Ch. 9 Gravitation

$$F = G\frac{m_1 m_2}{r^2}$$ Universal Gravitation

G ↓ Universal Gravitational constant $= 6.67\times10^{-11}\ \frac{Nm^2}{kg^2}$

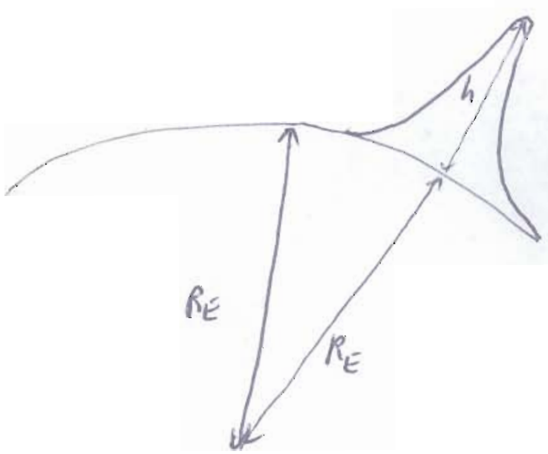
↳ Force of attraction b/w 2 objects of mass m_1 & m_2 separated by r

Why $g = 9.81\ m/s^2$? Gravitation by our Planet $\begin{cases} M_E = 5.97\times10^{24}\,kg \\ R_E = 6.37\times10^{6}\,m \end{cases}$

An object on the surface : $r = R_E$

$$F = G\frac{M_E m_2}{R_E^2} = m_2 g \rightarrow \boxed{g = \frac{GM_E}{R_E^2}} = \frac{6.67\times10^{-11}\times5.97\times10^{24}}{(6.37\times10^{6})^2}$$

$$= 9.81\ m/s^2$$



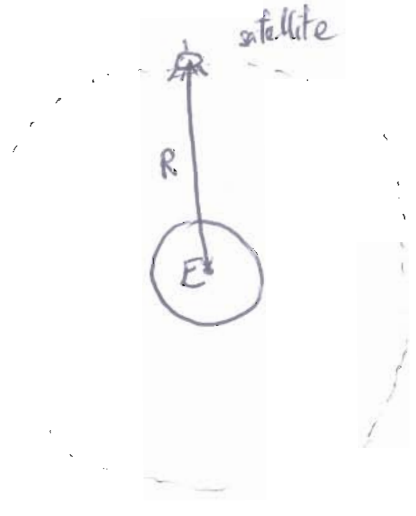
Mt Everest

$$g' = \frac{GM_E}{(R_E+h)^2} < g$$

Orbital Motion :

M_E

Period of a satellite at orbit R (Uniform Circular motion)



Agent that provides radial acceleration for this UCM is gravitation

$$\frac{GM_E m_s}{R^2} = m_s\frac{v_s^2}{R} \rightarrow v_s = \sqrt{\frac{GM_E}{R}}$$

