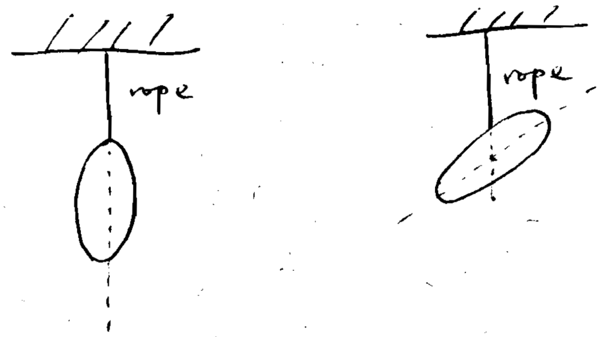
Now $\tau = \dfrac{\sum m_i \vec{r}_i}{M} \times M\vec{g}$, $M = \sum m_i$

$$= \vec{R} \times M\vec{g}$$

$\vec{R} = \dfrac{\sum m_i \vec{r}_i}{M}$ — center of mass.

Therefore, the net torque on an object due to gravity is just that of the gravitational force $M\vec{g}$ acting at the center of mass. In general, the point at which the gravitational force seems to act is called the <u>center of gravity</u>. We just proven that the center of gravity coincides with the center of mass when the $\vec{g}$ is the same everywhere, which tends to be the case for most of problems that we deal with.

How to determine the center of gravity ?



The intersection point of the two lines extended from the two strings attached to the edge of the object gives the position of center of gravity.

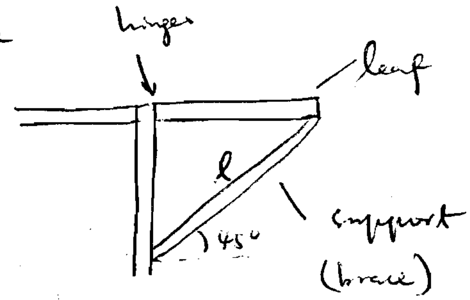
For most uniform object with regular shape, it is simply the geometrical center of the object.

14.3 Examples of Static Equilibrium.

A typical problem involves forces that lie in a plane, then the force equation ($\Sigma \vec{F}_i = 0$) has two components, while the torque equation still has only one, so we end up with three equations to solve for three unknowns.

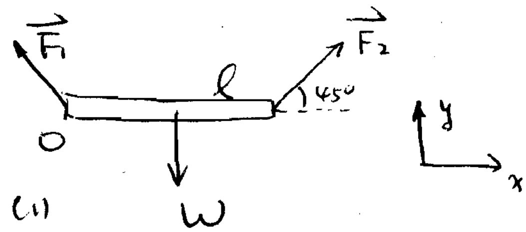
Example: A Drop – leaf Table

maximum load: 120 N on the center of leaf.



Find the force exerted by the hinge that connects the leaf with the table

Solution:



$$\Sigma \vec{F_i} = 0$$

x: $F_2 \cos 45^\circ + F_{1x} = 0$ (1)

y: $F_2 \sin 45^\circ + F_{1y} - W = 0$ (2)

$$\Sigma \tau_i = 0$$

Pick the pivot point at the position of hinge

$$-\frac{1}{2} l W + F_2 l \sin 45^\circ = 0 \quad (3)$$

From (3) $\quad F_2 = \dfrac{W}{2 \sin 45^\circ}$

From (1) $\quad F_{1x} = -\dfrac{W}{2} \operatorname{ctan} 45^\circ = -\dfrac{W}{2}$

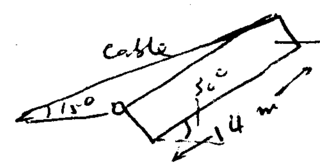
From (2) $\quad F_{1y} = W - \dfrac{1}{2} W = \dfrac{1}{2} W$

Magnitude

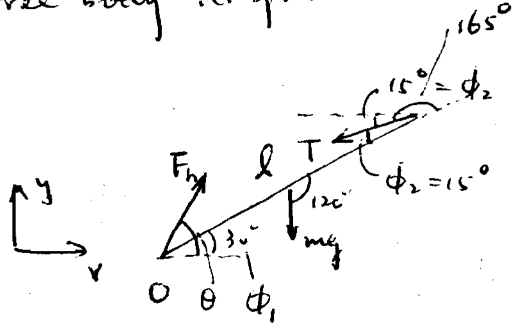
$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2} = \frac{W}{\sqrt{2}}$$

Example: 14-3

The raised span of the draw bridge has its mass of 11,000 kg distributed uniformly over its 14 m length. Find the magnitude of the tension in the supporting cable and the force exerted by the hinge.



Free body diagram



Force equations

$$F_{hx} - T\cos\phi_2 = 0 \qquad (1)$$

$$F_{hy} - T\sin\phi_2 - mg = 0 \qquad (2)$$

Torque equation about point O:

$$-mg\frac{\ell}{2}\,\overset{\sin 120^\circ}{\cancel{\cos\phi_1}} + T\ell\,\overset{\sin 165^\circ}{\cancel{\sin\phi_2}} = 0 \qquad (3)$$

From (3) $$T = \frac{mg\sin 120^\circ}{2\sin 165^\circ} = \frac{mg\sin 60^\circ}{2\sin 15^\circ} = \underline{\underline{1.8\times10^5\ N}}$$

From (1)

$$F_{hx} = T\cos 15^\circ = 1.74\times10^5\ N$$

From (2) $$F_{hy} = mg + T\sin 15^\circ = 1.54\times10^5\ N$$

$$F_h = \sqrt{F_{hx}^2 + F_{hy}^2} = 2.3\times10^5\ N$$

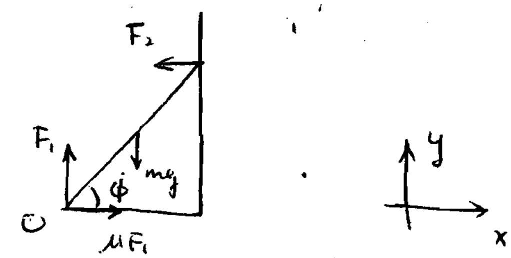
$$\tan\theta = \frac{F_{hy}}{F_{hx}} = 0.887, \qquad \theta = 41.6^\circ$$

Example 14-4 Leaning A Board

A board of mass m and length L is leaning against a wall. The wall is frictionless, and the coefficient of static friction between board ~~the~~ and floor is μ. Find an expression for the minimum angle ϕ at which the board can be leaned without slipping.

Solution:

A frictional force at the floor whose magnitude maximum possible value μF_1 (F_1: normal force) corresponds to the minimum board angle



Force:

x: $\mu F_1 - F_2 = 0$ (1)

y: $F_1 - mg = 0$ (2)

Torque (about O)

$$L F_2 \sin\phi - \frac{L}{2} mg \cos\phi = 0 \quad (3)$$

From (1) and (2) $F_2 = \mu m g$

Substitute into (3)

$$\tan\phi = \frac{1}{2\mu}$$

Does this make sense?

As $\mu \to \infty$, ϕ decreases

As $\mu \to 0$, ϕ increase

If $\mu = 0$ no friction, only vertical position is possible