Ch 12 Rotational Motion

Constant acceleration

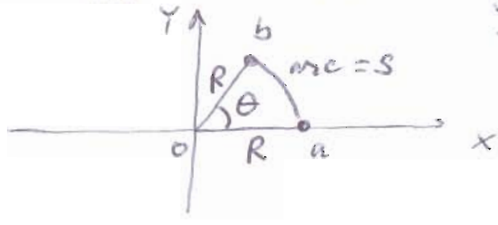
Linear motion		Rotational motion
$\bar{v} = \frac{v_0 + v}{2}$		$\bar{\omega} = \frac{\omega_0 + \omega}{2}$
$v = v_0 + at$	(1)	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2} a t^2$	(2)	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$\frac{v^2 - v_0^2}{x - x_0} = 2a$	(3)	$\frac{\omega^2 - \omega_0^2}{\theta - \theta_0} = 2\alpha$
$F = ma$		$\tau = I\alpha$

ω = "omega" : angular vel.
α = "alpha" : angular accel.
θ = "theta" : angle
τ = "tau" : torque
I = Moment of inertia

Angular velocity: ω

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \;; \quad \omega = \frac{d\theta}{dt} = \frac{1}{R}\frac{d(\text{arc})}{dt} = \frac{1}{R}\underbrace{\frac{ds}{dt}}_{v} = \frac{v}{R}$$

$$\boxed{\frac{\text{arc}}{R} = \theta}$$



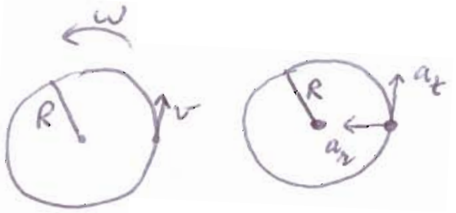
Unit: $\frac{1}{s}$ or $\frac{\text{rad}}{s}$ or $\frac{{}^\circ}{s}$ $\quad$ ($\underbrace{2\pi \text{ rad} \leftrightarrow 360^\circ}_{1 \text{ rev.}}$)

rpm (or # of revolution per minute)

Angular acceleration:

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t} \; ; \quad \alpha = \frac{d\omega}{dt}$$

Units: $\frac{rad}{s^2}$; $\frac{°}{s^2}$; $\frac{rev}{min^2}$



$\left[\begin{array}{l} v = \omega R \\ = R\omega \end{array} \right]$

$$a_r = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = R\omega^2$$: Radial accel.

$$\left[a_t = \frac{dv}{dt} = \frac{d(\omega R)}{dt} = R\frac{d\omega}{dt} = R\alpha \right]$$: Tangential accel.

Torque : $\boxed{\vec{\tau} = \vec{r} \times \vec{F}} \equiv rF \sin\theta \, \hat{\tau}$

θ : angle b/w $\vec{r}$ & $\vec{F}$

$\vec{r}$: position

$\vec{F}$: Force

$\times$: "cross-product" or a product of two vectors

Unit: Nm

$\hat{\tau}$: a unit vector normal or perpendicular to the plane formed by $\vec{r}$ & $\vec{F}$

Example:

Pivot point, system rotates w.r.t. this point.

$$\vec{\tau}_{Total} = \vec{\tau}_M + \vec{\tau}_m = \vec{R} \times M\vec{g} + \vec{r} \times m\vec{g}$$

In balance: bar is horizontal : angle b/w $\vec{R}$ & $\vec{g}$ is 90°
same with $\vec{r}$ & $\vec{g}$

$$0 = \vec{\tau}_{Total} = \underbrace{RMg \, \hat{\tau}_M + rmg \, \hat{\tau}_m}_{\text{provides opposite torques}} = (-RMg + rmg)\,\hat{\tau}$$

$\left\{ \begin{array}{l} CW = + \\ CCW = - \end{array} \right.$

$RM\cancel{g} = rm\cancel{g} \rightarrow \boxed{\frac{M}{m} = \frac{r}{R}}$

Analog of Newton's Law:

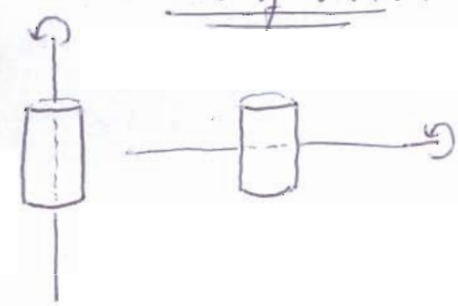
Moment of inertia

$F = ma \longrightarrow \tau = I\alpha$

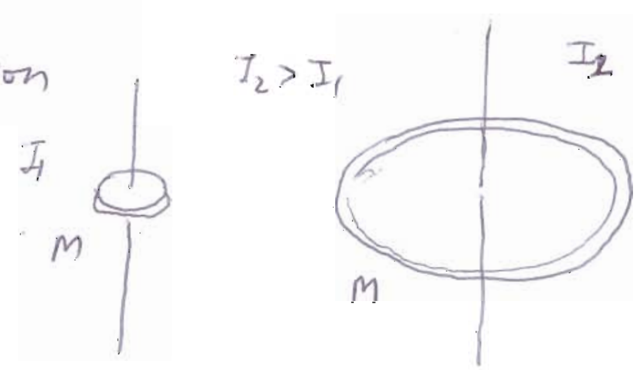
Moment of inertia $I = \sum_i m_i r_i^2$

location/position of component i w.r.t. axis of rotation

$\boxed{I = \int dm\, r^2}$

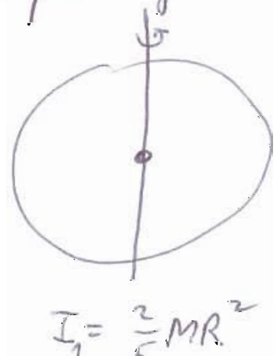


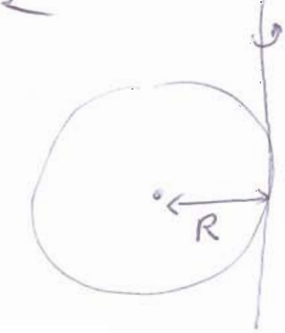
Not just the mass, also its position w.r.t. axis of rotation



Parallel-axis theorem:

Solid sphere of mass M, radius R

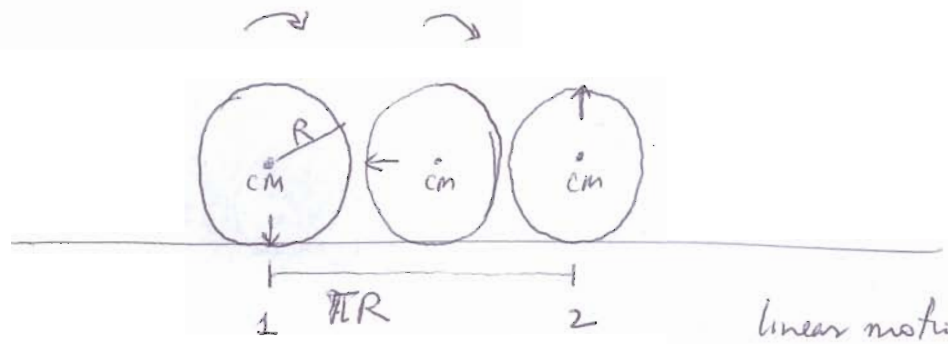




$I_1 = \frac{2}{5}MR^2$

$I_2 = I_1 + MR^2 = \frac{7}{5}MR^2$

Rolling Motion: non skidding



linear motion $1 \to 2 = \pi R$

$$\downarrow$$

$$v_{CM} = \frac{\pi R}{\Delta t}$$

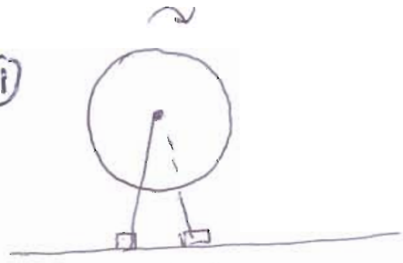
$$\downarrow$$

(SI unit for θ is radian or rad)

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\pi}{\Delta t} = \frac{v}{R} \longrightarrow \boxed{v_{CM} = \omega R}$$

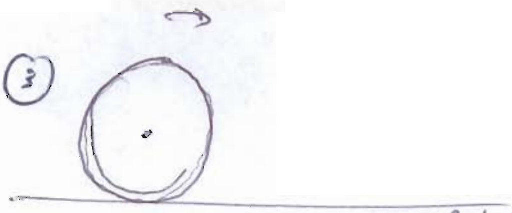
What is the contribution to K.E. from rotational motion?