$$\boxed{T = \frac{2\pi R}{v_s} = \frac{2\pi}{\sqrt{GM_E}}R^{3/2}}$$

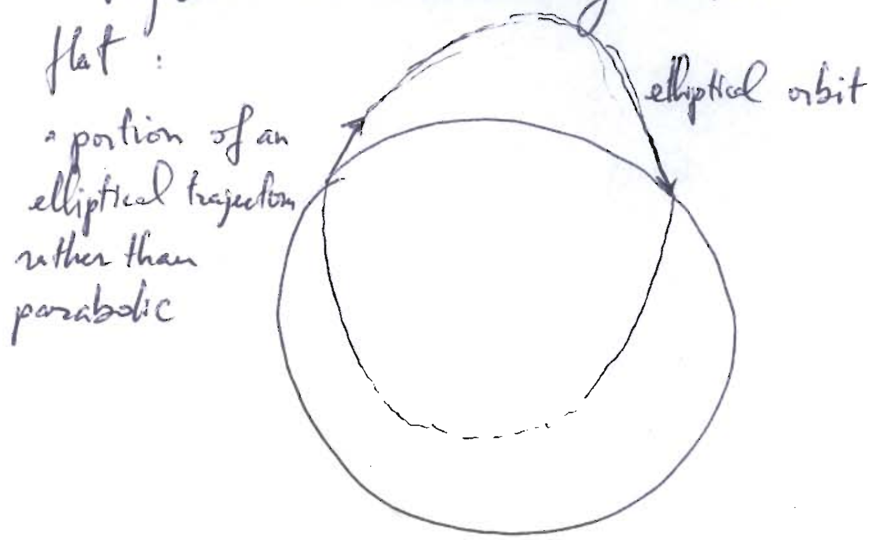
A satellite at: $R = 250\ km \rightarrow \begin{cases} v_s = 7.75\ km/s & = 27900\ km/h \\ T = 5400\ s = 1.5\ h. \end{cases}$ $1.26 \times 10^6\ km/s$

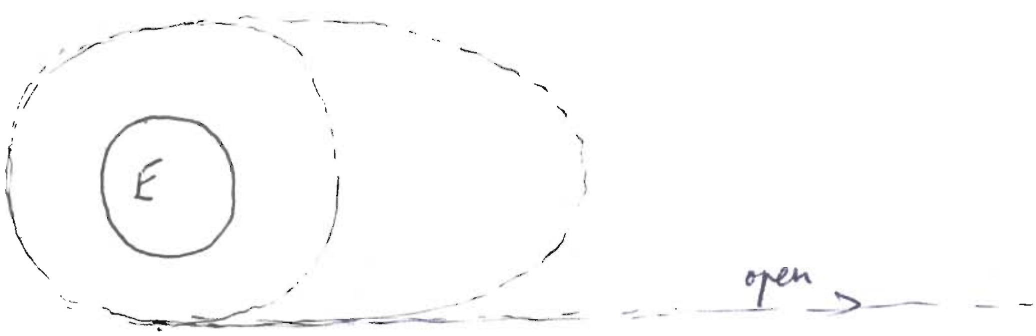
$R_E = 6370\ km$

$$R = 6370\ km + 250\ km = 6.62 \times 10^6\ m$$

→ In general orbits under gravity by round objects are elliptical

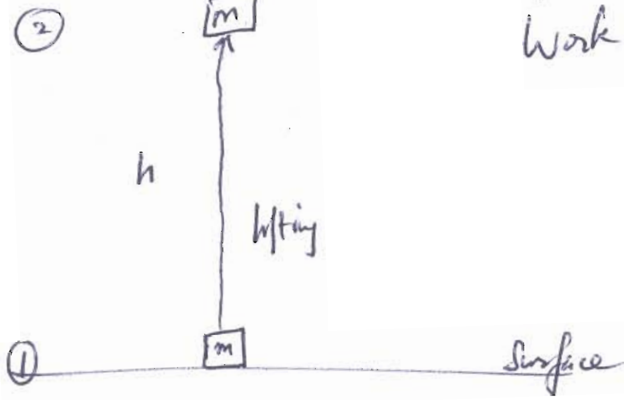
Projectile motion over long distance: → Earth's surface is not flat:





Gravitational Potential Energy

$$\text{Work} = \int_1^2 \vec{F}\cdot d\vec{r}$$



Gravitational energy for m at ② is <u>higher</u> compared to at ①

$$\underset{(U_2 - U_1)}{\Delta U} = \int_1^2 \frac{GM_E m}{r^2}\, dr = GM_E m \left(-\frac{1}{r}\right)_1^2 = GM_E m \underbrace{\left(\left(-\frac{1}{R_E+h}\right) - \left(-\frac{1}{R_E}\right)\right)}_{\frac{1}{R_E} - \frac{1}{R_E+h}}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \; ; \quad \int \frac{dx}{x^n} = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1}$$

$$\int \frac{dr}{r^2} = -\frac{1}{r}$$

(if you define $\Delta U = U_1 - U_2 \Rightarrow$ use $F = -\frac{GM_E m}{r^2}$ (sign to indicate attraction) so the RHS is +)

$$\Delta U = GM_E m \left(\frac{1}{R_E} - \frac{1}{R_E+h} \right) = GM_E m \left(\frac{h}{R_E(R_E+h)} \right)$$

