INTRODUCTION
Braiding Mathematics, Language, and Thinking

[The] human ability—to imagine the future taking several paths, and to make adaptable plans in response to our imaginings—is, in essence, the source of mathematics and language. . . . [T]hinking mathematically is just a specialized form of using our language facility.

—Keith Devlin, *The Math Gene*, 2000

THREE KEY PIECES

I have said to many elementary school teachers over the past fifteen years, "If you have learned ways to help kids to think effectively and understand ideas in reading and language, use them in math and you won't be disappointed." These teachers know quite a bit about reading and language. This suggestion is a good one, but it is only part of the story. Certainly, if you use what you know works in reading, language arts, and writing to teach mathematics, you will get some good results. Teachers enthusiastically report significant involvement and understanding from their students. However, teachers can go beyond simply applying aspects of reading comprehension to mathematics. I have taught hundreds of teachers (and even reading specialists) in scores of courses and workshops how to do more through *braiding* together mathematics, language, and thinking.

If you set out to integrate mathematics and language (or problem solving and reading), what would you put together? What aspects of language fit nicely with mathematics? How about vocabulary? Or writing in math journals? However, the real question is what guides you in determining exactly what the kids should do and how you teach them. Asking the language arts people will only get you what they do in their knowledge domain. Concepts in mathematics are very different from concepts in language. They don't really know the domain of mathematical knowledge or the process of mathematical thinking.

Nonetheless, cognitive processes underlie both language and mathematics. In fact, some theorists think that the same kind of cognition underlies both. Therefore, I see several key principles from cognitive psychology that can guide us. I have used the term *braiding* to indicate that thinking, language, and mathematics can be braided together into a tightly knit entity like a rope that is stronger than the individual strands. When these three important processes are braided, the result is stronger, more durable, and more powerful than any one could be by itself.

For many years, the teaching of mathematics and especially problem solving has suffered from insufficient attention to thinking and language. If you want students to understand mathematical ideas, they must use both language and thought. Trying to put more thinking into the math curriculum or one's teaching without attention to language will be fruitless and so will trying to use language without thinking. The term *braiding* here suggests that the three components are inseparable, mutually supportive, and necessary.

DEATH, TAXES, AND MATHEMATICS

There are two things in life that we can be certain of . . . everyone knows the answer . . . death and taxes. Add another item to that list: at least half of our nation's fifth graders hate story problems. Actually, they dislike math in general, but story or word problems hold a special place of loathing in their souls. This percentage may vary a bit from classroom to classroom. Research has shown that most children start kindergarten with some fairly good ways to solve mathematical problems in the sandbox or with toys and games. However, during their first four or five years of school, they abandon their previously successful ways of dealing with problems involving mathematics.

Have you ever watched three children trying to figure out how to split up a bunch of candies? If the candies are identical, they just give one to each, then another, then another until they run out. Obviously influenced by Long John Silver of *Treasure Island* fame, they refer to this process as "divvying" the candies up. The children have not memorized division facts; they just have a way to do what is necessary. If the candies do not come out evenly (they often check to make sure everyone got the same number of candies), they may use some probabilistic device (e.g., flip a coin, odds or evens of total fingers displayed, paper/scissors/rock) to see who gets the extras. Children can be remarkably resourceful when no adults are around to tell them what to do.

No. I am not advocating *Lord of the Flies*. The question is: what forces are at work to dampen our children's inherent awe and wonder, their excitement about learning, and their facility with mathematics? The early years of schooling present children with a torrent of messages that they must *do* math in one particular way, that there is one right answer, and one right way to find it. They are told what to memorize, shown the proper way to write down problems and answers in symbolic notation, and given a satchel full of gimmicks they don't understand. No matter that they don't understand what they are trying to memorize. Does anyone ask if the symbols make any sense to them? Does anyone notice that they have ceased to trust their own reasoning or intuition borne of experience? Many children (and adults too) see mathematics as arcane, mystical. They believe that understanding mathematics is beyond most mere mortals. Can the children get the answer quickly, efficiently, and accurately? That's all that matters.

Of course, I believe that it is necessary for children to learn the basic math facts of the four operations. As much as I love problem solving, I know that students will be hampered in their problem solving (especially in estimation and determining reasonableness) if they don't know basic arithmetic facts. The question is not *if* those facts are learned, but how and when. All students should understand and be able to use number concepts, operations, and computational procedures. There are several critically important processes, each with a critical cognitive component, that lead to understanding, proficiency, and fluency that need to be developed. When students have many successful experiences using these processes, remembering math facts becomes a simple matter.

These processes are: *counting* (building one-to-one correspondence and number sense); *number relations* (decomposing and recomposing quantities to see relationships among the numbers); *place value* (creating sets of ten with objects and beginning to understand the base ten, positional notation); the *meaning of the operations* (creating mental maps of different situations and realizing that operations have multiple meanings); and *fact strategies* (thinking strategies for learning the facts for the operations).

Acting along with some erroneous beliefs about computation is another perhaps even more sinister force. Most people deny the importance of language in the world of mathematics. An exaggeration? Textbook publishers are very sensitive to the feedback from teachers whose message has been crystal clear for years: too many words on the page will make learning too hard for the kids who can't read well. "Johnny is not a good reader. Math is the only subject he likes (or does well in). Just let him work with the numbers." So what happens when Johnny has to read the story problem? The teacher is ready with a magic trick: just look for the *key* word (*cue* word) that will tell you what operation to use. If you see *sum* or *all together*, you add the numbers. If you see *take away* or *difference*, then subtract the smaller number from the bigger number.

What is the fundamental message the kids get when told to look for the *key/cue* word? Don't read the problem. Don't imagine the situation. Ignore that context. Abandon your prior knowledge. Who cares about metacognition, metaphors, metamorphosis, metatarsals, whatever? You don't have to read; you don't have to think. Just grab the numbers and compute. After all, you've got a 25 percent chance of randomly selecting the correct operation.

This situation, all too prevalent in U.S. schools, discourages kids from thinking. That makes no sense. Both reading and mathematics require thinking. Teachers should use every means possible to encourage students to think, reflect, question, imagine. And how do we do that? With language—with expressive language (speaking and writing) and with receptive language (listening and reading). And they all fit together in a child's life and in the classroom.

