2 MAKING CONNECTIONS

*"There are more things in heaven and earth, Horatio,
Than are dreamt of in your philosophy."*

—William Shakespeare, *Hamlet*, act 1, scene 5

THE NATURE AND FUNCTION OF CONCEPTS

Even a cursory glance at the research and theory in the areas of reading, mathematics, and thinking would reveal the central role of *connections* in each. Connections build conceptual understanding. The more and the stronger the connections are among related ideas, the deeper and richer the understanding of a concept. There are dozens of different psychological theories concerning the connections that humans make. Rather than trying to summarize, compare, and contrast the different theories, I want to briefly address several ideas that are central to the braiding of language, thinking, and mathematics. After all, the purpose of braiding is to make connections. One difficulty in talking about these ideas is the terminology used by various psychologists and educators.

Lest you think I'm exaggerating, psychologists use terms (actually concepts) such as *interiorization*, *condensation*, *reification*, and dozens of others to explain human development and understanding of concepts. Teachers don't need to make all the excruciating distinctions that psychologists do in their books and articles for scholarly journals. What is the difference between *reflective abstraction* and *reflected abstraction*? Instead of answering, let's examine what concepts are and why they are so important.

Concepts are abstract ideas organizing a lot of smaller bits of information (facts) in a somewhat hierarchical fashion. We can see a set of concepts, subsumed under a macroconcept (an even bigger idea). For example, in language arts we encounter concepts such as hyperbole, synecdoche, metaphor, or metonymy that are examples of a bigger concept, *figures of speech*. Each of these fairly abstract ideas explains particular expressions encountered in literature or poetry. In mathematics, the science of patterns, we have branches of mathematics devoted to the study of specific types of patterns such as shape, dimension, change, uncertainty, and quantity. These are certainly big ideas or macroconcepts that can organize a lot of information. Subordinate to quantity we'd find the concept of multiplication, one that subsumes a great many facts.

Consulting the dictionary for the definition (*denotative* meaning) of a concept is a lot like eating non-fat plain yogurt. Concepts are rich and complex, filled with deeper *connotative* meaning. I get nervous when someone talks about students needing to "know" particular terms or vocabulary words. I don't want kids to memorize a definition. I want their lives enriched by deeply experiencing the *context* that surrounds the concept. For instance, the term *revolution* certainly should be defined, but when we examine the "revolutions" in some of the English colonies in North America in 1776, in France in 1789, and in Russia in 1917, these examples in their rich contexts breathe life into the concept. The same issue of rich, meaningful concepts applies to mathematics. How do third graders conceive of multiplication? Something you do to make the amount you've got get bigger? If so, there may be a real problem when the student encounters multiplying by fractions or decimals that are less than one. The product is smaller than the multiplier and mutliplicand (now you don't hear that word every day!). The concept of multiplication and its relationship to division continues to grow more complex each year for about six years as the operation is performed with different kinds of numbers, then with variables, with matrices, with vectors, and so forth. The concept of multiplication can grow richer and more elaborate and more abstract as you experience it in different contexts.

Conceptual understanding is not like an on-off light switch: you don't *understand* a concept in an all-or-nothing fashion. Initially we grasp some aspect of the concept and build upon it, adding and elaborating our understanding. I like to think of it as building a snowman. First, you find some good snow for making a snowman—not too wet and slushy, not too dry and powdery. You make a snowball with your hands and roll it in some good snow. The ball gains size as more snow sticks to it. You do this to make a big sturdy ball of snow for the foundation. You repeat this process for other parts of the snowman. But you must continue to roll it in the right kind of snow; the wrong snow, or worse, rolling it on grass, will not accumulate more snow. In general, the more connections of the right kind, the more examples in different but relevant contexts, the more elaborate the networks of ideas and relationships—the deeper, richer, more generalized, and more abstract is our understanding of a concept.

SCHEMA THEORY, THE FOUNDATION OF READING COMPREHENSION

Since the 1980s most reading researchers have found schema theory to have extraordinary power in explaining how proficient readers understand text, store their knowledge, and remember what they have read and learned. "Teaching children which thinking strategies are used by proficient readers and helping them use those strategies independently is the core of teaching reading" (Keene and Zimmermannn 1997, 53). Harvey and Goudvis (2000) consider the making of appropriate connections of paramount importance and make it the launching point in their approach. They use the apt metaphor, "building bridges from the new to the known" (p. 67). All of their examples of stories are designed to help kids use their personal and collective experience to enhance understanding.

There are several related devices used by the Public Education Business Coalition (PEBC) folks from Denver. The major device is *explicit modeling* of making connections while you read. Starting with young children (beginning readers), they urge teachers to read aloud to the kids, occasionally stopping and telling the kids what they are thinking.

Debbie Miller does extensive "Think Alouds" with her second graders. For her, preplanning is essential. Simply grabbing a book and reading it aloud to children, assuming you will spontaneously come up with wonderful connections, won't work. "Explicit modeling requires thoughtful planning . . . 'winging' it to model our thinking as we read is difficult to pull off" (2002, 54).

Teachers should think carefully about what connections to make. The teacher identifies important concepts and key themes, thinks about how her own experiences relate to the themes, and notes where in the reading of the text to pause and to think aloud about the text, all the while thinking about how to share understanding of her thinking strategy. What are the key concepts in the text that are critical for students to get in order to understand the story? Keene and Zimmermann (1997, 69) report that students often comprehend the words, but lack a schema for the *setting*, which may be critical to understanding the key themes of the book.

Harvey and Goudvis (2000) describe how they begin the strategy instruction in making connections with stories that are similar to the lives and experience of the children. When the students have had experience with a substantial number of stories and narratives, they begin to connect themes, characters, and issues from one book to another. The teacher then tries to broaden their horizons to consider themes and issues of the larger world. When the students move to new and unfamiliar topics and broader issues, some students really struggle. Students with background knowledge have a much easier time. "Our responsibility is to help build students' background knowledge so that they can read independently to gain new information" (p. 75).

Teachers think aloud and model how connections can help activate schemata. Miller and company want the students to relate unfamiliar text to prior knowledge and/or personal experiences. In general, they ask students to think: what does the text remind you of? More specifically:

- Does anything in the text relate to yourself—relating characters to oneself, when something in the story reminds you of your life?
- Does anything in the text relate to other texts—finding common themes in different books by the same author; comparing characters, their personalities, and actions; comparing story events and plot lines; comparing lessons, themes, or messages; across different authors, comparing how different authors handled the same theme; comparing different versions of familiar stories?
- Does anything in the text relate to the world—what is going on in the world, real-world issues or problems—natural disasters, poverty, war, crime, technology?

With beginning readers, teachers read, making a big chart with the connections. After modeling the "think aloud," the students practice thinking aloud, and teachers record their connections on the big chart.

Older students developing proficiency at reading, read the text for themselves and put coded sticky notes onto the pages of books: T-S for Text to Self, T-W for Text to World, and T-T for Text to Text. The students might also code R for Reminds me of. They also may jot down some brief connections on the sticky notes.

These devices help activate schemata in the midst of the reading. But how relevant are they to comprehension? Once kids start seeing connections, they may see them anywhere, regardless of how meaningful they are to the understanding of the text. (*Amelia Bedelia Goes Off on a Tangent!*) Harvey and Goudvis (2000) remind us, "We need to read student work carefully and listen well to conversations to see that kids are making meaningful connections" (p. 77). "We watch for authentic connections that support understanding. Kids are terrific teacher-pleasers and may think that any connection is better than no connection at all" (p. 78). "Although children may initially have trouble articulating more significant connections, with teacher and peer modeling and plenty of time, they gradually begin to refine and limit their connections to those that deepen their understanding" (p. 80).

Miller (2002, 67) describes a postreading activity in which she and the second graders went back to earlier connections they had made, marking the ones that had helped them understand and why. This activity is one of Miller's "Anchor Experiences" (highly effective minilessons used as anchors for students to remember specific strategies). "When I begin to teach children how to think out loud, I have the same expectations for them as I do for myself. I want their think-alouds to be genuine, their language precise, their responses thoughtful. My goal is to give them a framework for thinking, as well as to help them build a common language for talking about books" (p. 55).

Katie George, a middle school math teacher in Lincolnshire, Illinois, identifies some of her more powerful activities as "math anchor lessons." She frequently refers the students back to the anchor lesson as a touchstone for clarity and understanding. Miller's ideas are perfect for mathematics classes: anchor lessons, genuine, precise, thoughtful, and a framework for thinking!

HUMANS ARE PATTERN-SEEKING CREATURES

Humans are pattern-seeking, meaning-making creatures. We have experiences. We encounter people, events, phenomena, circumstances, thoughts, ideas, symbols, music, art, emotions. And what do we do with these things? We classify, organize, sort, group, pull apart, look at little pieces, grab a whole handful of pieces and put them back together. We even look for the pattern in tea leaves, ashes, and chicken bones. We see faces, animals, and many strange shapes, in clouds. For example, we drive down the highway, see a "vanity" license plate, and try to decipher its meaning, [IMAQT2] or [ZUP2U]. More to the point, we see patterns in license plates where none was intended (although I knew [XAG756] was randomly generated, I could not help thinking Xylophones Are Great). Humans of all ages are remarkably equipped to make connections.

Perceiving patterns is essentially an *inductive* process: the child examines a bunch of particular examples and derives a pattern. These perceptions can't be forced. Consider the following sequence of numbers that would challenge any adult to discern the pattern since it is expressed abstractly with no context. It begins 1, 6, 11. What would come next in the sequence? When I ask this question of fifth or sixth graders, I get answers such as: 66 or 16. Some people just say that we need more data, we don't have enough examples. I do not at this time ask them to describe the rule for generating the pattern. The sequence is 1, 6, 11, 4 . . . Now what comes next? Some again say they need more examples. I have a hunch that they can think of possibilities but they do not want to be told they are wrong. Others say, 9 or 24 or negative 3. Here are the next four numbers in the sequence: 1, 6, 11, 4, 9, 2, 7, 12 . . . What comes after the 12?

We are in a "pure induction" process: a bunch of examples with very little feedback (only yes or no, are your guesses correct) and no real-life context to give meaning to the numbers. Have you ever heard the saying, "Deduction is going from the general to the particular and induction is going from the particular to the general"? I must have heard that saying in every math class from sixth grade up through twelfth, but I didn't really understand it until I was 30 (and had my first midlife crisis). No, I finally figured it out in high school.

Pure induction can be amazingly challenging and motivating, if the example or the context is conceivable for the student. However, it can be very frustrating to others. Here is the sequence again: 1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8, 1, 6, 11 . . . and it continues to repeat. What is the pattern? What is the rule that governs this sequence? Note that you have no hints, no partial explanation, no scaffolding. Yet some people love to intellectually struggle with this pure induction. If you are such a person, stop reading now and try to figure out the rule. However, most humans like to have some scaffolding or hints, or the explanation. Okay.

After 8 it went back to 1 and then continues in an endless *cycle*. What is the smallest number in the sequence? What is the largest number? Are all the numbers between 1 and 12 present in the sequence? But they are not in numerical order. What does create the order? At this point or sometimes earlier someone may say "The pattern is add 5, add 5, subtract 7, add 5, subtract 7, add 5, add 5, subtract 7, add 5, subtract 7, add 5, subtract 7." I ask the one who offered this rule, "How do you know when to add 5 and when to subtract 7?" I explain to the class that what the student recited is an excellent *procedure* for generating the sequence accurately. But it does not explain why. Concepts do that.

What object in your immediate surroundings has all twelve numbers and continues in an endless cycle? A clock. What is the pattern or the rule? Something happens every five hours starting at one o'clock. I didn't trick you: I merely gave you the sequence in its most abstract form, divorced from the real-life example that made it so clearly understandable.

Contrast the way you handled the inductive sequence with what most of us experienced in math class most of the time. Can you remember your math teachers who gave brilliant lectures, explaining the procedures, the principles, and the concepts? They'd explain the rules, the formulas, the theorems and then expect us to apply them, using deductive reasoning. The problem with pure deductive teaching is that most of the time an explanation of the principles does not *connect* to anything in kids' heads because most of the time, most humans (especially young children) need examples. In most cases of mathematics in the elementary and middle school, *simply telling* does not work.

As a teacher, you provide your students with a mixture of *examples* and *explanations*. If you taught purely by an inductive process you'd give the kids lots of examples for them to figure out the rules, principles, concepts by discovering them. But pure inductive experiences, with no feedback, can be frustrating, and many kids just never "discover " what they are supposed to learn. Typically, human beings need both examples and explanations. We all need examples to build the meaning of the concept, principle, theorem, or rule. Examples can clarify what the explanation meant. We construct our personal understanding of the concept through an interaction of inductive examples and deductive explanation.

The key question is, not whether or not, but rather *when* are each of these kinds of teaching and student thinking done?

Most traditional mathematics textbooks start the lesson with an explanation, or a definition of some kind. Then the teacher explains the explanation, developing the main ideas. Finally the students do guided seatwork where they work on the exercises or problems related to those ideas that the teacher and book have shown. This sequence puts the explanation in the wrong place, making it difficult for students to make connections. The students have no experiential referent for it; no schemata are activated; there is nothing to connect it to. During guided practice the teacher bounces back and forth like a ball in a pin-ball machine with the kids hitting the flippers. "Ms. Jones, I don't get it! Show me what to do again." A good rule of thumb for effective math teaching is to make sure that *every* symbol has a concrete reference (an anchor) in their experience that you and they can refer back to when dealing with abstract symbols.

There is a better way than the highly deductive aproach. Here are five phases:

Situation. The teacher presents the problem to the students. Key concepts are embedded in a real-life situation which exists within a **context** that is familiar or imaginable to the students. The KWC is used with some enhancements (described shortly) to help imagine and understand the situation.

Representations. The students create representations of the problem by using one or more of the representational problem-solving strategies.

Patterns. The teacher asks the students to look for patterns in the representations. Some students are able to discern patterns, solve the problem, make all the connections the teacher hoped they would. But some do not.

Connections. The teacher leads the students in a debriefing discussion to ascertain who has understood the problem, who has a good way of looking at the problem, and who has made the right connections. It is *now* that a cogent explanation of the major concepts can be effectively done! Why now? The teacher can tailor-make her explanations to use the conceptions that the *students* have generated. She can *connect* the mathematical concepts directly to what the students have just done, just expressed, and just realized. She explicitly builds bridges between ideas, the new and the known. She explicitly makes connections between and among concepts. She helps them *crystallize* their understandings.

Extensions. As the teacher has just witnessed the students wrestling with these ideas, she now has a feel for what needs to be *differentiated* for whom. For most of the students the appropriate extensions would be doing more of the same challenging problems. For some the problem may have been too challenging and they need extensions that circle back to build some foundation. For others the problem may not have been much of a challenge and they will need to work on more advanced extensions.

These five phases require students to do some hard thinking. In the first two phases, Situation (in context) and Representations, the students are trying to understand the problem. They will activate relevant schemata through asking questions. The teacher should provide her students with a good balance of examples and explanations at the right time. Next we'll examine the power one gets when using examples from real situations that live in a context.