(General)

Now $h \ll R_E$ (h up to 1000 m) $R_E + h \simeq R_E$

$$\Delta U = \underbrace{\frac{GM_E}{R_E^2}}_{g \text{ for } h \ll R_E} m\, h = mgh$$

Escape speed:

Total energy of object under gravity is 0

$\hookrightarrow$ KE + PE $= 0$

$$\frac{1}{2}mv^2 - \frac{GM_E m}{r} = 0 \rightarrow v_{esc} = \sqrt{\frac{2GM_E}{r}}$$

$$r = R_E \rightarrow v_{esc} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6}} = 11.2 \frac{km}{s}$$

$$= 40320\ km/h$$

(8.27) $k = 890\,N/m$ $\quad m = 80\,kg$ $\quad \Delta x = 2.6\,m$ (spring compression), No friction.

a) $\vec{v}_2 = ?$ $\qquad$ b) KE_f in spring.

$$\tfrac{1}{2}m\cancel{v_1^2} + mgh_1 + \underbrace{\tfrac{1}{2}k\Delta x^2}_{\text{Elastic potential energy}} = \tfrac{1}{2}mv_2^2 + mgh_2$$

$\quad 0$

$$mg(h_1 - h_2) + \tfrac{1}{2}k\Delta x^2 = \tfrac{1}{2}mv_2^2$$

$$v_2 = \sqrt{2g(h_1-h_2) + \frac{k}{m}\Delta x^2}$$

$$= \sqrt{2\times 9.81\times 9.5 + \frac{890}{80}\,2.6^2}$$

$$= 16.1\,m/s$$

b) $$\frac{\tfrac{1}{2}mv_2^2}{\tfrac{1}{2}k\Delta x^2} = \frac{80\times 16.1^2}{840\times 2.6^2} = 3.44$$

Final speed from both elastic pot. energy AND gravitational potential energy:

$$\frac{\tfrac{1}{2}k\Delta x^2}{\tfrac{1}{2}mv^2} = \frac{1}{3.44} = 0.29 \text{ or } 29\%$$

(9.59)

$$\left.\begin{aligned} v_{esc} &= \sqrt{\frac{2GM_E}{R_E}} \\ v_{esc}' &= \sqrt{\frac{2GM_E}{R_E'}} \end{aligned}\right\} \qquad \frac{v_{esc}}{v_{esc}'} = \sqrt{\frac{R_E'}{R_E}} \rightarrow \frac{11.2^2}{30^2} = \frac{R_E'}{R_E}$$

$$\rightarrow R_E' = \left(\frac{11.2}{30}\right)^2 R_E = 887\,km$$

Ch. 10 System of Particles

Center of Mass: average position of components weighted by their masses, $\vec{R}$

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M} \quad ; \quad M = \sum_i m_i \text{ (total mass)}$$

So far, 2nd Newton's Law: $\boxed{\vec{F}_{net} = M \frac{d^2\vec{R}}{dt^2}}$

Can we justify this, from the point of view of individual particle?

Particle i: $\quad \vec{F}_i = m_i \frac{d^2\vec{r}_i}{dt^2}$, sum over particles:

$$\underbrace{\sum_i \vec{F}_i}_{\vec{F}_{total}} = \sum_i m_i \frac{d^2\vec{r}_i}{dt^2} = M\frac{d^2}{dt^2}\underbrace{\left(\sum_i m_i \vec{r}_i\right)\frac{1}{M}}_{\vec{R}}$$