There is yet another critical reason to braid language, thinking, and mathematics. When math is taught with the language pruned or purged, who is immediately penalized? Those who use language as their primary means of processing ideas; those who develop their language facility early. *Girls*. Of course, all children can profit from discussing, verbalizing thoughts, talking mathematics, but girls develop language strengths earlier than boys and, when encouraged, can use them effectively to build mathematical understanding. Girls understand the value of braiding language, thinking, and mathematics; they even get the metaphor.

A LITTLE FORESHADOWING

Consider the following ideas, each culled from the literature on reading and language learning and well known to most elementary school teachers. Each of these ideas has a solid foundation in cognition. I have simply played those ideas through mathematics and problem solving. Like the television game show, *Jeopardy*, I have worded the idea in the form of a question.

1. Are students expected to *construct their own meaning* in mathematics?
2. Are students encouraged to have *ownership of their problem solving*—to choose to use mathematics for purposes they set for themselves? What would ownership look like?
3. Are students encouraged to do problem solving for *authentic purposes*? What would *authentic* mathematics look like?
4. Are students encouraged to do *voluntary mathematics*, selecting tasks for information, pleasure, or to fulfill personal goals?
5. How is mathematics instruction *scaffolded*?
6. Does the school help teachers and students build a *rich, mathematically literate environment* or community?
7. Are students encouraged to see the *big picture, important concepts, vital connections* versus isolated pieces of mathematics?
8. Is *forgiveness* granted to students in mathematics? Is making *mistakes a natural* part of learning? Is doing mathematics seen as a dynamic process that incorporates *planning, drafting, revising, editing, and publishing*?

Is it heresy for an unapologetically passionate teacher of mathematics to believe that we could do a far, far better job of teaching children how to understand and love mathematics if we did all of these things?

READING COMPREHENSION

What is reading? Sounds like a silly question, but if you have been following the "reading wars" in the past twenty years (and you may have also followed the "math wars"), there are a whole lot of people who believe that reading *is* decoding, phonics, and word attack skills. No respectable educator would argue that these things are not part of the process of reading and learning to read, but they do not *define* it. By analogy, arithmetic computational proficiency and math facts are part of mathematics, but they do not define it. Mathematics *is* the science of patterns. Neither reading nor math is a collection of skills or subskills. I have no intention of addressing all aspects of reading and language in this book, nor should I. Instead, I will draw selectively from the experts whom I admire.

Reading is the process of constructing meaning from written language. *Reading is thinking*. Constructing meaning does involve decoding, but in greater measure, it is a very dynamic process requiring some very special *thinking* about what one knows already (prior knowledge) and one's experiences (especially with language). Readers *interact* with what they read. They do not passively receive its meaning, they construct it. They use what they know about the content of the text, about the context being described, about how texts of this kind are structured (their format), and about the particular vocabulary (including specialized terms). They must continually draw inferences about the meaning of the words. They must make assumptions about missing pieces, things implied but not there on the page. For proficient readers, this is all done effortlessly and largely unconsciously. As complex as all these processes are, throw into the mix that these things are greatly facilitated by the readers' metacognitive monitoring of what they are doing and a metacognitive awareness of their own ways of operating, being able to reflect on their own ways of thinking (as if looking at your mind in a four-dimensional mirror).

Some people ask: "Do children really do all those things? Don't they just learn phonics and listen to people talk and they figure out how to read?" In psychology, they call that kind of thinking "the black box." We don't know what goes on inside kids' heads. It is a black box. We can't read their minds. We can only go by their behavior. Such an approach is dangerous. And yet, I hear it frequently in both reading and math. A significant portion of adults in this country believe that if they could just memorize the math facts, they'd be fine in subsequent mathematics courses. Ironically, some people who are involved in one of the biggest "reform" curricula in math have said that if teachers just give the kids some good math to do, by "osmosis" they will construct meaning. OSMOSIS!

Those who believe in such osmosis may lack an understanding of cognition and metacognition, and how a teacher can facilitate meaning-making. Teachers model, show, ask questions, make suggestions, and create a safe, supportive, rich, literate environment in which students can explore ideas and interests. Teachers also can mediate between the larger world and the world of the child. Sometimes they explain things. Language, especially oral language, is used continually throughout these processes.

Reading Comprehension Strategies

Research on reading has identified several highly effective cognitive strategies for students to use in reading comprehension. Specific teaching techniques for helping students with these strategies have been developed. With minor differences in terminology among experts in the field, they are:

- *making connections* (activating relevant prior knowledge, linking what is in the text to their own experiences, discerning the context; relating what is in the text to other things they've read, things in the real world, to phenomena around them);
- *asking questions* (actively wondering, raising uncertainties, considering possibilities, searching for relationships, making up "what if" scenarios);
- *visualizing* (imagining the situation or people being described, making mental pictures or images);
- *inferring and predicting* (interpreting, drawing conclusions, hypothesizing);
- *determining importance* (analyzing essential elements);
- *synthesizing* (finding patterns, summarizing, retelling);
- *metacognitive monitoring* (actively keeping track of their thinking, adjusting strategies to fit what they are reading).

When teachers focus on these cognitive strategies in a variety of different text genres, students can learn to use those strategies independently and flexibly. The cognitive strategies are taught most effectively in a reading workshop that includes (1) crafting lessons with direct, explicit instruction and modeling by the teacher, (2) students applying the content of the crafting lesson, and (3) students reflecting at the end of the reading workshop (Public Education and Business Coalition [PEBC] 2004).

During crafting lessons, teachers introduce and explain a new strategy. They think aloud as they read, modeling their own use of that strategy for their students and carefully explaining how they are applying the new strategy to the text. After the crafting lesson, students spend large amounts of time applying the content of the crafting lesson to their own reading experiences. During this time, students might meet in small, needs- or interest-based groups, or read independently. Teachers spend this time guiding small groups of students as they negotiate a common text or a common instructional need, or conferring with individuals as they work to make sense of their reading materials. At the end of the reading workshop, students regularly share their insights about the content of this reflection can vary, depending on purpose. The teacher participates in the reflecting, offering observations and recording the individual and group needs generated by this process. The goal is for students to internalize these strategies and use them easily.