In Chapter 1 we mentioned the four-phase model of problem solving that Polya introduced in the 1950s and is still fairly popular today. Some current descriptive labels are given in parentheses. They are:

- understanding the problem (reading the story)
- planning how to solve the problem
- carrying out the plan (solving the problem)
- looking back (checking)

How are the five phases different from Polya's four phases? Why use five? How do the five relate to the four? Good questions I am frequently asked. Polya's four phases describe a very general approach that the students should follow. Although the five do include things that the students do, their orientation is what the teacher does to facilitate the students' problem solving. Both are necessary and they fit together nicely, as you can see in the summary outline. I have also incorporated the key moves from the reading comprehension strategy, Asking Questions, from Chapter 1. This is the beginning of the Braid Model of Problem Solving. I will continue to add pieces to it throughout the book.

The Braid Model of Problem Solving for the Students

Understanding the problem/Reading the story
Imagine the SITUATION
Asking Questions (and Discussing the problem in small groups)
K: What do I know for sure?
W: What do I want to figure out, find out, or do?
C: Are there any special conditions, rules, or tricks I have to watch out for?

Planning how to solve the problem
What REPRESENTATIONS can I use to help me solve the problem?

Carrying out the plan/Solving the problem
Work on the problem using a strategy.
Do I see any PATTERNS?

Looking back/Checking
Does my answer make sense for the problem?
Is there a pattern that makes the answer reasonable?
What CONNECTIONS link this problem and answer to the big ideas of mathematics?

BUILDING CONNECTIONS IN MATHEMATICS

Two basic types of connections serve somewhat different functions for students in organizing knowledge. These are *context connections* and *concept connections*. Bear in mind that our goal in teaching mathematics is understanding. Making connections, organizing knowledge, and understanding concepts are three things that braid nicely.

As we start by asking questions with the KWC, we enhance the K (What do I Know for sure?) with questions that stimulate students' thinking about the situation or context of the math problem. These are context connections: Math to Self and Math to World. Other questions call for concept connections (Math to Math), which tend to be the most difficult of the three. Of course, this process should engage their prior knowledge and activate relevant schemata. It is fairly easy when the students have used the coding T-S, T-W, T-T in their reading of text. The teacher models it herself. Typical questions are:

Math to Self (connecting to prior knowledge and experience; connecting to preconceptions and misconceptions)
What does this situation remind me of?
Have I ever been in any situation like this?
Math to World (connecting to natural or created structures, events, environment, media)
Is this related to anything I've seen in social studies or science, the arts?
Or related to things I've seen anywhere?
Math to Math (connecting the math concepts: to other math concepts [e.g., big ideas], within and across strands of mathematics; to related procedures; within and across contexts and representations)
What is the main idea from mathematics that is happening here?
Where have I seen that idea before?
What are some other math ideas that are related to this one?
Can I use them to help me with this problem?

For younger students a somewhat different list of questions is done orally and the teacher records their responses on chart paper. These are all essentially Math to Self, but responses could be coded either M-S, M-W, or M-M.

What do you think about that/it?
Tell me about this situation.
What do you know about it already?
What do you think will happen?
Is there anything weird or strange about this?
Does anything surprise you?

The students who are in third grade and up are for the most part able to read and write down connections on a graphic organizer of some kind. Most of the teachers I work with simply incorporate these questions in with their KWC, rather than having a separate graphic organizer.

Local Concept Development

Students are motivated to think when the *context* of a problem appeals to them. Initially they are much more interested in the particular examples, the situation, and the context than they are in the mathematics. Working in a meaningful context can help students build an initial understanding of a concept. When the student considers a bunch of examples from a particular context, an inductive process is at work to create meaning, to derive a pattern and create a particular, and perhaps context-specific, version of a concept that describes or explains the pattern.

A number of educators use the term "internal model" to explain what is going on here. We humans interpret our experiences by comparing them to internal models that are based on our past experiences. These internal models filter, construct, and create how we conceive of the new experiences. Students' knowledge is generally organized around their experiences, not around the abstract concepts of the discipline of mathematics. Similar experiences are grouped together in their internal models. Does this sound like schemata?

Students build up concepts gradually. First they come to understand the concept in a very specific context or situation (i.e., "local"). Their initial understanding is very much grounded in a set of examples in that context. It is not global, not generalized to other contexts. They create a kind of model that explains a particular problem-solving situation. With more experience in somewhat similar situations and with facilitation by the teacher, more elaborate understandings can be built up by experience and inductively derived. Heavy doses of deductive explanations trying to get them to generalize across contexts and to think abstractly about the concept will *not* likely have much effect until they have had experience with those other contexts. We cannot do Mr. Spock's Vulcan mind meld and make a student conceive of a concept the way we do. It does not work that way.

The good news is we know how to build the snowman. We can use the innate, pattern-creating, meaning-making, inductive reasoning that students bring to school. We can provide many examples of the target concept in a particular situation so they develop a solid initial local version of the concept. Then we can deliberately provide experiences of the same concept in a different context. We can help them build a strong local conceptualization of the second situation or context. We can help them discern similarities across the two contexts. We can help them build bridges between the two. The process of generalizing can be facilitated, but not deductively forced.

Working in One Context

Here is an example of helping students in one context to build a very solid *local* understanding. In Beverly Kiss' third-grade classroom, she was helping the students learn the KWC and they had been exploring multiplication as equal groups. She had all the students together as a whole class. She had given them the following problem to read.

> Imagine that you work on a ranch that has 24 horses. The owner of the ranch tells you that you must put all of the horses in corrals. You can fence off the corrals many different ways. The owner says you must put the same number of horses in each corral. What is one way you might do this? How many different ways to do this can you find?

She asked her students what they knew about ranches. Had they ever been to one? Several had. Several said they had been to farms where there were horses. She asked them what was the difference between a ranch and a farm? She did a brief compare/contrast chart on the board. She asked, "Are there any words that you are not sure what they mean?" Three students said they did not know or were not sure what a corral was. Instead of asking the others in the class, she went into the K of the KWC and said to them, "Ask yourself, 'What do I know for sure?'" They responded fairly directly with the information in each sentence.

"There is this ranch with 24 horses." "I work at the ranch."

The teacher asked, "What do you do?"

"I move the cattle around." "We are supposed to put the horses in corrals." The teacher said, "It does say that, but remember that we have a second question. What is it?" The students said, *"What do I want to find out or figure out?"* The teacher said, "Yes and sometimes that might mean, 'What do I want to DO?'" *"Put the horse in corrals."* One of the kids shouted out, *"There are lots of ways to make corrals."* The teacher turned to one of the three who wasn't sure about corrals and asked, "What do you think corrals are?" He wasn't sure. She said, "Put your finger over the word *corral* and read one of the sentences aloud that uses that word. When you get to that word just say 'blank.' Then do the same with the other sentence." He did. And then she asked, "What are corrals made of?" The student said tentatively, *"Fences?"* She asked, "Then what are corrals?" He replied, *"A bunch of fences to keep horses in one place."* Then she asked the class, "What is the second question?" They chimed in, *"What are we trying to figure out or find out?"* "How would you answer that question?" She called on a student who said, *"We have to figure out how to put all the horses in corrals so that every corral has the same number of horses in it."*

Beverly asked if anybody had other things we had to do or find out. No one said anything. Then she asked them about the third question. *"Is there a special condition that I need to look out for?"* She asked them to quietly on their own read the problem again. They did. After about a minute, one of the students said, *"I think we could get lots of different answers."* "Why do you say that?" *"Because the problems says, 'How many different ways to do this can you find?'"*

This dialogue took about five minutes.

Next the teacher gave each pair of children a collection of 24 Unifix cubes. She said that each Unifix cube was going to represent a horse. She asked them to please put them into the groups with the same number in each group. Let's see if everybody can find one way. She circled around the classroom and checked what each pair had done. They all had found a way to do this.

She gave each group a piece of poster paper about 12 inches by 14 inches that had been folded into four rectangular sections on each side. Her instructions were that every time someone found a solution, to draw the 24 "horse" cubes in one of the rectangles of the paper. Make it and draw it. She gave them about ten minutes. Some kids asked if they could turn the paper over. She said, "Yes." Others asked, "Are there eight ways to do this because there are eight sections of the paper?" She said, "We'll find out!" She also told them to draw the corrals before removing the cubes.

They put their names on their papers and labeled their pictures, very carefully, in the following manner. There are 3 corrals with 8 horses in each corral. She displayed an overhead transparency of 24 squares, 8 inside each of three roughly drawn circles. See Figure 2.1.

Each of the student pairs found several ways. As the teacher circulated about the room, kids asked her questions like, *"Can we put them all in one big corral? Do we have all of the ways?"* She just said, "We'll see."

After about ten minutes she asked them to stop and to put away the cubes. "Let's see how many different ones we found as a class." She pulled out a sheet of newsprint that had a big T-chart on it. She asked one pair of students to tell us one way they found. One of the kids said, *"Six horses in each."* She asked him, "How many corrals would you need?" He said,

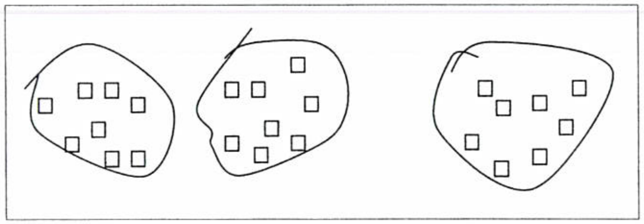


FIGURE 2.1

"Four." She then said, "Can we say it like we did with equal groups last week? How would we say it?" Someone volunteered, *"We have 24 horses in 4 corrals with 6 horses in each corral."* She acknowledged him and nodded to another student who said, *"Four corrals with 6 horses in each could hold 24 horses."* Beverly then wrote on the column headings of the T-chart: **[number of corrals] [number of horses in each]**. She explained that as each pair of students described their possible arrangements of horses, she would record it in the T-chart, but they had to tell her what to write by saying it in the order that was left to right. She said, "Say it the way we did in groups. I have 4 groups with 6 in each group, or 4 groups of 6." As pairs responded, she entered their solutions. After the first four, pairs started saying that we had all the ones they had found. Others in the room said there were more. She paused when the chart looked like the table in Figure 2.2. Then she asked one of the students to ask her question again.

The girl asked, *"Can we have all the horses in one big corral?"* "What do you think, class?" A debate began. Some students were adamant: *"The problem said 'put them in corralSSSSS,'"* with great emphasis on the S. (No doubt these children were strict constructionists of the Constitution as well.) Others maintained that you could have one group, so why not one corral? Finally, one of the students asked me (they knew me as Beverly's professor). I suggested that the solutions they had so far were "recorded" but not organized, not in any order, and asked, "Which solution would you put first?" They were about evenly split between 1 corral of 24 and 2 corrals of 12. I suggested they enter both in the chart, but use a different color for 1 corral of 24.

We went down the first T-chart looking for what would come next. After 4 corrals with 6 horses, they paused and I quickly said, "Oh, oh. We missed one. What about 5 corrals?" Some were still thinking while others blurted out, *"You can't do it! You can't have 5 corrals!"* I said, "Sure I can." I drew 5 circles on the chalkboard and put 5 squares into 4 of the circles and 4 squares into the last circle. "There you are. Five corrals, 24 horses." You could hear them on the other side of Lake Michigan: *"The groups aren't equal."* I said, "Oh, you mean we have a SPECIAL CONDITION?" At least a couple of kids giggled. Most just had expressions that seemed to say, *"Well, duh. Isn't that what you've been teaching us?"*

number of corrals	number of horses in each corral
6	4
4	6
3	8
2	12
8	3
12	2

FIGURE 2.2

number of corrals	number of horses in each corral
1	24
2	12
3	8
4	6
6	4
8	3
12	2

FIGURE 2.3

They used reasoning similar to 5 corrals when we got to 7, 9, 10, and 11 corrals. They stopped at 12 corrals with 2 in each. See Figure 2. 3. I asked, "What would come next?" Some thought we were finished. I asked, "Do you see any pattern in this table or T-chart?" One said, *"One side goes up, the other goes down."* I asked, "Why?" Several kids started to answer but stopped. Finally one said, *"If you've got more corrals, you don't need to put as many horses in each one."*

We went back to the T-chart. And I asked, "Can we have 13 corrals?" They said, "No." I went through 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23. They were chuckling. "What about 24 corrals?" Someone said that would be too much work. "But could you?" The same kids who objected to 1 corral, now objected to 24 because, *"The problem says to put the same number of horseSSSSS in each corral."* Beverly and I let the kids argue briefly and then one student interjected, *"Let's just put it in the table in the same color as 1 and 24."* The class liked this idea. I asked them again about the patterns and this time they immediately saw that the same numbers on the left side were repeated on the right, but as one kid said, *"Going the other way."* See Figure 2.4.

I must tell you of an incident a few years earlier in a different classroom when doing an analogous problem. I asked if they saw any patterns. They mentioned the ones cited here. Then one child said, *"The top half is like the bottom half except like upside down and looking in a mirror."* Well, I almost fell off my chair and had him repeat it and show us what he meant (I knew what he meant). See Figure 2.5. He drew a horizontal line

number of corrals	number of horses in each corral
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

FIGURE 2.4

One factor	The other factor
1	72
2	36
3	24
4	18
6	12
8	9
9	8
12	6
18	4
24	3
36	2
72	1

FIGURE 2.5

separating 8 9 from 9 8, and explained that when you get to the middle it *"flips"* and there are four sections that are like mirrors to one another. Thereafter that class referred to *"hitting the flip—that's when you know you've got them all."* For sixth graders, this idea comes in handy because the top half are all the factors of the number and little problems like this reveal another way (besides tree diagrams) to generate all the proper factors. For example, to find all the factors of 72, I would not give the kids 72 manipulatives. That is too many to handle without major trouble. But once they have built up their number sense through these kinds of activities, they can generate the organized table in Figure 2.5.

Do you see the middle of Table 2.5 where it flips from 8 9 to 9 8? We can draw a horizontal line and all 12 numbers above the line in the table are the factors of 72.

Notice in this example of a single problem how easily the students moved across representations. They read the story, and then talked about it (language). They used objects to represent the horses. They drew pictures to record the different solutions that they found with the objects, they labeled the pictures with written language, and then used symbolic notation when generating a T-table. These representations were roughly in sequence from more concrete to more abstract. Yet they were not big intellectual leaps for the kids, they were manageable. *Every representation was explicitly connected to the others.* They were all done in the same context (corrals and horses). Most of the students could easily have done extension problems in this same context, but with different numbers. How about 32, or 40, or 60 horses, but don't use cubes? Can you do it with just a table? Would it help to draw 60 little circles and partition them into corrals? Same context, different examples.

A simple yet powerful way to *differentiate* your instruction is to have a variety of extensions ready to go. You can easily provide the kids with the right kind of next experience. Plan for many of the same kind of problem for practice, some much more challenging, and some quite a bit easier. You can vary the context and the kinds of representations to provide other opportunities for kids with different abilities, cognitive styles, and preferred learning modalities to enter the world of mathematics in a way that makes the most sense to them.

Handling Multiple Contexts

Let's return to Beverly's classroom a few days after she did the 24 horses in corrals. We did a kind of round-robin in the classroom. Beverly had 27 kids in her class. They usually did problem solving (PS) in groups of 3 and she had used this fact to help them see that the 9 groups of 3 were 27 (repeated addition 3 + 3 + . . . nine times). She had spent some time helping them learn how to work together in small groups, for instance, developing social skills of how to listen to one another, how to disagree in a way that did not offend the other person, how to respect one another, how to share materials. She also emphasized task roles for small-group problem solving: supplier, recorder, reporter are three that can fit most problems. As they practiced these roles by using them in problem solving from the first day of school, they became increasingly more successful in cooperative learning tasks.