1



just rotation on wheel

3



Rolling motion $\begin{cases} \text{Rotation w.r.t CM} \\ + \\ \text{Linear motion of CM.} \end{cases}$

2

skidding, no rotation $\{$ linear motion of CM only

$$KE \begin{cases} \text{- linear motion} : \frac{1}{2} m v_{CM}^2 & (1) \\ \text{- Rotational motion} : \frac{1}{2} I \omega^2 & (2) \end{cases}$$

Rolling motion: $KE = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \omega^2$ (3)

additional inertia due to rotation

~~Rolling motion~~: $= \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \frac{1}{R^2} v_{CM}^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right) v_{CM}^2$

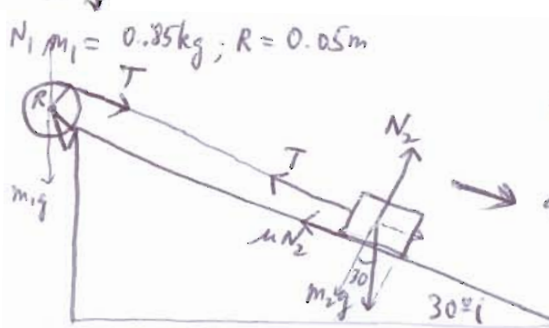
Round & symmetric object: $I = \alpha M R^2 \rightarrow \frac{I}{R^2} = \alpha M$

α: a coefficient

Rolling motion of round & symmetric object $\rightarrow$ $KE = \frac{1}{2}(m + \alpha m) v_{CM}^2 = \frac{1}{2}\underbrace{(1+\alpha)}\, m v_{CM}^2$

12.42

Solid drum



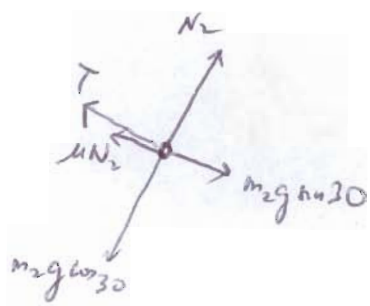
$m_2 = 2.4\,kg$

$a = 1.6\,m/s^2$

μ?

$I = \frac{1}{2} M R^2$

Solid drum thru its axis ~~thru center~~



$N_2 = m_2 g \cos 30$

$\rightarrow$ Newton's law:

(1) $\boxed{m_2 g \sin 30 - \mu m_2 g \cos 30 - T = m_2 a}$

look at solid drum rotational motion for T:

$\tau = I\alpha$ (Analog of Newton's Law)

Tangential is perpendicular to the radius

$RT = \frac{1}{2} m_1 R^2 \left(\frac{a_t}{R}\right)$

$\boxed{a_t = \alpha R}$ same a_t as for rope tangential accel. & block

$\boxed{a_t = a}$

$\downarrow$

$\boxed{T = \frac{1}{2} m_1 a}$

in (1): $\mu = \dfrac{m_2 g \sin 30 - \frac{1}{2} m_1 a - m_2 a}{m_2 g \cos 30} = 0.36$

Ch 13. Vectors in rotational motion: Angular Momentum

Linear	Rotational
x, v, a	θ, ω, α
$F = ma$	$\tau = I\alpha$
↓	
$\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$ (linear momentum) ⟷	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$ (angular momentum)

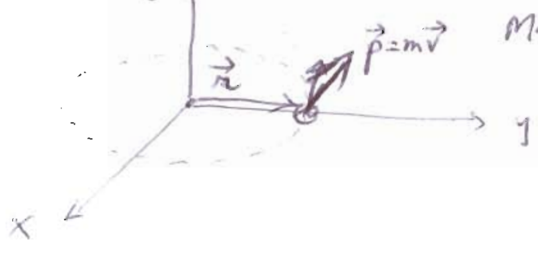
$$\vec{L} \equiv \vec{r} \times \vec{p}$$

↓ cross product b/w two vectors

Cross-product: $\vec{A} \times \vec{B} = \vec{C}$

$$\vec{C} \begin{cases} C = AB\sin\theta \quad (\theta \text{ is angle b/w } \vec{A} \,\&\, \vec{B}) \\ \text{Direction is perpendicular to both } \vec{A} \,\&\, \vec{B}\text{, given by Right Hand Rule} \end{cases}$$

$\vec{C}$: coming out of screen. (turn fingers from $\vec{A}$ to $\vec{B}$, thumb indicates direction of $\vec{C}$)



Mass turning around on a horizontal plane that is perpendicular to screen.