These seven strategies are fairly broad and incorporate quite a few pieces. Reading experts have also developed more focused strategies. For instance, K-W-L (Know-Want to know- Learn) is a method of having kids think about key ideas before, during, and after reading. QAR (Question-Answer-Relationship) is a method of asking questions while reading. Each chapter of the book addresses one of the broader reading comprehension strategies listed above. It is not my intention to be comprehensive. There are many marvelous books cited in the references list at the end of the book that provide a wealth of examples. In each chapter, I have tried to include only the key elements of reading and language, identified by principles of cognition, that can be braided with mathematics. The chapters will be both cumulative and recursive. Later chapters will incorporate previous ideas into the main strategy being discussed and also will revisit previous ideas to build a deeper understanding of them, in light of the new strategy.

MATHEMATICAL THINKING AND PROBLEM SOLVING

The National Council of Teachers of Mathematics (NCTM) developed a set of principles and standards for mathematics curriculum, teaching, and assessment in 1989. NCTM produced a revised version in 2000 with standards addressing five broad strands of mathematics K–12: *Number and Computation; Algebra; Geometry; Measurement; Data and Probability*. Each of the strands is chock full of powerful mathematical concepts. In order for students to learn those concepts with deep understanding, the NCTM Standards address five processes in which students must be engaged: *problem solving, connections, reasoning and proof, communication,* and *representations*.

Despite their complexity, I infer that the NCTM standards are based on three big (that is, foundational) ideas:

1. Math is the science of patterns; it is much more than arithmetic. Every strand of math has certain patterns that we look for. Probably every concept in mathematics is a pattern of some kind.
2. The goal of mathematics teaching should be understanding concepts, not merely memorizing facts and procedures. Therefore, we must use what we know about cognition.
3. For children to understand mathematical concepts, they must use language, the quintessential characteristic of human cognition.

The five content standards have provoked a lot of dialogue about the concepts of the elementary school curriculum. Less progress has been *doing* mathematics and as a powerful vehicle for building understanding of mathematical concepts. This would be a shocker for most of my old math teachers, for whom problem solving was an afterthought, something that the kids did after they'd been taught the concept or procedure. Today we can see that by using well-constructed problems, worthwhile mathematical tasks, the use of good strategies, and with the teacher's facilitation, students can construct deeper meaning for concepts by actually using the mathematics they know.

Granted, the majority of classrooms in the United States may be using textbooks that are hanging on to a conception of problem solving as determining what computational procedure to use. The problems may be reminiscent of the settlers leaving Missouri bound for Oregon. For example, "Hattie needs 5 and 7/8 yards of muslin at 27 cents per yard. What will it cost?" They are generally called "routine" problems or "translation" problems (translating the description of the situation into an equation). Somewhat more complex story problems usually entail a string of consecutive computational procedures in order to find the correct answer.

In the 1950s George Polya helped to broaden our sense of problem solving by describing heuristics or strategies that college students could use in their mathematics classes. By the early 1980s the idea of strategies found its way into the school curriculum and most textbooks introduced students to a dozen or so problem-solving strategies. A problem was seen as a task for which the person confronting it wants to find a solution, but for which there is not a readily accessible procedure that guarantees or completely determines the solution. Consider the case of a student confronted with the question, "Sacks of flour cost $4.85 per sack; how much would you pay for ten sacks?" If this student understands that there are ten sacks and each one costs the same amount ($4.85) and realizes that the answer can be readily determined by repeated addition or by multiplication, is this a "problem" for her or him? No, it is a thinly disguised drill exercise, perhaps valuable, but not a problem.

A number of math educators have infused textbooks with problems that require students to do more than merely determine which operation(s) to use, moving beyond what educators see as translation problems. They created "nonroutine" problems, what some called "process" problems, in which a good process of thinking was required. For instance, "How many different ways can a person make change for a quarter?" There is no obvious computation procedure to invoke. You have to figure it out. Try it.

If you tried it, did you find twelve different ways? But how did you work it? Did you make a list? Did you draw a picture? Did you make a table? If so, you chose a problem-solving strategy. Strategies can help students find solutions. They may also help them understand the problem.

In the 1980s some educators came to think of problem solving as an "art" in which mathematicians (as well as regular humans) worked on perplexing problems. They placed problem solving at the heart of math-

from an "initial state" to a "goal state" and strategies were to be used when one was "stuck" and did not know what to do next to move toward the end goal.

A serious drawback to this view is that it treats problem solving as a process independent of content. Strategies tend to be seen as generic, applicable to anything, and able to be mastered, like a skill or a procedure. For me, the term *problem-solving skill* is an oxymoron. Skills are physical in nature, requiring a certain amount of innate ability and massive amounts of practice, but with minimal thought or reasoning. In contrast, problem solving is clearly a cognitive venture. How you think and what you think about are intimately related. Analyzing a poem and analyzing a spreadsheet of data are very different processes. That they are similar in that both "break things down" becomes completely irrelevant when one is immersed in the task.

There is yet another way that some math educators are conceiving of problem solving and strategies. Some use a perspective referred to as *modeling* or *creating models* in which problem solving serves primarily to *interpret* the problem. Similar to what some would call *task definition*, this broader approach to problem solving emphasizes the need for interpretation, description, elaboration, and explanation of the nature of the problem. This perspective recognizes the importance of the context, content, and the concepts of the problem. The solution to problems is often the building of a model using particular concepts that are still being developed by the students. In this view, the purpose of the strategies is to help students refine, revise, and extend their ideas, especially through interaction with others.

The point is the kids have to *do* the math. How does a student get better at solving problems? What is the best way to get better at reading? By reading more. Of course, a third grader can't simply pick up Kant and make any sense out of him. (Come to think of it, I can't either.) Problems should be challenging, but not overwhelming.