I have noticed fewer teachers in recent years emphasizing these kinds of skills and roles and actively helping kids learn how to work in groups. Then when the teacher asks them to work in pairs or triads, some of the students fail to share, help one another, and work cooperatively. It is predictable. I also believe that a good structure like the KWC gives them some guidelines on exactly what to do or talk about when in PS groups.

We put them into the 9 regular PS groups. We had 3 PS groups go to math tables we had set up in the northwest corner of the room (NW station). We had another 3 PS groups go to the northeast corner of the room (NE station), and the final 3 PS groups went to the math tables in the southeast corner (SE station). Each station had 3 sets of materials so that the 3 PS groups could work on the problem independently of the other 2 PS groups at a particular station. Each corner station actually took up about one-fourth of the classroom so that the PS groups would have plenty of room to work. At each station there were recording sheets specifically formatted for the task.

Before we started, we designated one member of each PS group as the recorder: one at the NW station, one at the NE station, and one at the SE station. We similarly rotated responsibility for roles of manipulator and clean-up batter. The recorder handled all data entry to the recording sheet. The manipulator was in charge of the materials for the group at one station. The clean-up batter was in charge of making sure that when they left that station, everything was set up for the next PS group exactly as the current group had found it when they arrived at the station. We prepared a chart that showed who was doing what at each station. Here are the roles (R, M, and C) for PS Group I. The other 8 PS groups were included in the chart. See Figure 2.6.

	SE STATION ROLES			NW STATION ROLES			NE STATION ROLES		
PS GROUP I	R	M	C	R	M	C	R	M	C
	AL	JO	KIT	KIT	AL	JO	JO	KIT	AL

FIGURE 2.6

The SE station had three zip bags each with 36 pennies, one for each PS group. The NW station had three sets of 40 Unifix cubes; each set was a single color. The NE station had three square pegboards that were about 26 inches on a side and zip bags of 48 golf tees. (In the past, I have also used Lite Brites: small, cylindrical plastic pieces). The pegboard leaned against a chalkboard, sitting up in the chalkboard tray.

We presented the students with very simple, situational story problems relating to the tasks.

Task 1

You have been given 3 dozen freshly baked doughnuts. What are all the different ways you could share them evenly with your friends? You have a bag of pennies to help you figure out this problem. Arrange the pennies into groups, with the same number in each group.

Task 2

Your PS group is working with a real estate developer (Arnold Grump) who wants to put up a lot of medium-priced condominiums. He has enough money to build 40 units as condo-towers. Find all the ways to build condo-towers using all 40 Unifix cubes, so that every tower has the same height.

In the preceding week, the students had worked on the concept of *array* and Beverly and I had brought in collections of things that came in arrays (e.g., egg cartons, racks of soft drinks in cans).

Task 3

You are part of a design team for a department store, assigned to find all the ways to arrange the expensive items into a rectangular array of 48 items in a display window. The display items will be represented by golf tees. You will put 48 golf tees into the pegboard holes to make rectangular arrays and then write down the possible solutions.

We gave each PS group about 7 to 8 minutes at each station. All 9 PS Groups moved to the next station clockwise at the same time. So all 9 experienced the 3 stations. All 27 students had the experience of being a recorder, a manipulator, and clean-up batter.

What might have appeared to an outside observer as a three-ring circus actually went very smoothly. Of course, the kids made a lot of noise, but they were amazingly on task. The three groups that had the pegboard arrays as their first station had more trouble getting started. This was a new manipulative for them and I think they wanted to get the feel for it and just play a little. This was a classic mistake on my part. I know full well how kids love to get a feel for a new manipulative by messing around with it. Consequently I always give them five minutes to explore the manipulative before they have to use it.

Probably due to a lack of familiarity with the manipulatives, the three groups did not find very many rectangular arrays at the SE station. I think another problem was that it just took longer to make the displays than it did to create the groups of pennies or the stacks of Unfix cubes; just the physical act of inserting golf tees into the proper holes in the pegboard took some time. They obviously enjoyed it; squeals of glee were frequent.

As they went through the stations, they did appear to get a bit more adept at doing the tasks. However, I rarely heard any talk that would suggest that they saw the inherent mathematical similarity in these tasks, probably because they were so intent on the materials and the specific task that was in front of them. They were not yet generalizing the math across contexts.

After about twenty minutes, all nine groups had completed three stations. We debriefed one station at a time with the nine recorders reading their solutions from their recording sheets. We started with the three recorders whose groups encountered the pennies that represented doughnuts first. I wrote their solutions on a large newsprint T-chart. They took turns giving me one solution from their group. When these three had finished, we asked the three recorders from the other six groups if they had found any solutions that had not been mentioned by the three groups.

We thoroughly discussed the doughnuts/pennies, repeated these steps with the condo/Unifix cubes, and then with the pegboard arrays. We debriefed the doughnuts/pennies first because this was the most familiar situation of the three and we assumed that it would likely be a good foundation for making connections to the other two. The newsprint allowed us to collect all the data from the nine groups and then to organize it. Two of the nine groups had found all the solutions in the brief time we gave them. In fact, all the groups did very well, getting nearly all. Several gleefully asserted that they had found them all. So I asked them the critical questions about combinations:

How many different ways (solutions, combinations, etc.)?
Did you check for duplicates/repeats? How?
Did you find all the ways?
How do you know that (when) you have found them all?

The kids answered that they had tried all the ways. I asked, "How do you know you tried all? Maybe you missed some." We are leading up to the very important mathematical idea of generating an organized list or table, but as with most ideas in mathematics, kids need to experience the power and meaningfulness of an idea (conceptual understanding) and not simply memorize how to do it (procedural understanding). Even the two groups that had found all nine solutions had not generated them in order. That was fine. We create a second T-table using all the solutions the class had found (see Figure 2.7). We went in order and we considered if some numbers of persons not listed were possible (e.g., 5, 7, 10, etc.). The kids were certain these were not possible and that this was all.

Beverly asked the students what patterns they saw in this table. As before, they noticed the numbers in the two columns were the same but

number of persons	number of doughnuts for each person
1	36
2	18
3	12
4	9
6	6
9	4
12	3
18	2
36	1

FIGURE 2.7

going in different directions. We introduced the words *ascending* and *descending*. A couple of students noticed that the middle answer had the same two numbers (6 and 6). Then others noticed that the answers above 6 and 6 were the same as the ones below it. Another student mentioned that one of these was a *"turn-around fact"* that they'd been talking about in class. Which one, I asked. She said, *"4 times 9 is 36 and 9 times 4 is 36."*

I cannot emphasize too strongly how important the debriefing of any problem is and the critical role played by language in debriefing. Teachers are pressed for time to cover massive amounts of content, but the better the debriefing, the more complete the crystallization of concepts will be, and less reteaching will be needed. There will always be some who don't get it, but that number is cut down dramatically when the five phases are done well by the teacher and the students: situation in context, representations, patterns, connections, and extensions.

Oral language is critical throughout the problem-solving process, and especially so in the debriefing process. When they are doing a KWC, when they are discussing and describing to one another what the picture they drew means to them, when they are telling others about the pattern they see, they are communicating their mental models through the powerful medium of language. Language representations are used to describe and communicate insights into all other representations. It is the first and the last representation we use. We start with KWC and end with students writing or talking about their writing.

As we were winding up the debriefing of the doughnuts, one kid piped up, *"I see something we didn't talk about!"* We asked him to explain. *"If there are 3 people, they each get 12 (that's a dozen); then if you have 6 people (that's twice as many people), they'd only get 6 (that's half a dozen)."* Some of the kids asked him to say that again. He did and then added, *"And if you've got 12 people (and that's twice as many as 6 people), they'd only get 3 doughnuts each (and that's half of the half dozen)."* (This kid is headed for MIT or Cal Tech, I figured.)

I asked him if he could say this in a pattern or rule. He wasn't sure, so I asked the class. No one volunteered, so I gave it a try, "How about this: if you have twice as many people as before, then each person gets half of what people got before. Or how about: twice as many, half as much, for short?" They thought that was pretty cool. Later in the year kids spotted other patterns in these multiplicative tables: three times as many, a third as much; four times as many, one-fourth as much. But they were just beginning to understand the meaning of multiplication and division and the twice/half pattern was a great one to start with. It made sense to most of them.

These are subtle relationships that need nurturing. Students need to truly grasp them in their own mental images and models. I would venture that very few of these third graders truly and deeply understood the pattern that this kid perceived, even after he and I told them about it. When a child shares what she or he sees/conceives, it is the sharer who benefits more than the recipient. I am sure that we do get initial foundational ideas from one another; when a person has to explain his or her reasoning, defend a thesis, justify a conjecture, it is he or she who crystallizes understanding.

Next came the 40 condos. The students had used Unifix cubes to model the condo buildings and the number of stories or floors in each. They considered both high-rise and low-rise buildings. Again, we collected the data from each group and then organized them into one table (see Figure 2.8). We discussed patterns and if they'd found all the solutions. They picked up on the previous insight and several said, *"Twice as many buildings, half as high."* They used 1 building 40 stories high compared to 2 buildings, 20 stories high, and 4 buildings 10 high, and 8 buildings 5 high. They also spotted a couple of turn-around facts.

Finally we got to the rectangular arrays with golf tees in the pegboard holes. Using arrays helps to build a good sense of rows (horizontal) and columns (vertical), size in two dimensions, and rectangles, which is where the teacher was headed next. Arrays are a marvelous bridge from *groups* to *area*. I am not in any way denigrating arrays, but time is precious and a teacher can get many more conceptual connections working with rectangles and area than with arrays. That is what we were doing here, building some bridges. Based on the previous week's work with arrays, the students were able to create arrays of golf tees in the pegboards. In the debriefing we talked about rows and columns.

The pegboards were about 26 inches square and would not accommodate a couple of conceivable arrays (e.g., 1 by 48, and 48 by 1). None of the nine PS groups found all the arrays. But in the debriefing when we listed what ones they had found, several students successfully discovered others that should be there. In this manner they generated the table in Figure 2.9.

number of buildings	number of stories for each building
1	40
2	20
4	10
5	8
8	5
10	4
20	2
40	1

FIGURE 2.8

number of rows	number of columns
2	24
3	16
4	12
6	8
8	6
12	4
16	3
24	2

FIGURE 2.9

The debriefing of the arrays was very similar to the other two in terms of recorders reporting, and so on. One thing that was a bit different was that the pegboard allowed us to easily rotate the array. For instance, when they had found the 6 rows, 8 columns solution, three of the nine groups who were at that station simultaneously then rotated it 90 degrees so it could become an array with 8 rows and 6 columns. The students also caught on that "*twice as many rows means half as many in that row*" (which is the same as the number of columns). We demonstrated it with pegboards.

The final part of the debriefing was about the connections among these three problems and the recent problems of horses in corrals. We asked students to compare and contrast the different problems. We put the sets of newsprint up: all the T-tables side by side. The students discussed the patterns in the tables. It was then that they began to move away from the particular materials, manipulatives, and context and started focusing on the more mathematical patterns, irrespective of color, size, shape, position, order. This process of ignoring some features, characteristics, or properties while attending to others is critical to mathematics. Earlier, when the MIT/Cal Tech-bound eight-year-old kid saw the pattern *twice as many people get half as many doughnuts*, he was creating an abstraction of this kind. He saw it long before others. In this cross-context debriefing we brought this pattern up for discussion. Then others began to see or perhaps to catch a true glimpse of why that worked.

Concepts in mathematics are about relationships; they are not really about concrete objects or contexts. British psychologist and mathematician Richard Skemp refers to *relational* understanding. "Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas and on the creation of new connections. . . . Understanding is never an all-or-nothing proposition" (Van de Walle 2006). Relational understanding thus has a rich web of interconnected ideas and relationships. I have been using the term *conceptual understanding* in this same way.

All concepts in mathematics, it seems, are about relationships in some way. The more abstract the ideas, the harder to grasp the relationship. For instance, in the soft drink problem in Chapter 1, one thing that was calculated was the gallons per person of soft drink bought. As we explored the data, we found a ratio comparing two quantities (technically a rate, because the two quantities are measuring different kinds of things). A ratio of boys to girls is a ratio because they are both humans. But comparing gallons of soft drinks to the number of people in a state is really a rate.

Beverly wanted to build the richest possible web of interconnected ideas about multiplication and division, so we planned extensive work in number relationships through numbers in problem solving. She started first with the equal groups examples and built on the repeated addition of equal groups begun in the second grade. We tried to elaborate the equal-groups conception of multiplication with the horses and corrals. But you may or may not have noticed that in the version that we did, the teacher initially held back from calling a problem a division problem or a multiplication problem. Her intent was to help the kids see that in this representation of a real situation it could be either or it could be both. Asking if you had eight groups with three horses in each, how many do you have?—that signifies multiplication. But if you ask, "How can I divide these 48 horses into corrals with six in each; how many corrals would I need to build?"—that is more obviously a division question.

We helped the kids work with groups and arrays in the round-robin. They had done some introductory array work prior to the three problems. The following week we gave the kids 24 one-inch-square tiles and asked them to make some kind of array using all 24 tiles. I displayed 24 overhead square tiles on an overhead projector something like Figure 2.10.

I asked them to describe my array in rows and columns. They described the arrays that they had made in a similar manner. "*I have 3 rows and 8 columns and 24 total squares.*" However, when Beverly first introduced arrays, she made connections to the equal groups they had just been doing. For instance, she had them say, "*I have an array with 3 rows with 8 in each row.*" She also said later, "3 rows of 8." After a number of those, she shifted to saying, "I have an array of 8 columns with 3 in each column (or 8 columns of 3)." After a few days, she shifted again to, "I

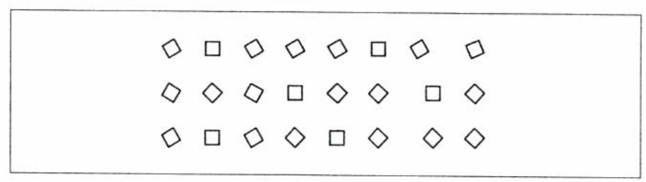


FIGURE 2.10

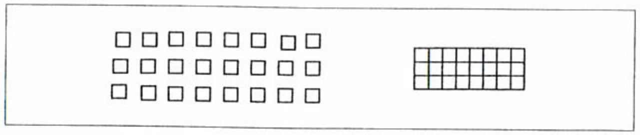


FIGURE 2.11

have 3 rows and 8 columns." There are two ideas here: (1) the language she's using links to the group language, and (2) she's giving them an overall strategy for moving back and forth flexibly between representations and between contexts.

I asked them to straighten out their arrays into rows and columns. I showed these on the overhead. Then I asked, "What would they look like if we pushed them all together?" And so I did (see Figure 2.11).

We then gave them graph paper with one-inch squares, the same size as the square tiles. We asked them to find as many rectangles as they could with the 24 square tiles. Whenever they found one, they had to draw it on their graph paper. We supplied plenty of graph paper sheets. Some of the kids wanted to make rectangles that would not fit on the paper. We asked them to explain what size these rectangles were and why they wouldn't fit. These questions allowed us to make an important transition and link. Some of the kids realized and stated that the rectangles they had made were 12 inches long and would not fit on the paper, which was 11 inches on its long side. They shared their insight and dilemma with the whole class. The teacher and I from then on talked about the lengths of the two sides as the distance from corner to corner, 3 inches and 8 inches. We explicitly connected the rectangle we called 3 by 8 to the array of squares that was 3 rows and 8 columns.