$$\Rightarrow \vec{F}_{total} = M \frac{d^2\vec{R}}{dt^2}$$

$\sum_i$: $\vec{F}_{total} = M \frac{d^2\vec{R}}{dt^2} = \vec{F}_{net,\,ext}$ ✓

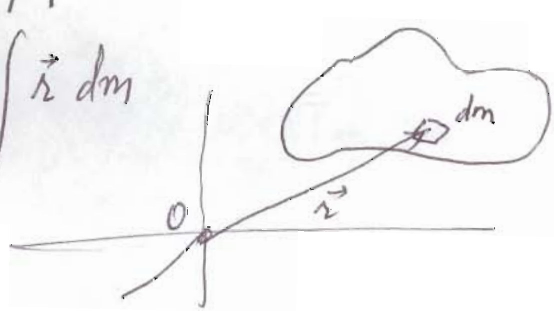
So far : $\vec{F}_{net} = M \frac{d^2\vec{R}}{dt^2}$

$$\vec{F}_{total} \equiv \underbrace{\sum_i \left(\vec{F}_i\right)_{net}}_{\text{(a) internal forces (b) external forces}} = \boxed{\sum_i (\vec{F}_i)_{int}} + \sum_i (\vec{F}_i)_{ext}$$

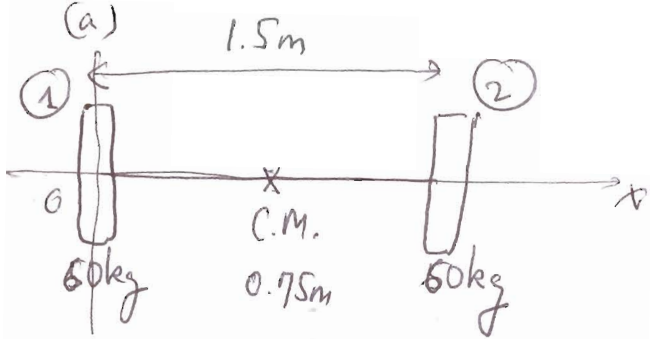
$\rightarrow$ 3rd Newton's Law $\rightarrow 0$

$$= \boxed{\vec{F}_{net,\,ext}}$$

C.M. $\begin{cases} \vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{M} \\ \vec{R} = \dfrac{1}{M}\int \vec{r}\, dm \end{cases}$

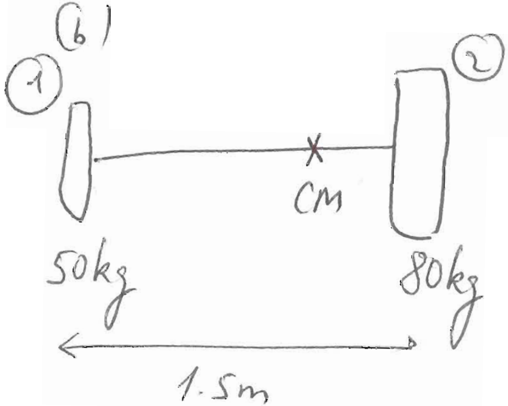


Example:



$$CM: \vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} = \frac{0 + 1.5}{2} = 0.75\,m$$

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M} = \frac{\cancel{m}.0 + \cancel{m}.1.5}{2\cancel{m}} = 0.75\,m$$



$$CM: \vec{R} = \frac{m_1 0 + m_2 1.5}{m_1 + m_2}$$

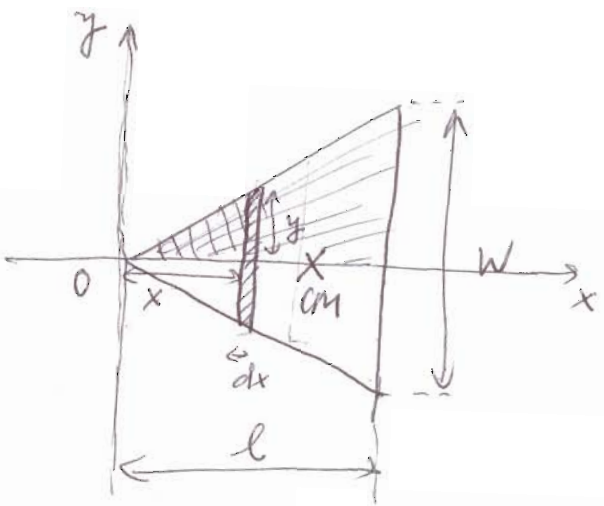
$$= \frac{80 \times 1.5}{130}\,m$$

$$= 0.92\,m$$

Example:

$$\frac{y}{x} = \frac{\frac{w}{2}}{l}$$

$$\rightarrow \boxed{y = \frac{xw}{2l}}$$



CM ?
↓
$(X_{cm}, 0)$

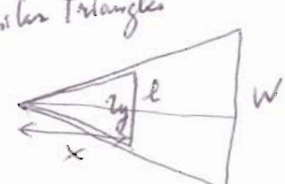
1) Symmetry axis $\longrightarrow$ x - axis (CM along this symmetry axis)

$$\vec{R} = \frac{1}{M}\int \vec{r}\,dm \longrightarrow X_{cm} = \frac{1}{M}\int x\,dm \quad (*)$$

2) $dm = ?$

Since need $\int$ over the whole triangle, along x-direction → need to use vertical strips.

similar triangles



$$\frac{w}{l} = \frac{2y}{x}$$

$$dm = A_{strip} \times \rho \quad \begin{cases} \rho = \dfrac{M}{\frac{lw}{2}} & \left(\text{mass density} = \dfrac{M}{A_{total}}\right) \\ A_{strip} = 2y\,dx = \dfrac{xw}{l}\,dx \end{cases}$$

$$\Rightarrow \boxed{dm = \frac{xw}{l}\,dx\,\frac{2M}{lw} = \frac{2M}{l^2}\,x\,dx}$$

$$(*) \quad \boxed{X_{cm} = \frac{1}{M}\int_0^l \frac{2M}{l^2}\,x^2\,dx = \frac{2}{l^2}\left[\frac{x^3}{3}\right]_0^l = \frac{2}{3}l}$$

Momentum:

$$\vec F_{net} = M\frac{d^2\vec R}{dt^2} = M\frac{d\vec V}{dt} = \frac{d}{dt}\underbrace{(M\vec V)}_{\vec P}$$

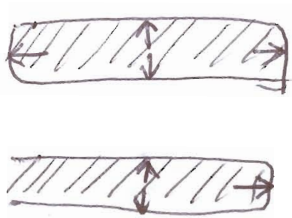
$$\boxed{\vec V \equiv \frac{d\vec R}{dt} = \frac{d}{dt}\left(\frac{\sum_i m_i \vec r_i}{M}\right) = \frac{\sum_i m_i \vec v_i}{M}}$$

Mom. of a system of particles is $\vec P = M\vec V = \sum_i m_i \vec v_i$

Conservation of momentum:

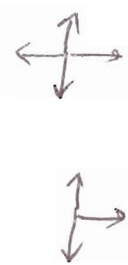
$$\vec F_{\substack{net\\(ext.)}} = 0 = \frac{d\vec P}{dt} \Longrightarrow \boxed{\vec P = \text{constant} = \sum_i m_i \vec v_i}$$

Rockets:

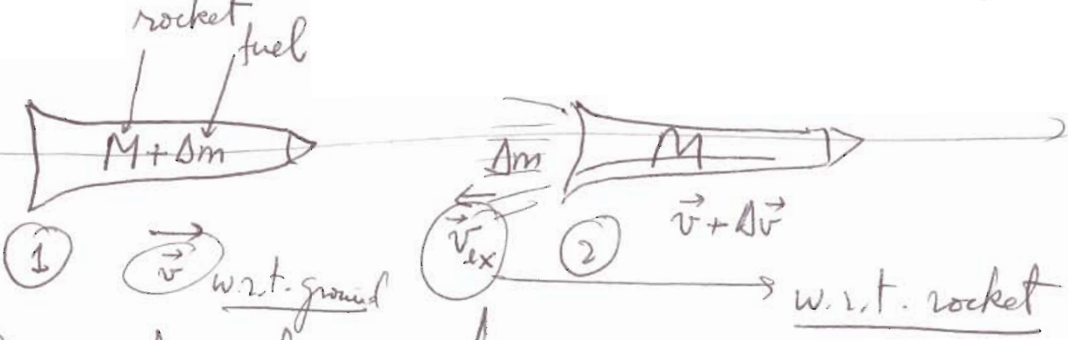


Focus on cylinder:

''



System of 2 components: rocket & fuel



$\vec{F}_{net} = 0 \rightarrow$ Conservation of momentum:

$$\vec{P}_1 = \vec{P}_2$$

$$M\vec{v} + \Delta m\,\vec{v} = M(\vec{v}+\Delta\vec{v}) + \Delta m(\vec{v}_{ex})_{\text{w.r.t. ground}}$$

1D $$(M+\Delta m)\,v = M(v+\Delta v) \underbrace{- \Delta m\, v_{ex} + \Delta m\, v}$$

$$\cancel{\Delta m\, v} = M\,\Delta v - \Delta m\, v_{ex} + \cancel{\Delta m\, v}$$

$$(v_{ex})_{\text{w.r.t. ground}} = (v_{ex})_{\text{w.r.t. rocket}} + v = -v_{ex} + v$$

$$\rightarrow \boxed{M\,\Delta v = \Delta m\, v_{ex}}$$ Consequence of Conservation of momentum of Fuel + Rocket system

$$\Delta m = -\Delta M \rightarrow \left[M\frac{\Delta v}{\Delta t} = -\frac{\Delta M}{\Delta t}v_{ex}\right]_{\lim \Delta t \to 0}$$

$$\downarrow$$

$$\boxed{M\frac{dv}{dt} = -\frac{dM}{dt}v_{ex}}$$

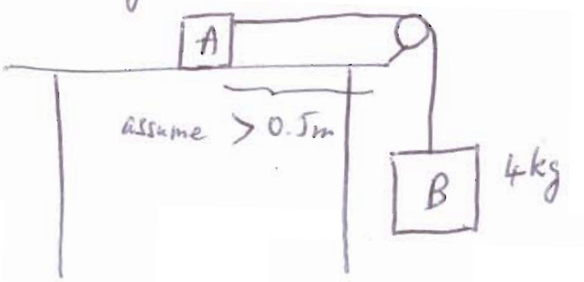
$$\downarrow$$

$$M\,dv = -dM\, v_{ex}$$

$$\int \left[dv = -\frac{dM}{M} v_{ex} \right]$$

$$\rightarrow \boxed{v_f - v_i = -v_{ex}\left[\ln M\right]_{M_i}^{M_f} = +v_{ex}\ln\frac{M_i}{M_f}}$$

8.35



No friction → Conserv. of Mech. Energy

Speed of these masses after they moved 0.5m

$$\underbrace{KE_1 + U_1}_{\text{rest}} = \underbrace{KE_2}_{\text{moved } 0.5m} + U_2$$

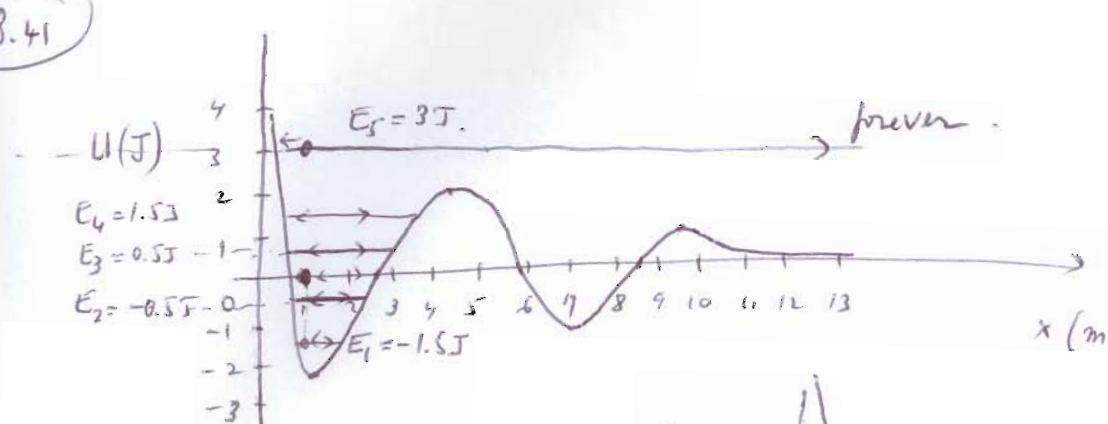
↑ isolate this for final speed.