The dilemma goes even deeper. The NCTM process standard termed *Connections* encourages a wide variety of links within mathematics. For decades the mathematics curriculum has consisted of little, bite-sized chunks of mathematical knowledge. I am not speaking only of narrowly defined skills (as in skill and drill), although some still cling to the erroneous belief that if children crank out a gazillion math facts they know how to do mathematics. I am concerned here with the fragmentation of concepts into isolated compartments that is contrary to the NCTM admonition that students need to see that mathematics is a coherent whole. Many concepts are connected to a multitude of others. Even when teachers go after conceptual understanding, the curriculum treats that concept in isolation from its related concepts. To make matters worse, the curriculum deals with a topic for two weeks and then ignores it for a year.

My wife and I always knew when it was April 1 because the fraction worksheets would come home to be stuck on the refrigerator with magcome, they just disappeared one day (usually at the deadline for filing income tax returns—I am not sure what the connection is).What are kids to make of this phenomenon? Fractions only exist during these two weeks? Nobody thinks about them or uses them at any other time during the year.

What about the highly touted "spiraling" curriculum, which does not expect mastery (or understanding?) the first time a child encounters a concept because it will be back two more times during the year? The issue is not how many times or how often one revisits a concept, but the nature and quality of the experience. Is it conceptually rich? Have the students built a solid initial foundation to begin using the concept in a way that is meaningful to them? Some spiraling math curricula are more like tornados. What happens when the tornado touches down? It briefly stirs things up, and then leaves for an indeterminate time. If its touchdown time is riddled with gimmicks, what do the kids have to show for their brief encounter with the mathematics of that moment?

Another NCTM process standard not fully developed in the United States is *representations*. It was barely mentioned in the 1989 version of the standards. Therefore, in 1991 my wife and I wrote *Mathwise*, in which we made a strong pitch for the critical importance of creating representations in doing mathematics. Furthermore, we asserted that of the ten popular problem-solving strategies, there are five strategies that are based on representations, two that were so broad as to be metastrategies that should be used all the time, and three that were fairly narrow and should be considered supplementary.

When using the five most powerful strategies, students *create their own representations*. Through this creation, they are truly *constructing meaning*. These five strategies are:

- Discuss the problem in small groups (language representations using auditory sense).
- Use manipulatives (concrete, physical representations using tactile sense).
- Act It Out (representations of sequential actions using bodily kinesthetic sense).
- Draw a picture, diagram, or graph (pictorial representations using visual sense).
- Make a list or table (symbolic representations often requiring abstract reasoning).

Language should be used throughout all five of these strategies.

The two common strategies of looking for a pattern and using logical reasoning *always* should be used in problem solving. Mathematics is the science of patterns; every branch of mathematics (e.g., numbers, geometry, measurement, data and chance, algebra) has characteristic patterns. Logical reasoning is essential to doing mathematics. But is it a strategy? Is it something one chooses to do instead of something else? Okay. What is the alternative? "Pay attention, kids: we've been reasoning illogically all

year long in math; now it is time for a new strategy. Let's use logical reasoning for a change!"

Children delight in seeing patterns in mathematics or truly understanding a concept. Consequently, I am concerned when children's books present seeing mathematical patterns everywhere as a "curse," mathematics as magic or as witchcraft practiced by the Number Devil. I believe it is a big mistake to tell children that mathematics is magical or incomprehensible while at the same time trying to help them believe in their own capabilities and that they can *expect* it to make sense through diligent work. The books may be cute, but can send a decidedly mixed message.

When students are taught how to look for patterns and reason logically in every activity along with the five representational strategies, the representations they create build understanding of the problem (and lead to a solution). In creating them, students are developing different *mental models* of the problem or phenomena. In rich, meaningful mathematical tasks, students may use several of these representations, moving from one to another to figure out more about the problem. Later they might draw on supplementary strategies (such as the three popular ones: guess and check, work backwards, simplify problem), but these cannot be used effectively unless one *understands* the problem. As students become more mathematically sophisticated, they are able to use more abstract and symbolic strategies (e.g., use proportional reasoning, apply a formula).

Obviously, thinking is critically important in reading and language as well as mathematics and problem solving. It is beyond the scope of this book, let alone this chapter, to adequately address cognition or all the related cognitive issues. Fortunately two wonderful volumes do a fine job of just that. They are *How People Learn* (Bransford, Brown, and Cocking 2000) and its companion, *How Students Learn* (Donovan and Bransford 2005), which uses three main principles to synthesize a tremendous amount of information about human cognition:

1. engaging prior understandings (using prior knowledge, confronting preconceptions and misconceptions)
2. organizing knowledge (developing a deep foundation of factual knowledge organized into coherent conceptual frameworks that reflect contexts for application and knowing when to use which information—referred to as conditionalized knowledge)
3. monitoring and reflecting on one's learning (developing metacognitive processes and self-regulatory capabilities)

HOW DO ALL THESE IDEAS FIT TOGETHER?

Fitting all these ideas together is not an easy task. May I phone a friend? In Figure I.1, I have simply placed the major ideas that need to be braided into three separate clouds. Imagine that each cloud was an overhead transparency that could be placed like a template on top of one of the