The next piece of the puzzle was to draw the big rectangles (2 by 12 and 1 by 24) by either taping together two sheets of the one-inch paper or by drawing on a new sheet of centimeter graph paper. Each square centimeter was to then represent one of the square-inch tiles. They made scale drawings of all rectangles they found. It was a little bit of a stretch for some, and we gave them the choice. About half wanted to try the centimeter paper. The kids worked diligently and everyone found at least four different rectangles. We asked them to label each rectangle by the lengths of its two sides, such as 2 inches by 12 inches.

In the days following, Beverly introduced the concept of *perimeter* as the distance around a shape like a rectangle. As a distance they could think of it like a rope that went around and then you'd pull it into a straight line to see how long it was. The kids were given some nonstretching nylon string and used it to measure the perimeter of a variety of objects, including cylinders. Beverly introduced the special name for the perimeter of such circular objects, circumference.

Context	Model	Objects	Visual/Pictoral	Symbolic/Recording
Cookies per person	(Equal groups)	pennies	draw circles	T-table
Horses in corrals	(Equal groups)	Unifix cubes	draw squares	Label picture/ T-table
Condo towers	(Equal groups)	Unifix cubes	———	T-table
Rectangular display	(Arrays)	golf tees/pegboard	———	T-table
Tiles	(Arrays)	1" square tiles	———	Label graph/ T-table
Rectangles	(Area)	1" square tiles	graph paper	Label graph/ T-table

FIGURE 2.12

One day I came in and asked the kids if they had seen my friend, Perry. They looked puzzled. I told them that Perry was from Greece and that he owned a Greek restaurant where he carefully measured all the ingredients. "He really loves to measure things, especially measuring around the outside border of things. His full name is Perry Meter. In Greek *peri* means around and *meter* means measure." Some kids laughed, most of them groaned, but all of them remembered what *perimeter* meant.

About a week later Bev also introduced them to the concept of *area*. Traditionally area has been seen as an application (and a somewhat procedural one at that) of multiplication facts memorized from working with the group model. It is usually introduced much later in the school year than she was doing now. This decision was based on my experience that some kids actually develop an initial local conception of what multiplication is better from working with area, than they do working with equal groups.

Before describing in some detail how area can be conceptualized, related to other topics, and used to teach multiplication, I want to summarize what Beverly has done (see Figure 2.12). She wanted the students to wrestle with examples from several different contexts for multiplication. She wanted to see local concept development of several contexts. She wanted them to generalize across these contexts. She helped them build a complex set of relationships among different models of multiplication (group, array, and area). She wanted to relate multiplication to division. In doing these things she not only used multiple contexts, she also used multiple representations in each context.

Frequently, parents, teachers, and administrators will ask a question, the essence of which is: Won't all these different contexts confuse the students? Not if it is done carefully by the teacher. Experiences should be savored, not hurried. Of course, a teacher can bombard the students with multiple situations so quickly that the kids' minds are reeling. On the other hand, good experiences, good questions, and steady movement without giant leaps to the abstract will pay off. Students will begin to see the basic relationship is similar in each example, and with help can make the necessary connections to generalize across contexts.

POINTS TO PONDER

As you plan for your kids to do problem solving there are several critically important things for you to consider. In the next section you will see some considerations related to the material in this chapter. There are many different ways to address these considerations, and I have given you some suggestions on how I do things. However, you always will modify and adapt anyone else's ideas to fit your own personality, your teaching style, your school circumstances, and the particular students you have. At the end of subsequent chapters I will add additional considerations that come out of that chapter. I will also include here a summary of the features of the Braid Model that have been addressed so far.

CONSIDERATIONS IN PLANNING FOR PROBLEM SOLVING

Cognitive Processes in the Context

How do I scaffold experiences for progressive development from concrete to abstract?

How concretely should I start?

How can I encourage initial play and exploration with the materials or ideas?

How can I make the experiences challenging, but not overwhelming?

What questions can I ask or terms could I use to help them visualize or imagine the context, situation, or problem?

Should they work in small groups and discuss the problem or concept in the specific context?

Grouping Structures to Encourage the Social Construction of Meaning

How can I vary the grouping structures: whole class, small group, individuals (with attention to small groups of 2–5)?

How can I enhance small-group discussions for students to develop, refine, and elaborate their thinking?

The Braid Model of Problem Solving
New entries from Chapter 2 are in italics.

Understanding the problem/Reading the story
- Imagine the SITUATION
- Asking Questions (and Discussing the problem in small groups)
  - K: What do I know for sure?
  - W: What do I want to know, figure out, find out, or do?
  - C: Are there any special conditions, rules, or tricks I have to watch out for?
- *Making Connections*
  - *Math to Self*
    - *What does this situation remind me of?*
    - *Have I ever been in any situation like this?*
  - *Math to World*
    - *Is this related to anything I've seen in social studies or science, the arts?*
    - *Or related to things I've seen anywhere?*
  - *Math to Math*
    - *What is the main idea from mathematics that is happening here?*
    - *Where have I seen that idea before?*
    - *What are some other math ideas that are related to this one?*
    - *Can I use them to help me with this problem?*

Planning how to solve the problem
- What REPRESENTATIONS can I use to help me solve the problem?

Carrying out the plan/Solving the problem
- Work on the problem using a strategy.
- Do I see any PATTERNS?

Looking back/Checking
- Does my answer make sense for the problem?
- Is there a pattern that makes the answer reasonable?
- What CONNECTIONS link this problem and answer to the big ideas of mathematics?

3 VISUALIZATION

*We would define **imagination** to be the will working on the materials of memory, not satisfied with following the order prescribed by nature, or suggested by accident; it selects the parts of different conceptions, or objects of memory, to form a whole, more pleasing, more terrible, or more awful than has ever been presented in the ordinary course of nature.*

—Webster's Dictionary, 1904

VISUALIZING WHILE READING

I remember being absolutely dumbfounded when my wife told me that some of her third-grade students did not form mental images when they read. They read the words and appeared to understand some of the meaning of text but did not see pictures in their minds. How can this be? The folks from the Public Education and Business Coalition (PEBC) are passionate about helping kids to create images "connected to the senses of sight, hearing, taste, touch and smell to enhance and personalize understandings" (PEBC 2004). Harvey and Goudvis speak of visualizing as "movies in the mind" (2000, 101).

In a workshop setting, extended periods of time are devoted to children reading and to sharing in small groups what they've read. They confer with the teacher and they discuss with their peers. Teachers ask them to process the words and their images, via oral language (to the whole class or in small groups), through writing prose and poetry, by drawing pictures, or through dramatization.

In her second-grade classroom, Debbie Miller uses anchor lessons to deepen children's understanding of the strategy of making mental images, creating and adapting images in their minds. She has children explore how images are created from the readers' own schema and words in the text. They listen to the teacher read aloud and the teacher asks them to consider what are the most vivid images. They each individually read the text and draw something that captures that image. Then they meet in small groups and share what they've drawn and discuss it. She asks them to talk about their images and the pieces of text that inspired that image (Miller 2002, 80–83).

Similarly, Keene and Zimmermann (1997) describe how even jaded junior high schoolers can respond to a teacher's think aloud of a vivid text. The kids initially offered only brief descriptions of their images evoked by the text and the teacher's think-aloud images, but the teacher gently probed, asking questions about the images the kids described. She

probed for more details, for the kids to imagine more and to elaborate on their images. "These kids showed us that images come from the emotions as well as the senses. Readers take the words from the page and stretch and sculpt them until the richness of the story becomes the richness of a memory replete with senses and emotions. Words on the page become recollections anchored in an unforgettable image of one's own making" (1997, 130).

They also may dramatize a piece of text, reenacting the story. Readers create images to form unique interpretations, clarify their thinking, draw conclusions, and enhance understanding. Images are fluid and readers adapt them to incorporate information as they read. They are influenced by the shared images of others.

Visualization works best when the text has rich detail or vivid language. When children immerse themselves in the worlds created by these words, visualization helps them perceive and conceive what the author is trying to share with them. "The detail gives depth and dimension to the reading, engaging the reader more deeply and making the text more memorable" (Keene and Zimmermann 1997, 141).

This kind of engagement seems most likely with fiction, biographies, autobiographies, or poetry. They need to be good stories, well told. I believe that we all especially appreciate stories that allow us to share universal human emotions. The PEBC folks urge us to help kids "attend to 'heart' images—feelings evoked while reading" (2004). Although such emotions are universal, our response to literature is intensely personal. The images we create belong to us. *What we own becomes our own*. Making it personal encourages us to persevere with challenging material. When students share their personal images, interpretations, and feelings in discussions or in writing or through drawing they tend to "revise their images to incorporate new information and new ideas revealed in the text. They adapt their images in response to the images shared by other readers" (Keene and Zimmermann 1997, 141).

As students grow older they "begin to censor and limit their images as they read. They focus on literal meanings—narrow, dictionary-type definitions of each word read . . . though when they were younger, their imaginations were intact and they were full of vivid images. Too often in school they've been conditioned to pay attention only to the literal interpretation of text" (Keene and Zimmermann 1997, 140).

THREE TYPES OF VISUALIZATION IN MATHEMATICS

There are two ways that students use visualization in mathematics that should come as no surprise: *creating mental images* as they read and *creating representations* of their mental images. The other way, *spatial thinking*, is quite different and I will review it first.

Spatial Thinking, or Visualizing Spatial Relationships/Orientations

We could say that mathematicians in the United States came late to the game. From the 1930s, Russian psychologists from the former Soviet Union investigated *spatial thinking*, creating spatial images and manipulating them (e.g., *mentally* rotating objects in three-dimensional space) and creating new mental images produced by imagination of something not yet seen.

Soviet psychology was focused on the human potential that *some* could attain under proper conditions. If they could demonstrate that some of their students could go far beyond the typical ability when taught in a particular way, this level of ability was available to humans (not necessarily all humans, but attainable by some nonetheless). In contrast, U.S. psychology has been more oriented toward statistical definitions of what the average students can do or if a teaching technique was statistically significant across a broad population of students.

Yakimanskaya (1991) summarized a vast body of Soviet research on spatial thinking. The value of spatial thinking lies in helping students to identify spatial properties and relations and to use them in solving problems of orientation in real space and theoretical geometric space. Though they found individual differences among school children, a major source of the differences came from the teaching techniques to which students were exposed. Furthermore, students do *not* make the transition from representations of real space to a system of graphical substitutes via maturation or development. Even those who were innately beyond the typical spatial ability (i.e., in U.S. parlance, "gifted") required particular teaching methods to acquire "a specialized conceptual apparatus" enabling them to use various frames of reference and methods of representation.

The training in spatial thinking, creating and manipulating mental images, nurtured several generations of highly proficient technicians, draftsmen, and engineers in applied math and science areas as well as many pure math and science people. These were the people who launched Sputnik (years ahead of the Americans), Soyez, their space station, and Major Tom.

Little attention is paid to spatial thinking in U.S. curricula, teaching, and teacher preparation. It tends to be seen as an interesting topic that teachers never get to in the school year. This trend is likely to continue because of the unfortunate confusion caused by the term *visualization* in one of the more prevalent educational theories about the teaching and learning of geometry (the Van Hiele model), in which visualization is the lowest form of geometric thinking. Their model contains sequential stages beyond visualization: *analysis, informal deduction, deduction, and rigor.* Visualization is merely where space is simply observed, where geometric figures are recognized by their physical appearance as a whole, and not for their properties.

U.S. researchers have investigated how *spatial thinking* compares and contrasts with *verbal reasoning* and how both are used in problem solving. They have studied *spatial visualization* (mentally moving, manipulating, twisting, or transforming a visual representation—rotating cubes or folding paper) and *spatial orientation* (changing only perceptual perspective for viewing an object, comprehending arrangements of elements within a visual pattern; understanding a visual representation or a change between two representations; and organizing/making sense out of visual information). In contrast to the Russians, findings have been mixed and inconclusive. Definite conclusions about these complex distinctions has been further complicated by the discovery of two different types of logical thinking processes: one characterized by step-by-step, analytical, and deductive thinking, often mediated by verbal processes, and the other by more structural, global, relational, intuitive, spatial, inductive processes. I designed a two-part activity for students to explore both kinds of reasoning with the same materials. Part one emphasizes intuitive, spatial, inductive processes; part two analytical and deductive thinking. I have done versions of this activity with four-year-old preschoolers, every grade up through eighth, and even with adults in my college classes, and I am certain that in this spatial thinking activity, as in so many others, everyone gets better with experience.

Twenty-four Shapes

I arrange the students into groups of four, five, or six. I give each group a zip bag containing some specially made geometric shapes that have

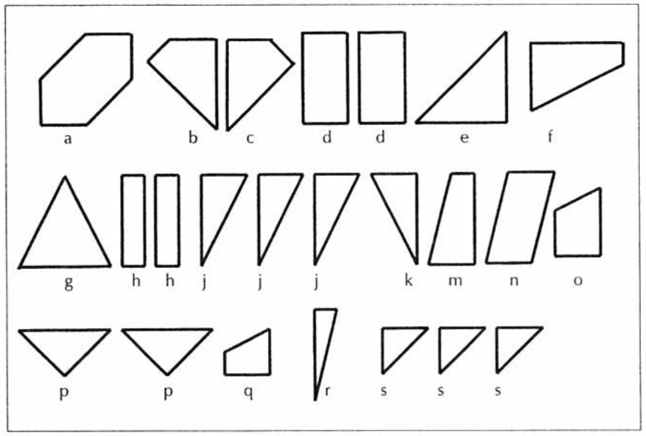


FIGURE 3.1

been photocopied onto bright pink paper, mounted to white, high-density foam boards (from an office supply store), and cut out with a utility knife. Initially, I will describe how I do this activity with the older kids (sixth grade), then how I have modified it for younger students. There are 24 shapes. See Figure 3.1.

Actually there are only 17 different shapes, or 15 if you are allowed to flip mirror pairs (**b** and **c**; **j** and **k**), over onto the other side. Shapes **d**, **h**, **p** are in duplicate; shapes **j** and **s** are in triplicate. As in previous activities, we do a modified KWC and ask them what they can tell me about the shapes. What do you know for sure? If they do not start describing properties very soon, I ask more focused questions about the properties, such as which shapes have right angles.

Then I introduce the first task: Each group has the same 24 shapes from which they are to create 8 congruent squares, that is, 8 squares of the same size using all 24 shapes. This is a challenging task involving structural, global, relational, intuitive, spatial, and inductive processes. We go through the questions of the KWC. Often sixth graders struggle with, "What do I write down for C," the special conditions? The main thing is the requirement that all 24 shapes be used. See Figure 3.2 for how it may be done. (I give a blackline version of these in 3-inch squares on the website: www.braidedmath.com.) The solution is not unique because one can substitute congruent shapes. These squares are approximately in the order that sixth graders and adults find them, which generally corresponds with their perceived difficulty. Square H is described by students as the most difficult. There are several reasons: very frequently students will combine shapes **o** and **q** to get Figure 3.3.