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{r}$ & $\vec{p}$ on horizontal plane → $\vec{L}$ is vertical $\boxed{\text{up}}$

Is $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$ true in fact ?

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i\right) + \left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right)$$

$$\frac{d\vec{r}_i}{dt} \times \vec{p}_i \to \vec{v}_i \times m_i\vec{v}_i = m_i\underbrace{(\vec{v}_i \times \vec{v}_i)}_{0}$$

$$\vec{r}_i \times \frac{d\vec{p}_i}{dt} = \underbrace{\vec{r}_i \times \vec{F}_i}_{\vec{\tau}_i} \quad \text{(definition of Torque)}$$

$\frac{d\vec{p}_i}{dt} = \vec{F}_i$ ← Newton's law

$$\frac{d\vec{L}}{dt} = \underbrace{\sum_i \vec{\tau}_i}_{\vec{\tau}_{net}\text{ (total Torque on system)}}$$

So yes! $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$ is truly the analog of Newton's Law for rotational motion.

→ $\boxed{\text{Clearly } \vec{\tau}_{net} = 0 \rightarrow \vec{L} \text{ is conserved.}}$

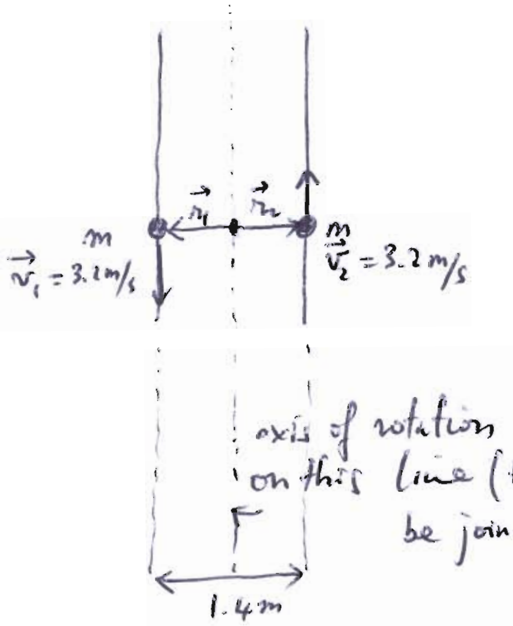
13.32 Two ice skaters on parallel paths at 1.4 m from each other will rotate when joining hands.

$$\vec{\tau}_{ref} = 0 \rightarrow \vec{L}_i = \vec{L}_f$$

$\vec{L}_i$ (before joining hand), $\vec{L}_f$ (after)

$\vec{\tau}_{ref}$ ↓ external to system of the two skaters

View from above: initial

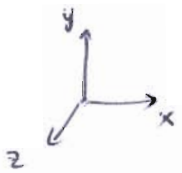


final

$$\vec{L}_i = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m(\vec{r}_1 \times \vec{v}_1) + m(\vec{r}_2 \times \vec{v}_2)$$

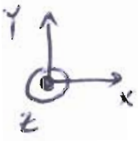
$$= (mrv)\,\hat{k}$$

$r_1 = r_2 = 0.7 m$, $v_1 = v_2 = 3.2 m/s$



XY plane of the screen.

z: out of screen.



$$\vec{L}_f = I_1\vec{\omega}_1 + I_2\vec{\omega}_2 = (I_1 + I_2)\omega\hat{k}$$

Rotation: CCW → by Right Hand Rule if fingers turn CCW thumb indicate direction of $\vec{\omega}$ ($\vec{\omega}_1 = \vec{\omega}_2 = \vec{\omega} = \omega\hat{k}$ they are connected)

$I_1 = mr^2 = I_2 =$

$$\rightarrow \vec{L}_f = 2mr^2\omega\,\hat{k}$$

Conserv. Ang. Mom.

$$\cancel{m}\cancel{r}v\,\hat{k} = 2\cancel{m}r^{\cancel{2}}\omega\,\hat{k} \rightarrow \omega = \frac{v}{r} = \frac{3.2}{0.7} = 4.57\,rad/s$$