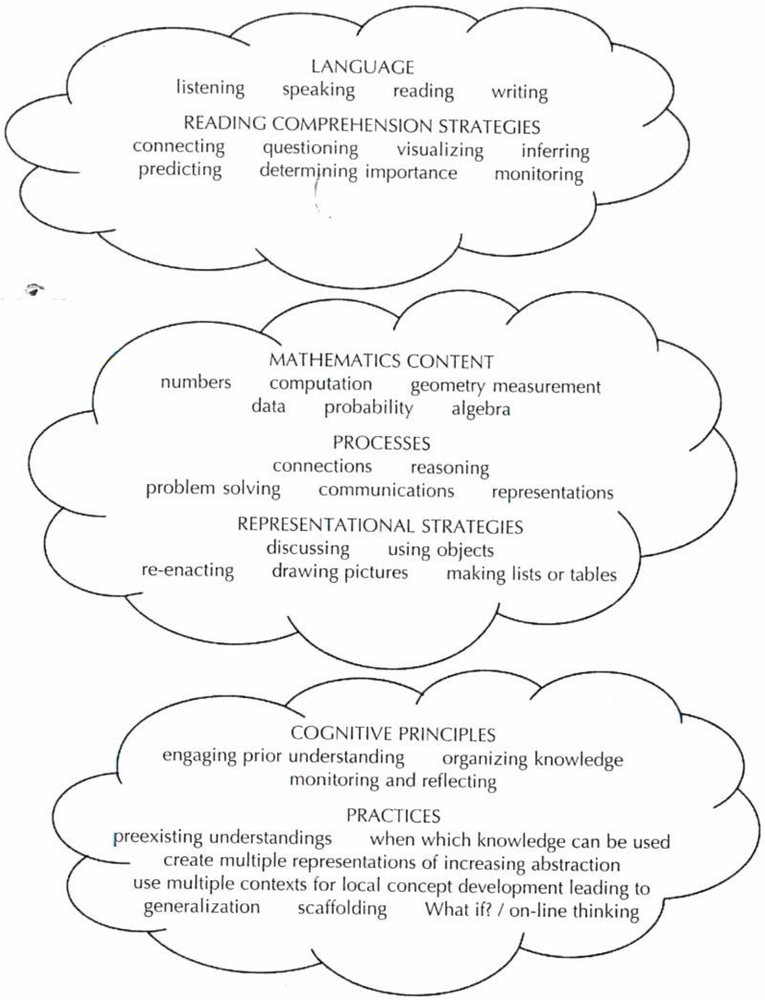


FIGURE I.1

Place the template of thinking over the mathematics and one can see the importance of connecting to prior knowledge, activating relevant schemata, building organized knowledge, and developing self-awareness and monitoring as a learner. Place the template of language over the mathematics and one can imagine some wonderful opportunities for language—expressive, spoken, written, responsive, interactive, dynamic, mercurial, creative, playful, clarifying, metaphoric, defining, and so on. All of these can be a part of the mathematical experience of all students.

From among many ways to organize the braiding of these ideas, this book devotes six chapters to the reading comprehension strategies (asking questions, making connections, visualization, inferring and predicting, determining importance, and synthesizing). The seventh strategy (metacognitive monitoring) involves students becoming increasingly self-regulating when doing the other six strategies. In Chapter 1 I discuss how the strategy of asking questions is essential to metacognition. With these six big ideas to organize the chapters, I explain what each of these strategies contains and then braid in first the cognitive ideas and then the mathematics. No doubt more clever minds than mine would have organized this material differently and better. I am also certain that astute readers will note ideas and relationships that I have overlooked. That is inevitable and I look forward to hearing about them.

Knowing what works is good, but even better is knowing *why*. That is where theory comes in. Two books by mathematician Keith Devlin, *The Math Gene* (2000) and *The Math Instinct* (2005), and one written jointly by linguist George Lakoff and mathematician Raphael Nuñez, *Where Mathematics Comes From* (2000), suggest some rather important linkages between mathematical thinking and language. I could not do these books justice in a hundred pages, let alone one or two. Let me simply say that as the quotation at the beginning of this chapter stated, Devlin believes mathematical thinking is a specialized form of our human language facility. Both developed in humans in parallel tracks from the same source—the ability to reason abstractly in a "What if?" mode. "[M]athematicians think about mathematical objects and the mathematical relationships between them using the same mental faculties that the majority of people use to think about other people. . . . The overall mechanism is the same: a mental capacity developed to handle things in the real world is applied to an abstract world that the mind creates" (2000, 262–63).

Lakoff and Nuñez (2000) have a somewhat different nominee for this mechanism. Humans conceptualize abstract concepts in concrete terms by means of *conceptual metaphor,* "a cognitive mechanism for allowing us to reason about one kind of thing as if it were another. This means that metaphor is not simply a linguistic phenomenon, a mere figure of speech. Rather, it is a cognitive mechanism that belongs to the realm of thought" (p. 6). In fact, they see it as the principal cognitive mechanism: ". . . much of the 'abstraction' of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor, often over the course of centuries" (p. 7).

If you want more theory, these authors will oblige. Bear in mind that they distinguish between arithmetic and mathematics. In our popular way of looking at learning, people often assume that language and mathematics are incompatible or just too different. Yet it seems each day, week, or year we discover ever more powerful and surprising human capabilities!

1 | ASKING QUESTIONS

INQUIRING MINDS WANT TO KNOW

Children are natural inquirers; they have a million questions. Most have no trouble asking anyone just about anything. At age four my daughter asked me, "Do some men work alone?" I thought the question a bit odd, but simply told her that some do. After getting the same question four days in a row, I finally asked her why she wanted to know. She said that she just thought men would probably smell better if they wore cologne.

If I had allowed her question to initiate a discussion, instead of just trying to answer her and be done with it, I might have learned sooner what she was thinking. Language and thought go hand in hand (or perhaps, synapse in synapse). The point is a simple one: the more you know yourself and how your mind works, the better able you are to solve problems. The more we teachers can stimulate our students to be aware of their own thinking and monitor it, the better readers they will become and the better mathematical problem solvers.

About a dozen years ago a sixth-grade teacher had attended an intensive week-long summer math problem-solving course I taught in her district. When I ran into her in the supermarket a year later, she said she had enjoyed the course, had read *Mathwise,* and had elected to do her master's degree thesis on problem solving. In her research project, she had found that the most important things that she did with her kids were the suggestions on just two pages of the book. What helped them the most was developing their metacognitive awareness and monitoring through what we are now calling the *KWC*. This little device, which we'll get to shortly, is built around *students asking questions*.

The research literature on metacognition in reading is voluminous, far more extensive than in mathematics. All of the reading comprehension strategies involve metacognition in some way or to some extent. Proficient readers have learned how to use or to adapt various strategies to different purposes. They are able to use strategies to "fix up" or "repair meaning" when they don't understand what they are reading. They are quick to recognize when they have encountered some obstacle to meaning. They stop, go back to clarify, and reread to try to construct the meaning. They *think* and they *ask questions*. They try to determine the author's purpose. They "interrogate the text and the author." They do not passively "receive meaning." They aggressively grab it. They wonder about the choices the author made when composing. They realize that one question may lead to others. They recognize that their questions are important. They believe that their questions will help them understand.