Once a group has made this rectangle, it is as if the two pieces had been glued together. It seems so logical and it is versatile. It is congruent with shape **d** or with the two **h**'s and several other combinations. It

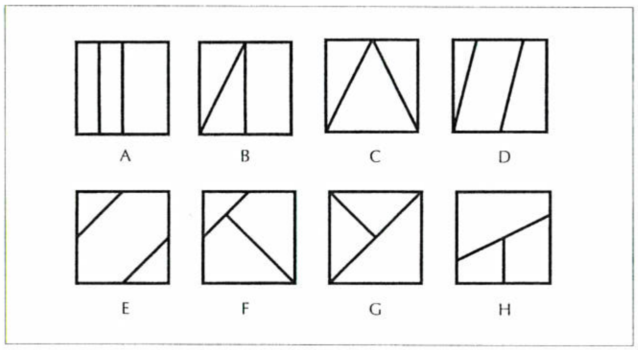


FIGURE 3.2

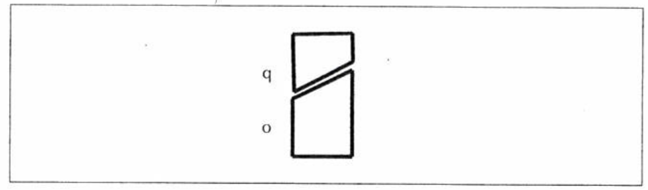


FIGURE 3.3

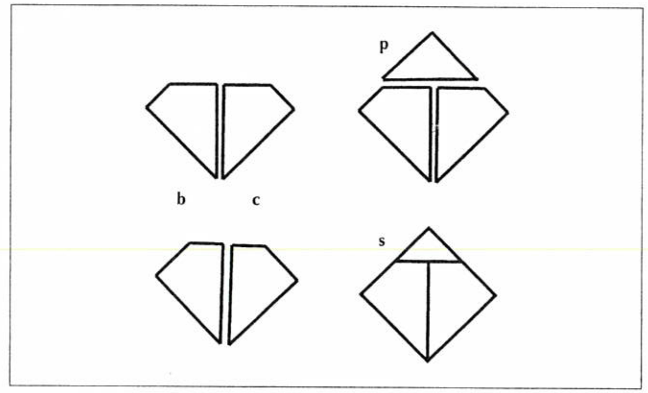


FIGURE 3.4

may be versatile, but it is wrong for this task and will block a solution to the 8 squares.

Similarly, if the mirror pair **b** and **c** are connected in the wrong way, they'll be stuck. Notice that shapes **b** and **c** are not kites; that is, they do not have pairs of adjacent sides congruent. Their long sides are congruent, but opposite them are two different lengths for the short sides. Therefore, **b** and **c** can be connected at their long sides in two ways. Students frequently connected them as shown in Figure 3.4 and then added onto it shape **p**, thinking they had made a square. Actually they have made a rectangle. The proper way to orient shapes **b** and **c** to make a square is shown in Figure 3.4 so that the small triangle **s** is joined to it.

The first task in this activity puts a strong premium on spatial visualization. Even though the students are in groups, they do not talk much about the properties of the shapes as a way to help them solve the task. They do a lot of trial and error, placing shapes next to one another. Most have not had much experience with such tasks.

When a group successfully completes the first part of the activity and creates the 8 squares, I give them a writing prompt to reflect on what they did, such as: which squares did you find the most difficult to make? Why? What did you do to get success? Were there any properties of the shapes that make it easier or harder to go with another shape to make a square? The group may discuss their ideas and answers, but I require individual written statements from every student.

When doing group activities, one group always finishes first and one last. That is logical. But in terms of management, I want the time between the first and last finishing to be as short as possible. So the writing prompt gives a little bit of a cushion. As the quicker groups finish (and quicker does not mean smarter), I have them write while the other groups keep on working. I may intervene a little in the groups that are taking longer, when, for instance, they have "glued" together two pieces in their mind. Or the group may be stuck on how big the square must be (this is more prevalent with younger students than with sixth graders). If so, I may scaffold them a little by providing a border/perimeter showing the size the square must be. It does not do the work for them, but it does help them focus on the gestalt of the square. They can then systematically try a different position for each shape, especially the big shapes. For instance, if you have the border and try to fit shape **a** inside, there is really only one way. Rotations don't count as different. Also, with this border they can check the arrangements shown in Figure 3.4.

I realize that in my way of doing this, the last group may have much less time to write. In fact, sometimes I will take the whole class into the second part of the activity and tell the last group to do the written prompt for homework. It is a trade-off I can live with.

If a group is stuck because they've mentally glued two pieces together in a way that will never work, I do not tell them what is wrong. Interventions are best done with questions that help them rethink and redo on their own. I usually ask questions such as, "Which of these squares that you have made can only be done the way you did it? Put them aside. Of the squares you made that did not have to be made that way, how could those shapes be combined with other shapes differently? Which would you now like to rethink?" Most groups respond well to these kinds of questions. However, if a group places one of those wrongly stuck examples, I may ask, "Are you certain that this is the only way those shapes may be used?" If they still don't get it, I may say, "I suggest you rethink this one also."

When they have found all 8 squares, I give them a paper handout with the solutions drawn for squares A through H. After a group has done some writing, I introduce them to the second part of the activity. I give each person a handout with the border of the square and ask them to look at the shapes and look at the square. "Show me a shape that is exactly half the area of the square." Invariably they show me shape **d**. (See Figure 3.5.) I ask for another, different half. Shape **e** is usually offered. Then I ask them to show me a shape that is one-fourth of the square. Shape **h** is usually the one they pick. "Your task is to take each

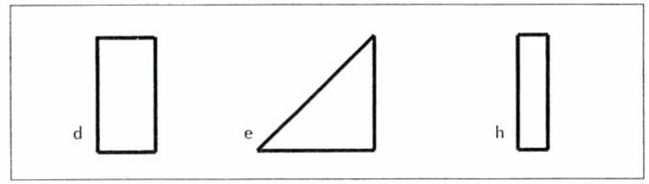


FIGURE 3.5

square it is." I encourage them to use shapes and solutions on paper that are the actual sizes of the shapes. I mention that drawing on the paper will probably help them. This second part of the activity requires more step-by-step, analytical, and deductive thinking. Both parts use the same physical materials, but the nature of the questions and tasks engenders different thinking.

Still in the same work groups in part two, the students talk more about this task (i.e., verbal mediation) than they did about the task in part one. They start with what they perceive to be the "easy" shapes—the halves and fourths. There is always an interesting moment when they consider shape **g** in square C. Some say it is half the square; others say, "No way!" Regardless of age or grade, some students have in their minds the misconception that one-half means two identical pieces that make up the whole shape. They missed the day the second- or third-grade teachers explained or showed how one-half can be the equivalent of half of the whole shape. Someone in the group figures out that shape **g** can be made from shapes **j** and **k** (which are congruent if you flip one of them).

Some younger children say the square C has been cut into thirds, because it is made of three pieces (even though they are not identical). Similarly, some students argue about shape **f**, saying, "It does not look like a half." The ones that look like halves to them are the familiar shapes **d** and **e**; **f**, **g**, and **n** are not often seen in the curriculum and it takes some reasoning for them to figure out that they too are halves. And that is one of the major points here. Part two requires *reasoning*, spatial reasoning, and part one is far more intuitive.

Familiarity with fourths readily allows students to see how two **h**'s as well as 2 **j**'s or **k**'s make shape **d**, the obvious half. Shape **p** always gives some students trouble as does square G. They try to put the right angle of **p** in the corner of the square. Usually one student draws the second diagonal of square G and everyone sees the four identical shape **p**'s.

Drawing lines on the pictures of the solutions (essentially doing geometric dissections) is necessary to finding the fractional parts of the more difficult pieces, **b**, **c**, **o**, and **q**. When doing this activity with younger children, third or fourth graders, I will leave out squares F and H. The way fifth or sixth graders find the shapes is by placing shape **s** (the shape in

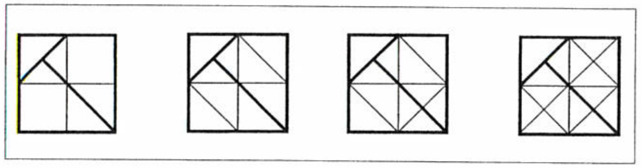


FIGURE 3.6

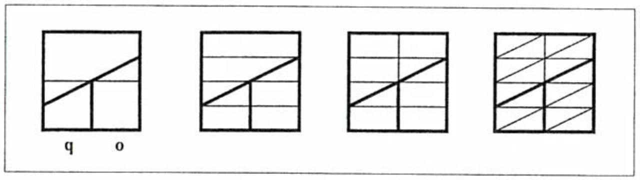


FIGURE 3.7

lines along the shape's sides. The kids say this is like *tracing*. Some kids draw more lines than others before they *see* that the shape they've been tracing is one-eighth. On the far left of Figure 3.6, *this* kid drew relatively few lines, just enough to reason that eighths, fourths, and sixteenths were there. On the far right is that of a child who needed to draw/trace lots of lines so that he could see the sixteenths. When the kids know that shape **r** and shape **s** are eighths, they can use these eighths to fill a square with drawn lines for eight of versions shape **s** and eight versions of shape **r**. When so doing with squares F and H, they see Figure 3.6 and they can reason that half of these eighths would be sixteenths.

In an analogous fashion, Figure 3.7 shows how square H can be dissected in order to find the relationship among the three shapes (trapezoids). I often encourage students to take a shape that they know and use it to help them draw lines. On the far left, a kid took shape **d** (a rectangle that is half of the square) and drew a horizontal line. That was all it took for him to see the one-fourth in the lower right corner. Then he reasoned that shapes **o** and **q** must be bigger and smaller (respectively) than one-fourth by the size of the small triangle. When I asked him what made him think this was true, he replied, "If you cut the little triangle off the top of **o**, you could put it on top of **q** and then they'd both be one-fourth." He did not know how big the little triangle was so he drew some more horizontal lines, which is what most others did. Others needed to take shape **h** and use it for dissecting and tracing lines. Most sixth graders drew lines like the middle two squares in Figure 3.7. Other kids kept drawing lines until they had the 16 sixteenths on the far right.

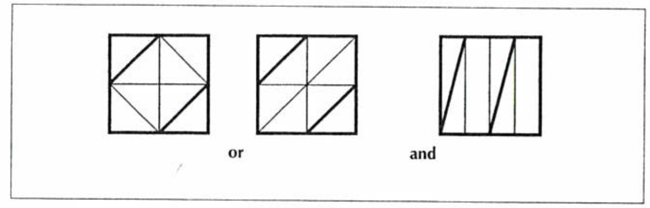


FIGURE 3.8

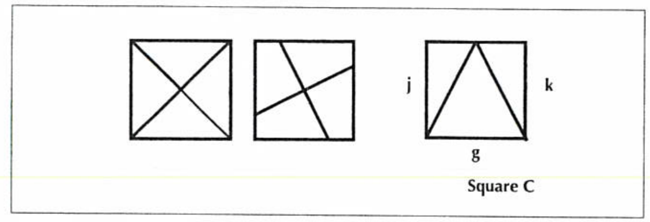


FIGURE 3.9

Dissecting existing shapes with straight lines works wonderfully with some of the easier squares and shapes so that even third graders can use reasoning to discern fractional parts. See Figure 3.8.

When working with kindergartners and first graders, I may use only squares A and B to establish halves and fourths and their relationships, and I also use four of shape **p** and four of a new shape that is half of shape **k** so that they experience the arrangements in Figure 3.9.

With good understanding of the relationship of halves and fourths, they'd be ready to tackle square C and understand that even though they do not have two of shape **g**, a second one can be made with the other two shapes, **j** and **k**.

Both parts of this activity are necessary in the development of good problem solvers and mathematicians. At the very least, the first part of the activity provided a strong experiential base of familiarity with the shapes and their relative size. Even more than "getting the feel" for the shapes, though inherently valuable, the first part develops awareness of some of the properties of the shapes and a motivation for analyzing them. The teacher can definitely weave in selected properties as a very natural extension of what they have been doing. For example, when debriefing the very difficult square H, the teacher can ask the class to compare and

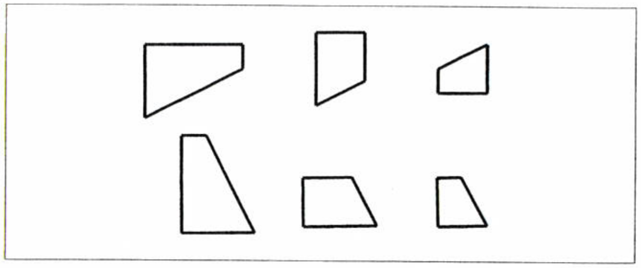


FIGURE 3.10

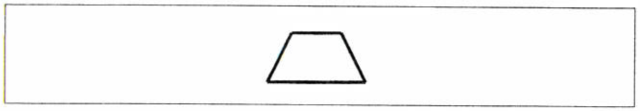


FIGURE 3.11

contrast the three shapes that constitute square H, shapes **f**, **o**, **q**. She might rotate them in a number of ways to make this visual task more difficult or easier. See Figure 3.10.

All three are trapezoids. At first, kids whose entire experience with trapezoids was with the red trapezoid of the pattern blocks (Figure 3.11) do not believe that these are trapezoids. In order to develop true understanding of the concept of trapezoid, the kids need to understand the concept of "parallel" and realize that the defining attribute of the shapes in Figure 3.10 is that each contains a pair of opposite sides that are parallel.

We will encounter the two different types of logical thinking processes continually in mathematics just as we did in this two-part activity. I don't want to pose these as being dichotomous or conflicting. They are different, to be sure, but they can be mutually supportive. It is also worth noting that these processes are not confined to spatial thinking or geometry. They permeate all of mathematics.

Answers to the fractions are: (**d**, **e**, **f**, **g**, **n** are 1/2); (**h**, **j**, **k**, **p** are 1/4); (**r**, **s** are 1/8); (**b**, **c** are 7/16); (**m** is 3/8); (**q** is 3/16); (**o** is 5/16); and (**a** is 3/4).

CREATING SENSORY IMAGES/VISUALIZING THE SITUATION

Students should be creating sensory images or using mental imagery whenever they read math textbooks, biographies of mathematicians, or story problems. Teachers should prepare beforehand specific passages, sentences, expressions, or words that are likely to prompt students to visualize the ideas in the text or problem. The teacher may do a whole-class KWC with each sentence separately shown on an overhead protector or written on poster board. As the students read each sentence the teacher can suggest that they imagine what is going on. Along with visualizing in the KWC are the questions about making connections from Chapter 2. These strategies (asking questions, making connections, and visualizing or imagining the situation) flow together easily and naturally as teachers develop their own way of orchestrating problem solving.

When most students read a story problem or hear someone describe a situation, mental images are generated. The words are catalysts for images and retain their imagery content. The PEBC folks mention that it is a good idea to picture story problems like a movie in the mind to help understand the problem. They should visualize concepts in their head (e.g., parallel lines, fractions). In fact, the more elaborate the images children have for mathematical concepts, the greater ease with which they can use them in problem solving even with what is probably the most hideous of all story problems. Let's see.

The Rendezvous of the Two Spies

Two spies decide to meet to exchange documents. One spy is in New York City (NYC); the other is in Indianapolis 700 miles away by train. They want to be together for only a few minutes at the train station. They consult the train schedules and find that there is one stop that will meet their needs. The Midwest Flyer leaves Indianapolis at midnight and arrives at NYC at 2:00 P.M. (14 hours later), covering the 700 miles at an average speed of 50 miles per hour. The Silver Streak leaves NYC at 2:00 A.M. and arrives at Indianapolis at noon (10 hours later), covering the 700 miles at an average speed of 70 miles per hour. How far from each city do the spies rendezvous and at what time?