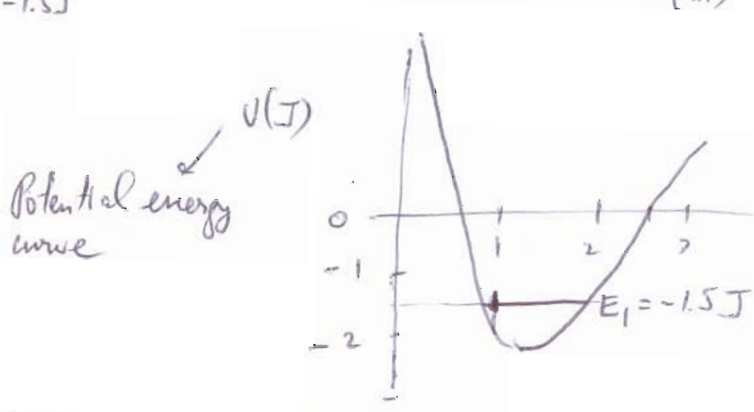
$$\underbrace{U_1 - U_2} + \underbrace{0} = \underbrace{KE_2}$$

$$\underbrace{(U_1 - U_2)_A}_{0} + \underbrace{(U_1-U_2)_B}_{m_B g\, 0.5} \qquad KE_{2A} + KE_{2B} = \frac{1}{2}(m_A + m_B)v_f^2 \quad \rightarrow \quad v_f = \sqrt{\frac{2 m_B\, 9.81 \times 0.5}{(m_A + m_B)}}$$

$$v_f = \sqrt{\frac{2 \times 4 \times 9.81 \times 0.5}{2+4}}\ \frac{m}{s} = 2.56\ m/s.$$

8.41





at $x = 1m$

$E_1 > U_1$

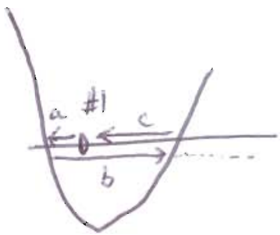
$E_1 = KE_1 + U_1$

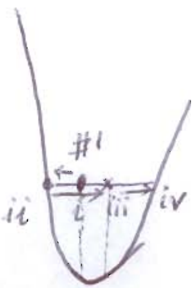
When particle #1 started at $x = 1m$: $E_1 > U_1$ (above the pot. energy curve)

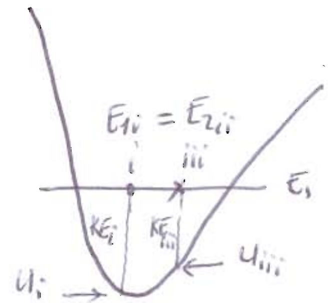
$\rightarrow KE_1 = E_1 - U_1 > 0 \rightarrow$ there is some KE_1 left to move left.

When particle #1 hits pot. energy wall : $E_1 = U_1 \Rightarrow KE_1 = 0$

$\rightarrow$ it will stop (and turns back)







$v_i > v_{ii} = 0$, $v_i > v_{iii}$; $v_i > v_{iv} = 0$

$KE_i = E_1 - U_i > KE_{iii} = E_1 - U_{iii}$