They ask questions about what they read before, after, and right in the middle of reading. They do this to help them understand, to construct meaning, to discover new information, to clarify what is going on, to check their inferences, to help them visualize, and a dozen other purposes, all of which require thinking and metathinking.

Related to the metacognitive purposes for asking questions is the value of consciously surfacing one's prior knowledge so that relevant information can be brought to bear on the text (or problem at hand). In reading circles this is known as "activating relevant schemata." Although many educators and psychologists use that term, they do not all mean the same thing. For some, schema means the knowledge and how it is structured. For others, it means both the way the information is organized and how it is habitually used (like a script). For our purposes here let's just say that a schema is the accumulated background knowledge and experience about something and how aspects of it are connected or organized.

The extensive use of asking questions to activate schemata or prior knowledge in reading contrasts sharply with the typical mode of children working on math story problems in the intermediate grades or middle school. Contrary to educational naysayers, most kids are capable of learning. By the time they get to fourth or fifth grade, they have figured out that the name of the game is memorization and quick recall of facts. The rules of the game are guess-the-operation (and we're not talking about tonsillectomies) and don't make any arithmetic errors.

They passively glance at the words of the story problem and perhaps they might see a cue/key word. The questions they ask are not especially metacognitive: "Can we do this for homework?" "Are we getting graded on this?" Usually followed by, "I don't get it." And then the truly important question: "What do I do?" Ironically, that is the right question, but by asking the teacher, they are asking the wrong person. Instead of asking that of the teacher in the hopes that she will come over and show them what to do (and maybe even do it for them), they should be asking that of themselves. The earlier in their school lives that teachers encourage students to stop and think, "Okay. What do *I* do? And why?" the better off they'll be in mathematics.

My personal belief is that as students experience more and more schooling in math, and continually find teachers emphasizing the one right answer obtained quickly, they consciously or subconsciously think that asking questions is a sign of not knowing. Therefore much of their energy is invested in covering up when they do not know. When I began my pilgrimages to elementary classrooms, I recall being very surprised at how quite a few students in the first grade would "tighten up" during math time. These were very capable students who usually did quite well in math. They were very intent on getting the right answers and even more intent on not making a mistake.

I remember one girl who despite my reassurances that she could simply erase the one thing that she had missed (a trivial error), with tears starting to form in her eyes, insisted on throwing away the entire paper and doing everything again. Over my protestations she crumpled up her paper, threw it in the trash, took a clean sheet of paper and began again. In a third-grade classroom, a teacher handed back a math quiz and complimented the boy who had done the best in the class. He had a nearly perfect paper. But the key word here is "nearly." He looked at his paper, began to cry, and ran into the coat closet. He pulled the door shut and would not come out. He had made a mistake. Were the behaviors of these two children anomalies? Are these two children in therapy today? No, I don't think so on either question. Though they may be at the far end of the anxiety scale, they have plenty of company.

How can we encourage children to ask questions in math class? We can establish a climate of acceptance, where mistakes are a natural part of learning, where successive approximation is valued. The classroom should be a place where students have initial ideas, write drafts, and then think some more, rewite, revise. I do not mean that we don't tell the kid that he's made a mistake or that anyone gets credit for doing it wrong. I simply mean that we promise to *forgive* our students. We let them know that everybody makes mistakes in math. They just need to move on and practice. Use whatever metaphor you want to help them get the point.

ASKING QUESTIONS OF THEMSELVES, THE TEXT, AND THE AUTHOR

The reading comprehension strategies interact and are not independent of one another. There is no absolute sequence implied. The math problem solving of most students by fourth grade suffers from a profound lack of thinking and questioning. Therefore, we think it makes sense to nudge them strongly toward a habit of mind of asking questions as the first strategy to consider. Good questioning by the students will greatly facilitate the other strategies.

There are several ways of helping students learn how to ask questions that the reading folks have found very useful. In general, the teacher does the kinds of things described in the previous chapter on crafting lessons. The teacher thinks aloud as she reads and models her own use of a particular strategy with the whole class. She explains how she is using the strategy. Then the students try doing it in pairs or small groups with the teacher conferring with them. She may also meet with individuals to guide them. The objective is for all students to individually internalize the strategy.

Some teachers read a text out loud and articulate questions that make sense to them as adults. They often record such questions on large sheets of chart paper. Keene and Zimmermann (1997) saw how important their own modeling was. Children needed to see and hear how their teachers read, especially the way they worded the questions they asked themselves while reading (p. 109).

Keene and Zimmermann advise teachers to talk to the children about why readers pose questions, how questions help them comprehend more deeply, and how they use questions in other academic areas.

Debbie Miller (2002) reminds us that through such actions, teachers help students reflect on whether or not the answers to their questions can be found in the text or if they will need to infer the answer from the text, their background knowledge, or some outside source. We need to help students understand that many of the most intriguing questions are not explicitly answered in the text, but are left to the reader's interpretation (p. 140).

These same teacher-moves are echoed by Harvey and Goudvis (2000) and they are always on the lookout for books that generate an abundance of questions from the students. Sometimes they have students list their questions on large chart paper. Sometimes they help students categorize these questions, such as: ones that are answered in the text, ones that are answered from someone's background knowledge, ones whose answers can be inferred from the text, ones that can be answered by further discussion, ones that require further research, and ones that signal confusion.

Taffy Raphael (1982) uses QAR (Question—Answer—Relationship) to help elementary school students see the value of different kinds of questions. She helps them ask "right-there" questions (literal questions that can be answered by finding the answer directly in the text). These contrast with "think-and-search" questions that are inferential, requiring students to put various pieces of information together.

Distinguishing between the literal meaning and inferential meaning is critical in story problems and in fact all of mathematics. Just as some students do not read much of the text in a story problem, others read very selectively and make inferences (and assumptions) without realizing that they are doing so. When questioned, kids often say, "I just figured that . . ." But if they are aware that they are inferring something not literally in the text, then they can ask themselves, "Is this inference accurate?" If they're not aware, they can't ask. Although distinctions among types of inferential questions are fascinating, for our purposes here we can say that we are thrilled when students can distinguish between the literal and the inferential in math problem solving.