It involves two trains going at different rates (different elapsed time to cover the same distance) and leaving at different times. One reason that it is a difficult problem is that there are multiple patterns. Each train has its own pattern of movement, essentially an average rate for covering a distance in a certain length of time. Added to these two are the patterns of the distances from each city and the distance they are apart, which narrows over time. It is inevitable that they will meet at some time (when the distance apart is zero). There is an overabundance of information.

For years I have used the expression and question, "Can you get your mind around it?" I think for this and other train problems, people cannot get their minds around this situation. They need help breaking it down. I almost always start with some version of a KWC. We carefully walk through "What do I know for sure?" I usually write down on the chalkboard abbreviated responses. Then "What do you Want to find out?" Most kids just say, "*Where will they meet.*" Other times someone will say that we also need to know what time. When no one does I try to get them to pick it up in the C question, "Are there any special Conditions?" If no one has focused attention on the passage, they might miss the need to know when—at what time.

This problem is ripe for visualization and for the math problem-solving strategy of *Act It Out* because there is a sequence of actions in the story. Using this strategy requires some creativity and a lot of common sense. I don't recommend it early in the year when you don't know the kids really well; some might take advantage of the opportunity to clown around. Also at the beginning of the year, some kids may be testing the teachers' limits, boundaries, and rules. An absurd example of how *not* to do this strategy would be to put the kids in pairs or triads and say, "Okay, go act it out."

I prefer the fishbowl, where the class is arranged roughly in a U shape and the actors are at the top of the U so everyone has a good vantage point. The teacher should be on one side near the front, so she can observe the actors, whom she will be choreographing, and the rest of the class, to whom she will address questions to keep them involved.

Because acting it out is a time- and energy-consuming approach, I use it sparingly. And since they cannot use Act It Out in a testing situation, I want them to use this bodily kinesthetic strategy to help them visualize the situation; I want to build their ability to visualize and wean them away from bodily actions.

So I choose two volunteers who go to the front of the room, one on the far left, the other on the far right, chalkboard between them. If the classroom has a U.S. map in the center of the chalkboard, I pull it down. I ask the class, "Which trains are each of our volunteers?" You may wonder why I don't have the two actors be the spies. In the past that has engendered some silly behavior. The class usually agrees that the kid on the right is the New York train and the left, the Indianapolis train. Why? Because on the map New York City (NYC) is east and on the right; Indiana (IND) is west and to the left.

How much choreographing or structuring of their behavior should you do? I try to ask questions of the actors and the class to keep the trains rolling. Some teachers put masking tape on the floor and put the two cities as far apart as they can in the room. Then they mark off the distance of 700 miles in increments of 100 miles, making a little scale model. I usually do not do that. I simply say, "It is now midnight. What is happening with them?" Many look puzzled. I ask, "Where are they? Have they left the station?" No, but the IND train is warming up. I ask the kid playing the train, "Aren't you going to work to warm up?" I make a chuga-chuga

"Okay. Now it is 1:00 A.M. Where are they now?" The NYC train is still doing nothing, but now the IND train has to move. I ask the class, "Where is the IND train now?" They say that the train is 50 miles away from Indianapolis. I turn to the kid playing that train and ask him/her to go forward what would look like about 50 miles. It really does not matter if the distance is exactly 50 miles of the 700 miles scaled by our masking tape at the beginning and end of the journey. They can just do a reasonable estimate. I ask the class, "How far away is the IND train from its destination?" (650 miles.) They need to visualize that the total distance of 700 is now partitioned into 50 and 650. As simple as this sounds, they need practice thinking this way.

"Okay, kids; it is now 2:00 A.M. What is happening?" They tell me the train from IND is now 100 miles along. The kid playing the IND train moves forward another couple of paces. "What is going on in NYC?" I ask. Invariably the kid playing the NYC train goes, "Chuga, chuga, I am warming up," and moves his arms like pistons. This time, I chuckle. I ask again, "How far has the IND train gone? How far away from NYC is it?" The class replies 100 and 600 miles. So far, so good.

When we go to 3:00 A.M., new things start happening. The IND train moves another estimated 50 miles and now the NYC train moves forward a few paces that the kid estimates to be about 70 miles. Sometimes the kid looks at the two pieces of tape and tries to figure out what one-tenth of the distance is. I reassure them all that the actors do not have to be exact, we just have to visualize what is going on—and that is the relationship between the two trains. I ask, "How far from Indianapolis is the IND train? How far from NYC? (150, 550.) "How far from New York City is the NYC train? How far is he from IND?" (70 and 630.) Now I ask the key question. "*How far are apart are the trains from one another?*" This time they need to really think and some kids in the audience grab paper and pencil. It is just so cool when they can visualize that the 700 miles can be partitioned into 150 miles away from Indianapolis and 70 miles away from NYC, which means that 220 miles out of the 700 miles have been covered, which leaves 480 miles between them.

Now the choreographer has to make a decision. Do the students all see all the elements in the problem? Do they truly understand what is happening? If so, then the Act It Out strategy has served its purpose. They can shift over to a more abstract strategy, such as drawing a picture or making a table to find a solution. How does the teacher/choreographer know? By listening to and observing everyone. If I think some are ready to solve with a more abstract strategy on their own, I ask them, "Do you feel like you understand the problem well enough now for you and your partner to use a different strategy to find the solution [pause] or would you like to continue acting it out a bit more?"

If they continue acting it out, then when they get to 4:00 A.M., I ask them these questions: "How far from where they started are they, and how far does each have to go to get to his destination?" (200, 500 and this time one of the kids usually spots a key piece of the puzzle. After 2:00 A.M. the distance apart will shrink by 120 miles every hour (70 from one train and 50 from the other).

Once again I can ask if they want to keep on acting it out or just figure it out for themselves. Often I will get a mixture of responses. I have tried a variety of ways to continue this choreography. I have continued to act it out with the whole class; some students were annoyed and, in a somewhat clandestine spy–like manner, figured it out for themselves. Other times I have bowed to their requests to stop the action and let them in pairs work out the solution. On these occasions, some kids got lost because they had not fully grasped all the pieces to the relationship. Consequently, they did not get as much out of this as they could have. The middle ground has proven best for students. It is a form of *differentiation*. I have prearranged some tables at the far back end of the room for those who want to quietly go back and work out the solution. The rest, which may be more than half of the class or may be only a handful (like five), get up closer to the chalkboard and we quietly continue the action.

I have a standing routine with students of all ages. If they figure out the answer way before others in the class, they tell me privately. They do not yell out the answer. I want the rest of the class to keep working on it. This is part of being respectful to others. And then I have an extension problem ready to go for them.

Revisiting the Two Spies with a Different Representation

The next section, written by Mona Tauber of Lincoln Elementary School in Wheaton, Illinois, describes how her high-track fifth graders took on the two spies on the trains problem.

My students and I have discussed that problem solving is an essential math (life) skill. The students know they are expected to solve problems "KWC" style. This means after we read the problem, they are to first identify (by highlighting) what they Know that will help them solve the problem. Then, they are to underline what they Want to find out (the question). After that, they are to choose a strategy that would be most effective in helping them solve the problem.

Early in the fall, my 5th grade high track students tried a problem that addressed a concept that was still fairly new to them: rate (miles per hour). They have had some experience with ratios and proportions. We began by reading the title, "The Rendezvous of the Two Spies." I asked them what they predicted this problem to be about. They immediately told me they expected this problem to be about two spies and that it would involve a meeting, as they understood what a rendezvous was.

Then we began to read the problem together. I asked the students to highlight what they "Knew." This problem is challenging in this aspect alone, because mirroring real life, it shares more information than necessary to solve the problem. Some students stated that they highlighted that the Midwest Flyer arrived at NYC at 2 pm and that the Silver Streak arrives in Indianapolis at noon. A couple of others quickly stated that this was not important, because they were going to meet somewhere along the way. They made an inference while reading that aided their understanding of the problem. I asked one child to explain how he knew this. He said it was because one only leaves two hours later and travels at a higher speed. All agreed that they understood because the problem already told us the speed for each train and when they left their respective stations. I was most impressed.

They had no trouble identifying what they needed to find out, but I was glad to hear that one child emphasized that there were actually two questions in one. We didn't just need to know at what time the spies met, but also how far each spy was from the original stations.

I asked them to think about whether what they knew and what they needed to find made them think of a strategy that might help them solve the problem. It was at that point one student suggested a table. The others all agreed, but I want students to share their reasoning, so I asked the first student to explain how he decided on this strategy. He told us that he knew that we could keep track of the time and distance for both trains in one table, which would be important to knowing when they met, and that a pattern would emerge since every hour they would cover 50 or 70 miles. When I asked how they knew this meant a table would be the best strategy so quickly, another student said that we had done enough problem solving for which tables were best. It was music to my ears!

The students were essentially telling me that they had made a *math-to-math connection*. They knew that tables are effective when we want to keep track of information in an organized fashion and for which we expect to use a pattern to complete it. [See Figure 3.12.]

I quickly went around to check their tables. All were correct. I asked the kids to share how they knew the answer. One said that he also tracked the total mileage and when the total reached 700, he reached the goal, since the two cities were 700 miles apart.

I concluded the day's discussion by discussing rate and its algebraic formula. My students already know some algebra, so I knew they could comprehend. I wrote $D = r * t$, while explaining along the way what each meant and referred back to the problem

A: The spies will meet 350 miles from each city at 7:00 A.M.

Time	miles covered Midwest	Silver
1:00	50	0
2:00	100	0
3:00	150	70
4:00	200	140
5:00	250	210
6:00	300	280
7:00	350	350
8:00	400	420

FIGURE 3.12

to apply the formula. I then asked them to share when they might use this strategy and knowledge in the real world. They saw the real world connections, like meeting someone else at a given location, just trip planning, etc. Their homework was to write an extension problem, so that the spies would meet at a different time and/or different distance from each city.

This lesson proved to be more than I could have asked for. I know the benefits of taking more time through the problem solving process and requiring students to use the same strategies they use in reading to benefit them in mathematics.

VISUALIZING AND TRANSLATING BETWEEN REPRESENTATIONS

Another problem that profits greatly from visualization and kids acting it out recalls those great six years of Michael Jordan and the Chicago Bulls basketball team. Six championships in eight years was a fabulous time for the High Five problem. The lights at the United Center dim to total darkness. The familiar theme music comes on, blasting at eighty decibels. The announcer's deep bellowing voice bounces off the rafters, "And now, your Chicago Bulls." The crowd goes berserk. He calls out the names of the starting lineup for the game. They jog out to center court and . . . Well, what do most basketball teams do? In the math class we simply say that they give one another a high five. I model this with one student. We both raise our arms and slap our two hands against each other's two hands. Shouldn't this be a high 20 or a high 10, some kids ask. We just agree that we'll call this "high fives" and we will not do any gymnastic moves like jumping up and bumping chests.

The math problem is: "When the Chicago Bulls come out onto the court at the beginning of a game, the starting five are announced. Each player slaps a high five to each other player. What is the total number of high fives slapped by the starting five players on the Bulls? You count one high five when two players slap high fives with each other."

The class does a relatively fast KWC. In prior years the rabid Bulls fans among the kids (pretty much all of them) thought we should name the starting five in order. It varied a bit from game to game and year to year, but for example, it was Harper, Pippen, Jordan, Rodman, and Longley. Most of the kids knew for sure the names of the Bulls. Most had no trouble visualizing the ritual that introduced the Bulls. Those who had never seen this opening ceremony were treated to various descriptions of it by classmates.

Every now and then when I did this problem with kids in grades three through five, there would be some who had been to live games, but many more had seen the opening ceremonies on television. The K was easy: there are five players. They come out one at a time and slap a high five to each of the other players once. The W was pretty clear also. What is the total high fives or how many high fives get slapped altogether? The C for conditions is always the most difficult because it is not obvious. When it is obvious, they catch it under the W. Sometimes in this problem, they could not think of anything. On other occasions someone might spot that when two players slap high fives, it should count as two because there are two people and they are both doing it. I always let them talk it out to try to reach a consensus. Sometimes that just will never happen. In this case most realized quickly without needing to be convinced that two players hitting one high five was the only way for it to make sense.

I select five students to be the players and send them up to the front of the room into one corner. I then require the five actors to get in a line and come out when I call their names (or the name of the Chicago Bull they are role playing). I call them out from the corner one at a time amid wild applause from the audience. About half the times I have done this with kids they are so into the action that no one remembers to count the high fives. I could build someone who counts into the structure, but I like for them to realize this for themselves and then act it out again with someone keeping count. I do not want to dampen their enthusiasm for a math problem, and I don't want to spend twenty minutes giving directions. I want the minimum amount of directions to achieve the success that allows them to get into the problem. The KWC slows the action, but that step is necessary.

I call another group of five up to the front (the first five go back to their seats), and we do it again. When they count they get 10 for an answer. I ask straight-faced, "Are you sure? Does anyone see a pattern that will help us be sure we have the correct answer?" Almost never at this stage does anyone see a pattern. I bring another set of five kids up to the front I ask them to "Act it out again and see if you can detect a pattern." Of the two times they counted 1, 2, 3, 4, 5 . . . 10, many will not see what is going on. But some do and I'd like them to explain to the class what the pattern is. They always do some version of the following:

- The first player comes out and has nobody to slap. That is zero.
- The second player comes out and he hits a high five with the first player. That is 1.
- The third player comes out and he hits high fives with the first two players. That is 2, 3.
- The fourth player comes out and he hits high fives with the three players. That is 4, 5, 6.
- The fifth player comes out and he hits high fives with the four players. That is 7, 8, 9, 10.

I then ask the student who said this, "Then what is the pattern? How would you explain this pattern to your parents?" In this case the kid usually repeats what he says, "You know 0, 1, 2-3, 4-5-6, 7-8-9-10. They get more slaps each time." But that last statement could mean so many different things. So I seize this opportunity for a sermon, *Carpe sermonatum.* "You know, kids, math is the science of patterns and the patterns that you see belong to you; they are all yours. People often look at the same thing, and each person sees a somewhat different pattern. *Or* sometimes they will both see the same pattern, but they will talk about it differently. They'll use different words to describe the picture that they see in their minds. Now, P.J. gave us a good description. Can anyone tell us about different patterns or use another way to describe it?" Often someone (like P.J.) will say something like, "*You add one bigger each time.*" To illustrate what this kid was describing, we usually act it out again with another five. By now just about everyone has had a chance to come up and act.

When they act it out this time I tell them to count how many players get high fives each time somebody comes out. So the count becomes 0, 1, 1-2, 1-2-3, 1-2-3-4. I ask P.J. if this is what she meant. She says, "*Yes, 0 + 1 + 2 + 3 + 4 = 10.*" At this point a rumble from the peanut gallery can be heard and H.L. pipes up, "*But the fifth player to come out was the only one who slapped high fives with all four of the other players. The problem said [I am waiting for the kid to say "and I quote"], 'Each player slaps a high five to each other player.'*" "And your point is?" "*The other players did not hit four other people when they came out onto the court.*" About half the class tries to tell H.L. that the others did slap high fives with all the players, because they came out on the court and hit high fives with anyone who was there before them and then they waited for somebody new to come out and hit them. Their proof is that the first person out hit nobody; does that mean that he never slapped any high fives? No. He got a high five from each of the other four when they came out.