To understand literal meaning requires making connections between what is there on the page and what is in your head (prior knowledge). But to understand inferential meaning (drawing an accurate inference) you are making connections among several things in your head and then going beyond the literal; making a connection among those ideas in your prior knowledge to realize something that you didn't see until you made the "leap" of inference. Actually, social scientists distinguish between low inference (small leap) and high inference (a big jump from what is literally there). In Chapter 4 we will address drawing inferences as an important aspect of reading comprehension.

A very well-researched and proven strategy for engaging students in informational text is Reciprocal Teaching, developed by Palinscar and Brown (1984), in which the teacher models each of four different strategies (summarizing, questioning, clarifying, predicting). When the teacher feels that the students have sufficient experience with the strategies, she turns the teacher role over to the students. They must guide the class in the continued use of these strategies with the text. The students learn to ask good questions of the text.

Students learn to think more about who has written a text and how successful the writer was for them as readers through the strategy of "questioning the author" (Beck et al. 1997). According to Blachowicz and Ogle (2001, 116), students "develop a dialogue with the author, just as they would with a person talking with them face to face." Such questions may be: What was the author trying to say? What could the author have said instead? What was the intent of the author? What is the point of view? How could something be stated more clearly?

Donna Ogle developed the K-W-L strategy to help students become engaged in reading informational texts. It is a "frontloaded" strategy, activating schemata to make sure students have the knowledge they need *before* they read (Daniels and Bizar 2005, 41). Prior to reading, the teacher essentially asks the students, "What do you know?" about a particular topic. She models and guides students through a group process of "brainstorming together what they *know* (the K in K-W-L) about the topic. The teacher guides students to probe their knowledge statements and to find conflicting or partial statements" (Blachowicz and Ogle 2001, 108). The teacher writes what the students say without evaluation or correction in the first column (K) of a three-column chart on the chalkboard, overhead transparency, newsprint, or computer. The brainstorming encourages students to think about the topic, to activate their knowledge, and to develop their interest before reading. The teacher can encourage more student engagement by continuing the thinking: "Does anyone know anything [more] about. . . . Can anyone frame a question that may help us find out more?" (Blachowicz and Ogle 2001, 108).

Inevitably, during the brainstorming/discussion process some questions and uncertainties come up. The teacher writes these down under a second column (W) as they signify things they *want* to know. The teacher next has to decide if the students can think more deeply about the topic;

she may ask them to think of ways that experts might organize this information that they have generated. They might also look at the information in the K column and try to find connections. These are recorded also. Next the teacher asks them to come up with real questions for the W column—what we want to know. The teacher can readily extend the questions in this column by asking "I wonder" questions. The third column (L) is for students to summarize what they have *learned* after reading the text. In a follow-up to the K-W-L, K-W-L + (plus) was developed to give students more responsibility for reorganizing the ideas involved in the K-W-L. Graphic organizer sheets are given to students so that they can record the K, W, and L information as it emerges in the brainstorming and discussions. Or they might create semantic maps of the key information. Ogle believes that it is extremely important for retention that each student write down his or her own ideas before, during, and after reading. Also the writing helps them monitor their own thinking and learning. This writing task is a concrete way for all students to continue to participate in the thinking (Blachowicz and Ogle 2001, 111).

SECOND GRADERS USE THE KWC

For more than a dozen years teachers and I have been refining an approach to engage students in understanding word/story problems. Initially we tried applying Donna Ogle's KWL to story problems. Although that worked well sometimes, we realized that it was not a simple application that was needed, but rather a transformation of the KWL into a mathematical tool, one that dealt directly with the essence of mathematical problems. We now use a KWC that becomes a framework for other strategies as well.

Probably the best way to understand what the KWC is would be to experience it as a student would. Come with me into Betty Hogan's second-grade classroom, where as soon as you enter the room your eyes are splashed with bright, vibrant colors. Dozens of posters fill the walls, books overflow their bookcases. And twenty adorable children there reside. Most noticeable are the long, twisted strands of crepe paper hanging in great boughs from the ceiling. We are in the Amazon rain forest. Betty is a veteran teacher with a master's degree in reading and language. Math is her least favorite subject to teach. But she has taken some courses with me and is willing for us to team on a KWC with her second graders.

Betty arranged the twenty kids into pairs and explained that I was going to tell them a story and then they were going to do some mathematics based on the story. We pulled out a large sheet of chart paper that had been rolled up. We taped it up on the wall and covered it with another sheet, revealing only the title, "The Freight Trains."

We asked the class a series of questions. What is a freight train? Have you ever been on a train? What kind of train was it? What is freight? What do you think this short story is about? The kids were eager to tell us things that we asked and to volunteer personal experiences with trains. One offered, "Yesterday we had to wait for a very real long time for a train to go past." I asked about the train and the cars that the engine was pulling and how long did they have to wait for the train to go by. We spent five full minutes just talking about trains, and the students had quite a lot of information to share. Of course, they are second graders and they repeated themselves. Sometimes their answers were a little off the mark, and I thought I was in a movie of an Amelia Bedelia book. But they were definitely trying to answer our questions. As Betty had predicted, most did not know what freight was, but some did and we asked them to explain.

We continued to slide the paper down to reveal the first full sentence:

At the train station there are many different freight trains.

Then the second:

They carry 3 kinds of freight across the US: lumber, livestock, and vegetables.

We again asked the children questions about the trains. They took about ten minutes discussing the three different kinds of freight. We went through each sentence one at a time, asking questions to get the kids to clarify what was meant. Here is the whole problem.

The Freight Trains

At the train station there are many different freight trains.

They carry 3 kinds of freight across the US: lumber, livestock, and vegetables.

Each train has some lumber cars, some livestock cars, and some vegetables cars.

Each train always has 18 freight cars.

There are never more than 10 cars of one kind.

Freight cars that are the same are always connected together.

How many different ways of making trains with 18 cars can you find?