But some kids are now whining, "I'm confused. I thought the answer was ten. You mean it's not?" I suggest that we act it out one more time, which makes five times, so just about everybody has been up front to act it out. This time I tell the actors to keep track for us of which people they had a high five with. They verify that each of the five players hit four other players once. Now I ask, "If five players each hit four players, why wouldn't that be twenty high fives?" Most catch on that if I did it that way, I'd be double counting.

A simple T-table can help students keep track of what is going on. See Figure 3.13.

Player	High Fives
One	0
Two	1
Three	2
Four	3
Five	4
Total	10

FIGURE 3.13

An interesting question is what does the column label on the right mean? Compare this table to the table in Figure 3.14, which labels the column on the right as "Total hi 5s, which could also be a running total or cumulative total. It is very valuable for kids to seriously think about the difference, especially so that they develop the habit of asking themselves, what do these numbers mean?

Player	Total Hi 5s
One	0
Two	1
Three	3
Four	6
Five	10

FIGURE 3.14

One of the most important messages in this book is this: for every symbol that students write, there must be a concrete referent in their heads of what that symbol refers back to. They must be able to conjure up a mental picture, an image of some sort, that this symbol is a symbol of something specific, or we might say that this symbol represents something specific. I routinely will stop a lesson and do a spot check. I will point to a symbol and ask the class about what it represents. It would be a good idea to keep the columns simple until the kids have the hang of it. Some textbooks are a little too quick to show the kids how to make a double T-table. See Figure 3.15.

Player	High Fives	Total Hi 5s
One	0	0
Two	1	1
Three	2	3
Four	3	6
Five	4	10

FIGURE 3.15

Why am I spending so much time on one activity? If they have to act it out at all, why not just one time? Do they all need so many examples? I am a strong believer in, "If you do it thoroughly and well the first time, all who have the prerequisite, prior knowledge can get it, and you'll do far less reteaching in subsequent weeks."

CREATING REPRESENTATIONS

The third major way children and adults use visualization in mathematics is to create representations of what they perceive in their minds, their mental images. Probably in every chapter of this book I will talk about the importance of children creating their own representations. And the

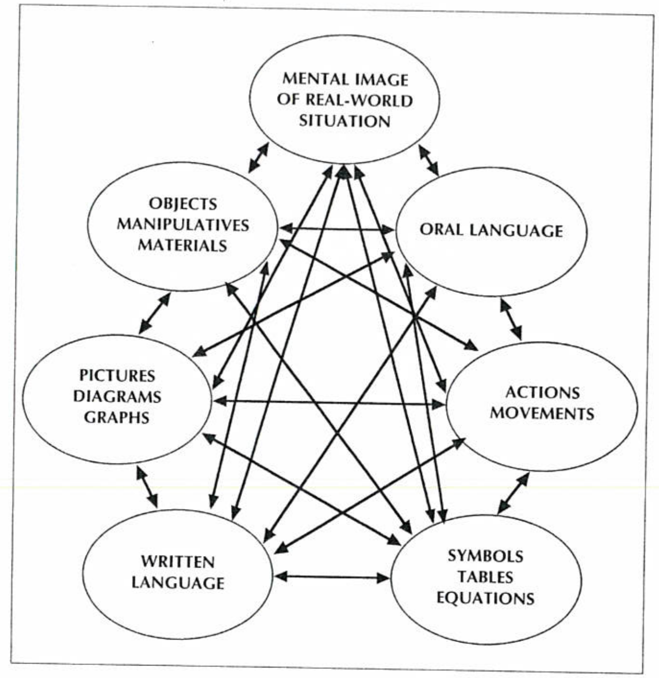


FIGURE 3.16

Of course, I feel strongly about the representational strategies being used in problem solving. Once students have a good "feel" for the problem from visualizing the situation, they should use one of the other representational strategies to work on it. In other words, they should try to represent the problem in a way that will help them to either understand it better, to understand it in another way, or to lead them to a good solution path. Zawojewski and Lesh (2003, 325–27) suggest that when students do math problem solving in small groups and work on rich problems they are using their representations to communicate their mental images to others. When students create and share multiple representations of the same problem or situation, they are continuing to keep their thinking alive. Multiple representations also may provide deeper, more elaborate understandings of the underlying mathematics, and fresh, new insights into the problem.

In the examples of different contexts in the previous chapter you

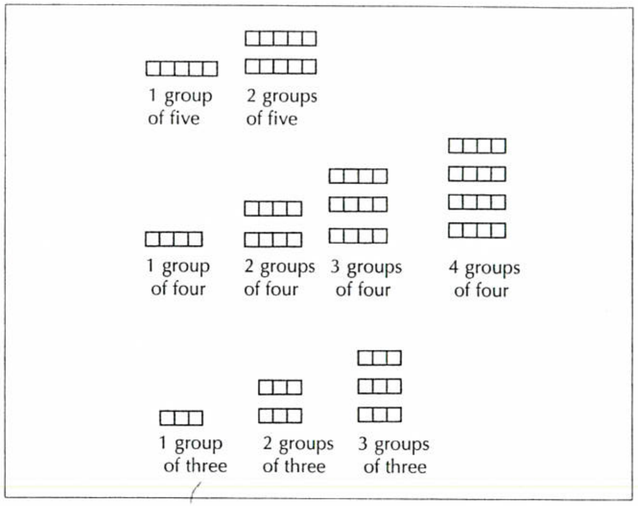


FIGURE 3.17

explore, we began fairly concretely and moved to more abstract representations. The teachers used situations that were discussed with a KWC enhanced by M-S, M-W, and M-M. They worked with objects (and language about the objects). Next, the students usually drew some pictorial representation (e.g., graph, picture, diagram) that I like to think of as a color-picture record. They often labeled the picture with symbols. Finally, they represented the data symbolically, first in the T-table, then some used an equation.

Figure 3.16 shows six major ways of representing the mental images in our heads. To build conceptual understanding to create multiple representations of a situation, and to be able to flexibly move back and forth across them, is critical. Teachers should ask students, "What does each representation reveal that the others don't, and what does each obscure?"

Our goal for mathematics teaching must be real conceptual understanding, and that means that at least some of the time, if not most of the time, students must work on complex, real-world problems, building mathematical models.

Models are mental maps, representations of relationships. They are ideas, constructs, schemata that have been generalized across a number of contexts. Perhaps this means that the problems students work on should be authentic to them in some way. But it definitely means they cannot thrive on a diet that consists only of the pablum of word problems. Students need experience with viable contexts to *mathematize* (a wonderful little word that signifies developing a way to conceive of or interpret a situation mathematically). To do so, they will need to be fluent in creating representations that capture the relationships in the situations.

Some excellent examples of this kind of "mathematizing" using multiple representations can be seen in the way one teacher builds an understanding of multiplication and division. Pam Regan, a third-grade teacher in River Forest, Illinois, had her students do a variety of problems and activities with equal groups. The kids used multilink cubes to create a physical model of all the multiplicative relationships they studied. For instance, when a story problem required them to think of a number of weeks, she would have them make chunks of seven multilink cubes to show the number of days (e.g., four weeks was represented by four chunks of seven cubes). They snapped the cubes together, threading strong nylon string through the holes in the hollow multilink cubes, and mounting it on a pegboard by passing the string through the pegboard's holes. See Figure 3.17.

Pam also worked with arrays much like Beverly did. Then she began a major unit on *area* during the first marking period. She used three major models (equal groups, arrays, and area) with activities designed to promote generalization of the concept of multiplication. For area, she started with a large number of real ceramic square tiles obtained from odd lots at a tile outlet. Ceramic tiles rarely come in whole number of inches for their side lengths. Therefore each size tile was a different nonstandard unit for measuring two-dimensional size (area). She gave each pair of children a box with about a dozen tiles. The dozen tiles in each box were identical in size, but colors did vary. There were three different sizes of square tiles across the pairs of kids in the classroom, $1\frac{3}{4}$, $2\frac{5}{16}$, and $4\frac{1}{4}$ inches. The students did not know these measurements. They decided to refer to these as small, medium, and large tiles.

Pam asked if they had ever seen tiles like this before. A few recognized them as being similar to what was in their bathrooms. Pam asked, "Where were they in the bathroom? Were they on floors, walls, counter tops?" Answers varied. She asked, "Does anyone have these in their kitchen?" One stated emphatically, "*Yes, but they are much bigger.*" " How big?" The child put two hands up and separated them by about six inches. Pam pulled out of a large manila envelope a light blue square tile that was 7.75 inches on each side. Some of the kids said, "Wow!"

Each kid had an identical desk with a flat rectangular surface on top. Pam asked them, "How many tiles would it take to completely cover your desktop? You'd have to make a rectangle out of your square tiles. How many would you need?" She taped a sheet of newsprint to the wall that had those questions on it for all to see. Although her students did not know it, the desktops were 16 inches by 25 inches (but they had rounded edges so the table top was a little less on each side).

True to form, many of the pairs within a few minutes complained that they did not have enough to completely cover the desktop. The teacher asked, "Did you go through our KWC?" Some kids looked a little sheepish. "OK, let's do it. What do I know for sure?" *"I have a bunch of tiles."* Long pause. She asked, "What can you know about the tiles?" *"They are ceramic." "They come in different colors." "Ours are all the same color." "Ours are not."" Ours are all the same size."*

"What is the next question?" *"What do you want to figure out?"* "Okay. What *do* you want to figure out?

"How many tiles we'd need to completely cover our desktop."

Some again complained that they didn't have enough tiles. Others said to them, *"You don't need to cover it to figure out how many you'd need."* Some thought about that for a minute and retorted, *"But it would be much easier."* The teacher asked, "Okay. What is the third question?" *"Are there any special conditions, or is there something weird about this problem?"* The teacher and the kids together had modified the third question, because she often used weird problems.

The kids thought about the question and finally one said, *"I don't think ours will fit right."* "Please explain what you mean." *"Well, we made a whole row of our tiles across the desk and they don't go all the way to the other side. If we put one more on it would fall off. So we can't completely cover the top."* The teacher made sure that everyone understood and then asked, "How would we *interpret* the problem in light of this information?" This question is a deliberate link to the modeling perspective. The kids thought about this for a while. One asked, *"Can we use parts of a tile?"* The teacher said, "In real life, when professional tile artisans are doing this, they may cut the tile to make it fit." They discussed what an artisan is. They talked a little more, but the teacher ended up suggesting that they rethink the problem to be "how many tiles will completely fit on the desktop." To help them agree to this wording, she brought out a small number of cross-like "spacers" and showed them how they would fit next to the corners of each tile. Each spacer only took up a little space, but when a dozen or so were used, one could not pack the tiles tightly together, and the tiles would come closer to the edge of the desktop.

The students proceeded to place as many tiles as they could in rows and columns, then added up the equal-sized rows (or columns) needed. The desktops were all the same size, but the students found quite different answers according to the size of tiles they used. Some groups found that it would take 104 of the small tiles. They determined that it would take 8 tiles on one side and 13 tiles on the other. But that left a lot of border around the edges that they could not cover. The medium tiles would have to be in a 6 by 10 arrangement and the kids with these tiles easily counted by tens to find they'd need 60 tiles. The large tiles could fit only 15 in a 3 by 5 arrangement.

As the teacher debriefed the class, her major question was, "Why was there such a big difference in the numbers of tiles needed: from 15 to 60 to 104?" The students realized that fewer large tiles could cover the desktop. The larger the tile, the fewer one would need. The focus of the debriefing was twofold: (1) on the basic principle of measurement and division: the smaller the unit, the more you need (and its converse); and (2) when measuring area one must think in terms of squares. This second point is a critical one because many textbooks and teachers deal with area only as $l \times w = A$. "Look at the picture of the rectangle. Take the two numbers and multiply them and then say square something afterward."

In contrast, Pam arranged this and several other activities to encourage them to think with squares when examining two-dimensional size. Over the next week Pam gave kids the opportunity to measure a wide variety of rectangular surfaces with these three ceramic tile squares. She cut out large rectangles of cardboard from appliance boxes. Like the desktop, they had to determine how many tiles of each size could fit.

They were "thinking in squares" with nonstandard squares. She then helped them see the value of working with standard square units. A key point is that with nonstandard tiles, you may know that the large tile is bigger than the small tile, but how much bigger?

Over the following weeks Pam had them working with one-inch square tiles as Beverly had her students do: making all the rectangles possible with 24, recording them on graph paper, making tables and writing the equations. Figure 3.18 shows three tables that each started with all rectangles or rectangular arrays that can be made from 24 square-inch tiles (rectangles of area 24 square inches.) It was a simple matter to transform those tables into equations. Teachers add the additional symbols in a different color to keep the integrity of the table, while emphasizing the inherent relationships of multiplication and division. There are many examples in the K–8 curriculum where one can help students understand mathematical relationships by first expressing them in a table and then allowing the patterns in the table to help them see the equation or formula (most abstract representations) that is readily created. After 24, they repeated the process with 20, 28, 32, and so forth.

horizontal	vertical
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

rows	columns
1	x 24 =
2	x 12 =
3	x 8 =
4	x 6 =
6	x 4 =
8	x 3 =
12	x 2 =
24	x 1 =

length	width
24 ÷ 1 =	24
24 ÷ 2 =	12
24 ÷ 3 =	8
24 ÷ 4 =	6
24 ÷ 6 =	4
24 ÷ 8 =	3
24 ÷ 12 =	2
24 ÷ 24 =	1

FIGURE 3.18

Both of these teachers' students spent lots of time working in one context with multiple representations: language, objects, pictures, lists, tables, equations. Then they had the students work in other contexts, again using multiple representations. In each context or situation, the children learned to move flexibly from one representation to another with full understanding and appreciation of what each representation shows.

USING MULTIPLE REPRESENTATIONS TO CONNECT CONCEPTS AND PROCEDURES

Some of the most difficult of the math-to-math connections are between concepts and procedures, made even more difficult because many people don't see the need to connect them. In the "how to teach mathematics" debate (or "war"), one of the battlegrounds concerns procedures or algorithms, ways people have developed to conduct efficiently some kind of operations or set of operations. Procedures tend to be general and therefore can be applied to a wide range of contexts. So when dividing a number by a fraction or mixed number, if you invert the divisor and then multiply, you will always get the right answer. This procedure will work for any divisor that is a fraction (any fraction) and any number, whole number, integer, fraction. It even works for decimals and mixed numbers (although you are supposed to turn them into an improper fraction or an immoral fraction, but it will never rise to the level of an indictable fraction).

But why does this work? Being able to memorize *how to do* the procedure to get the right answer and understanding *why* are two very different kinds of knowledge. (To my way of thinking, knowing why includes knowing how, even though you may not have practiced it a thousand times.) Many parents in the United States have given up on ever knowing *why* things work in mathematics. So when their kids can get the right answer by using a procedure, regardless of conceptual understanding, they are satisfied. One problem is that for most of us, this lack of conceptual understanding is cumulative and it all eventually catches up with us.

A second reason to be wary of memorized procedures can be seen when a student invokes a procedure in the wrong situation. Part of conceptual understanding is knowing *when* to use a particular procedure. For instance, ask a class of fourth or fifth graders, What is the mean? They will likely say something approximating, "It's when you add up all the numbers and divide by how many numbers you added up." But that is not what the mean is; that is the procedure for calculating the mean. Sometimes to illustrate this point I will ask all the students at one table or on one side of the room to one at a time tell me their phone numbers (seven digits). On my calculator I add up the phone numbers, divide by the number of people, and recite to them what is their *average* phone number. I take out my cell phone and tell them I am calling the average person who is just like them. They get a kick out of that, but it gets them

thinking about the differences between numbers that are on an equal interval scale and numbers that are just markers for something else.

Psychologists are now seeing that people who have conceptual understanding and organized knowledge are able to "conditionalize" what they know. They understand the conditions (contexts or situations) under which it is appropriate to use it, when to apply it, or where it works. Conditionalized knowledge does not come from memorizing things you don't understand. It comes from generalizing from a variety of similar contexts, representing one's conceptions, creating mental models, communicating with others, and consciously reflecting on what you are doing.

A third reason for being cautious about procedures comes from research with young children first learning mathematics. Connie Kamii and others' research (Kamii and Dominick 1998; Kamii 1994; Mack 1990) shows that the premature imposition of standard, traditional, efficient, general procedures and algorithms actually is harmful. Why? How could that be? Children come to school with some fairly good ways to figure out what they need to do in many real-life situations involving mathematics. Their home-grown ways make sense to them. However, in school we tell them to forget about the way they did it on the street and "learn the right way" to compute. This may be harmful in two respects: (1) they stop relying on their own reasoning and sense-making and (2) most of the procedures for multidigit computation taught in school require that the child ignore the base ten, place-value structure of our number system. Just at the time when they are building this crucial knowledge, they are required to abandon it. "Write down the 2, carry the 1." See Figure 3.19.

```
   1
   1 4
 x 1 3
 -----
   4 2
 1 4
 -----
 1 8 2
```

FIGURE 3.19

This action is incomprehensible to most kids. However, it can be understood by using multiple representations of the partial products and the rectangle/area model. Situations abound in students' lives where they encounter rectangles and need to find the areas.

I was in two fourth-grade classrooms helping the teachers try a different approach to multidigit multiplication. I gave each kid a zip bag of base ten blocks with 1 one-hundred flat, 10 ten-sticks, and 20 little unit cubes. By exchanging ten-sticks between two base ten sets, one yellow and one blue, I was able to give each kid blue ten-sticks while the other two sizes were yellow. The bag also contained yellow and blue crayons, one of each. The teachers in third grade and these two teachers in fourth had done a great job of helping the kids understand area. In fact, one of them had made up the language device of referring to the "Area Sod Company" and the "Perimeter Fence Company." In this suburb of Chicago, Glen Ellyn, the kids are very familiar with large rectangles of sod.

I told the kids that I was planning to convert a room at home to a home office. I asked them to help me figure out the area of the room in square feet because I wanted to cover the floor with one-foot-square tiles. I held up a one-foot square of cardboard. I gave them each a piece of centimeter-square graph paper. "The room is a rectangle, exactly 14 feet by 13 feet. Please draw the perimeter of this rectangle on your graph paper. Remember the scale is one centimeter equals one foot and one square centimeter equals one square foot." They did.

"We are going to use our base ten blocks to help us figure out the area of this 13 by 14 rectangle in square centimeters and then we'll know how many square feet of tiles I'll need. Please fill this 13 by 14 border with your base ten blocks." From previous work, the kids knew that the base ten blocks were made of cubic centimeters so that when one lays them flat on a table or paper the surface of the blocks can be measured in square centimeters. When everyone had filled their rectangles, I showed them a way to group all the ones together in one corner opposite the hundred. I told them that this way will make it easier to see what is going on. See Figure 3.20.

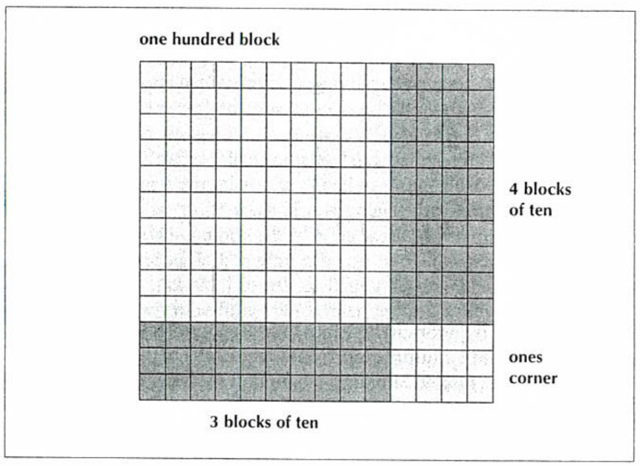


FIGURE 3.20

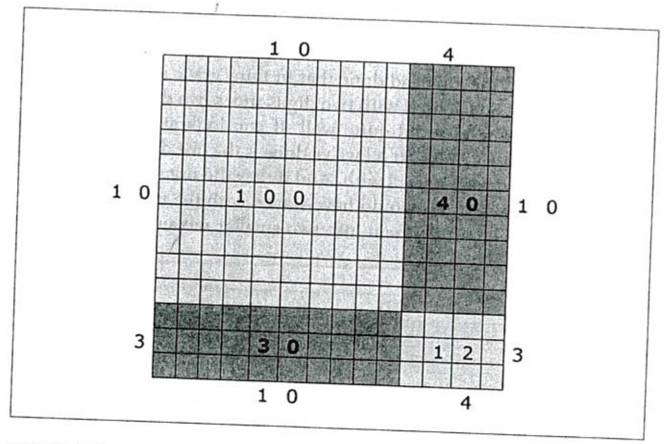


FIGURE 3.21

Then I asked the kids to remove the base ten blocks and use the appropriate color crayon to color in where the blocks were. Next I asked them think about how one big rectangle had been cut into four smaller ones and then write down how long the sides of these rectangles were. See Figure 3.21.

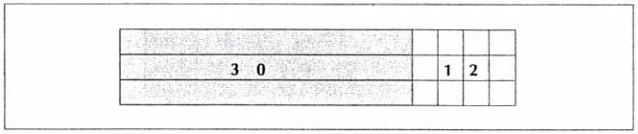
I asked, "How can we figure out the area of the big rectangle?" A dozen kids chimed in, *"Figure out the areas of the four smaller ones and add them up."* And so they added 100 + 40 + 30 + 12 = 182. I wrote on the board and said, "3 times 4 is 12. Do you see a 3 by 4 rectangle? Write down the 12 square centimeters of its area." Next, I pointed to the 1 in the 14 and asked them what that number meant. Several volunteered that it meant one ten. Another said it comes from 14 being 10 + 4. See Figure 3.22.

I told them the next thing we'd do was to multiply the 3 times 10. "Do you see a 3 by 10 rectangle?" They did. "What is its area?" They said, *"30."* I showed them where to write it down.

```
  1 4                      1 4
x 1 3                    x 1 3
-----                    -----
  1 2    3 x 4 = 12        1 2    3 x  4 = 12
                           3 0    3 x 10 = 30
```

FIGURE 3.22

I have a confession to make. My usual way to introduce these partial products is starting with one digit times two digits. But in this case, the two teachers said the kids had done that in third grade and they had reviewed it already. So I started with two digits times two digits to see what would happen. The kids did quite well. If you look at the two partial products in Figure 3.23, so far all we've done is the one digit by two (that is 3 times 14) that we'd started with. And if they did not remember doing this in the previous year, I would have stayed with the two partial products of 3 times 14, cutting them into two smaller rectangles and adding their area. See Figure 3.23.



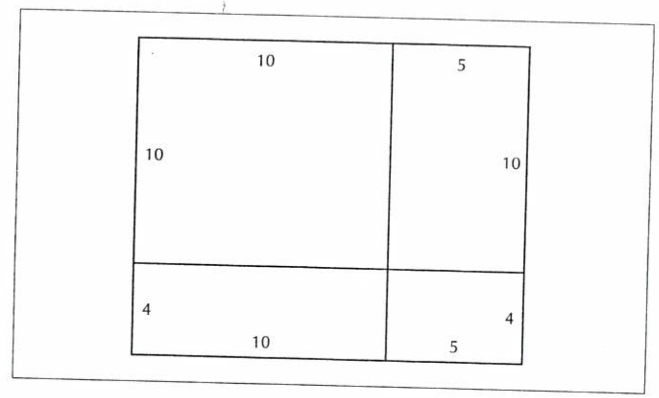

```
    1 4
x     3
-------
    1 2
    3 0
-------
    4 2
```

FIGURE 3.23

Let's go back to the original problem of 13 by 14. The four rectangles would give us four partial products. The tricky maneuver is the 10 times 4. It is much easier to "see" why it is 10 times 4 by looking at the four rectangles. I asked, "Do you see a 10 by 4 rectangle? What color is it?" The 10 by 10 is of course the hundred flat, which is easy for them to see. Then they added the four partial products to find the overall product of 182. See Figure 3.24.

```
    1 4
x   1 3
-------
    1 2        3 x  4 = 12
    3 0        3 x 10 = 30
    4 0       10 x  4 = 40
  1 0 0       10 x 10 = 100
-------
  1 8 2
```

FIGURE 3.24




```
    1 5
x   1 4
-------
    2 0        4 x  5 = 20
    4 0        4 x 10 = 40
    5 0       10 x  5 = 50
  1 0 0       10 x 10 = 100
-------
  2 1 0
```

FIGURE 3.25

We did another example in that math period in the same way. The two teachers followed up in the next few days with a sequence I showed them that I have done many times. Here it is. The next day, I usually come in and act very surprised to find there are no base ten blocks. I tell them that I'll bet they learned it so well yesterday that we can use graph paper without the blocks. "I am thinking of a rectangle that is 14 centimeters by 15 centimeters. What is its area?" I walk them through (1) drawing the perimeter on the graph paper, then (2) drawing a vertical line and a horizontal line separating the two dimensions at their place value. We have our four rectangles. Finally, (3) they merely find the areas of the four to find the total area. See Figure 3.25.

I usually have them do two or three more of these and on the next day, I tell them in mock horror, "I forgot the graph paper! What will we do? I have an idea. You have seen your teacher and me just quickly draw rectangles on the board. It doesn't have to be to scale. Let's try it. Take out a clean sheet of paper. I am thinking of a rectangle that has dimensions 19 feet by 28 feet. What is the area? Draw the perimeter.

Just make one side longer than the other. Lable the lengths of the sides. Now put in the cross bars. Where do they go? Break the sides between the tens and the ones." See Figure 3.26.

Even with an ugly drawing like this one, the kids can conceive of what is going on. And they say the area is 532 square feet. When I ask them how they calculated it, some say they added 200 + 80 + 180 + 72. Others added these same four numbers (the partial products), but in a different order. Some kids become incredibly adept at the mental math of doing this in their heads. I have yet to find a kid who, if taught the partial products in this "multi-rep" way, fails to understand what is going on in multidigit multiplication. Everyone gains immensely from this combination of representations.

Some teachers basically stop here because the understanding is so good and just use calculators from this point on. Others, including me, use this base of conceptual understanding to help them see why the traditional algorithm works. It takes about twenty minutes for them to make that connection. Essentially, the four partial products that are made explicit by the rectangle model are folded into two partial products in the traditional algorithm. I have found that it is worth having the kids connect the procedure to the concept and not just use calculators. First, someone is going to show it to them anyway, and they won't try to make it as sensible as we can. Second, it helps students believe that math is understandable.

Each of these representations is building understanding, building part of the snowman; they should not be skipped. Some textbooks show the four partial products without the rectangles. To my way of thinking that defeats the power of the approach. Some skip the step when the kids make the rectangle with base ten blocks. They assume that this step is unnecessary. They think the kids can get the idea from just looking at pictures, like those in this book. I can assure you that anyone and everyone profits from making the representations of the physical model and

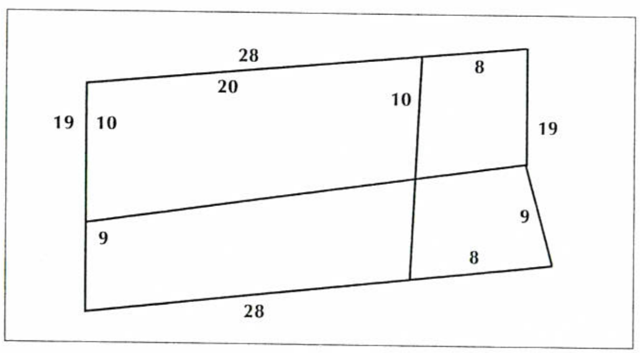


FIGURE 3.26

the drawings. Some kids are in desperate need of those sensory modalities to spark initial understanding. Others who are less in need of them will appear to be able to move to more abstract representations or even the mental computation, but they may be rehearsing something they have memorized.

I know many people who believe they understand some aspect of mathematics, but actually have only procedural knowledge. I also know many people who were taught only procedures, but by their own interest or persistence kept trying to make sense of what was going on and, in effect, they inductively derived a fairly good conceptual understanding on their own. Now they think that it is fine to teach only procedures because the kids "will pick it up later. It will make sense later." This is a dangerous assumption. Even more dangerous are those who believe that the optimal order is to use algorithms first because one cannot understand concepts until they've done a lot of work with algorithms. Working a bunch of similar problems with an algorithm is *not* the same as working with examples of concepts in a context. The former avoids the concept, the latter provides good examples of the concept.

CONSIDERATIONS IN PLANNING FOR PROBLEM SOLVING

Language Representations

How do I talk about the concept or ask questions to reveal connections or promote reflection?
How can I model thought processes, strategies, practices to encourage both cognitive and metacognitive processes?
How can I incorporate reading, writing, speaking, and listening into the activities?
How can I help the students use journals to document, reflect upon, and refine their thinking?
How can I help them explain their representations (orally/in writing)?

Other Representations

How do I scaffold experiences to move from concrete to abstract?
What manipulatives or physical objects can help students see what is going on?
Should they draw a picture of objects or of the situation/problem as they imagine it?
Does the situation contain a sequence of actions that students might act out?
Should they record information in a list and later organize it into a table?
What symbols are essential for them to understand?
How does each symbol specifically relate to the situation, objects, or pictures?

The Braid Model of Problem Solving

New entries from Chapter 3 are in italics.

Understanding the problem/Reading the story

Visualization

Do I see pictures in my mind? How do they help me understand the situation?

Imagine the SITUATION

Asking Questions (and Discussing the problem in small groups)

K: What do I know for sure?

W: What do I want to know, figure out, find out, or do?

C: Are there any special conditions, rules, or tricks I have to watch out for?

Making Connections

Math to Self

What does this situation remind me of?

Have I ever been in any situation like this?

Math to World

Is this related to anything I've seen in social studies or science, the arts?

Or related to things I've seen anywhere?

Math to Math

What is the main idea from mathematics that is happening here?

Where have I seen that idea before?

What are some other math ideas that are related to this one?

Can I use them to help me with this problem?

Planning how to solve the problem

What REPRESENTATIONS can I use to help me solve the problem?

Which problem-solving strategy will help me the most in this situation?

Make a model	*Draw a picture*	*Make an organized list*
Act it out	*Make a table*	*Write an equation*
Find a pattern	*Use logical reasoning*	*Draw a diagram*
Work backward	*Solve a simpler problem*	*Predict and test*

Carrying out the plan/Solving the problem

Work on the problem using a strategy.

Do I see any PATTERNS?

Looking back/Checking

Does my answer make sense for the problem?

Is there a pattern that makes the answer reasonable?

What CONNECTIONS link this problem and answer to the big ideas of mathematics?