Those familiar with second graders and their level of cognitive abilities may believe this problem too difficult for second grade. Betty thought they would be challenged by it, stretched in a good way. They had done a few KWCs before. They had done multiple addends in math. It was May in the school year, so they were "mature" second graders.

Like the KWL, the KWC expects the students to express their ideas under three big headings.

K: What do I Know for sure?
W: What do I Want to do, figure out, find out?
C: Are there any special Conditions, rules or tricks I have to watch out for?

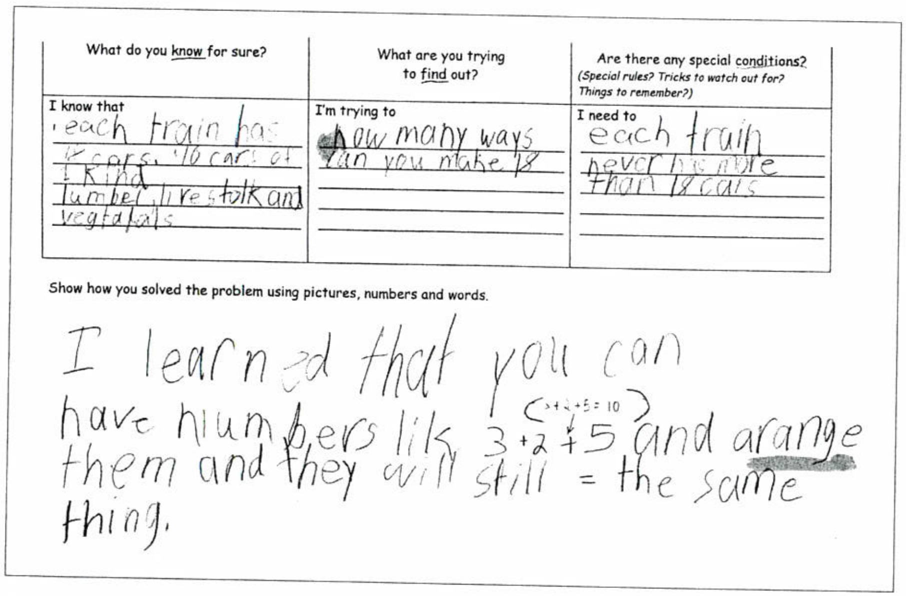
The students wrote down their thoughts and answers to the questions.

For the K, most copied parts of the original problem (despite Betty gently prodding them to "Say it your own way"). There were a few kids who rephrased information for the K column, such as: "Only 18 cars" or "10 cars per kind." In the W column sixteen of them simply copied the wording of the question asked by the problem. Four were a little different. Two wrote, "Find out how many different ways I can make a freight train." Two wrote, "Find out how many ways I can make 18."

In the C column (special Conditions), most of the students wrote down specific pieces of information from the problem, sometimes things they had mentioned in K, sometimes others things from the problem. Several were reminding themselves, "Remember, only use 18 cars." Sixteen of them referred to either using only 10 of each or 18 altogether. Four were a bit different. One wrote, "I need to remember to put the same kind of cars together." Another wrote, "Not take all the same kind." Two girls who sat and worked together both wrote, "Don't do the same thing twice." See four student papers in Figures 1.1–1.4.

Betty handed out to each group 10 Unifix cubes of three different colors that would represent the three kinds of freight cars (green for vegetable cars, yellow for lumber cars, and red for livestock cars). Each group got one sheet of legal size paper with the outlines of four trains of 18 cars the same size as the Unifix cubes. The students took out markers, crayons, or colored pencils of those three colors and began. We encouraged them to make the train, check to make sure it met the conditions of the problem, then color it in. One child ignored these instructions and immediately colored in an all green (vegetable) train. I asked her to look at her KWC sheet and see if that was okay to do. She said, "Oh, oh!" and drew a big "X" over the train, and grabbed the Unfix cubes to make her next try.

The kids charged ahead famously, making trains that met the criteria, and soon asked for more sheets to make more trains. When most of the groups had completed two sheets, Betty told them to write under the train the number sentence that fit with the number of each kind that were

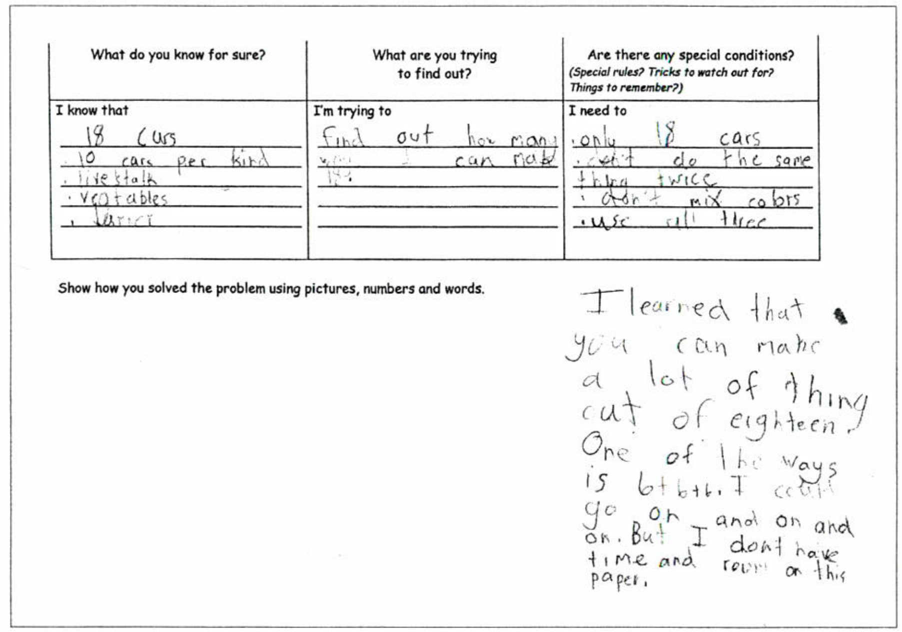


What do you know for sure?	What are you trying to find out?	Are there any special conditions? (Special rules? Tricks to watch out for? Things to remember?)
I know that each train has 18 cars. 10 cars of 1 kind lumber, livestolk and vegtabals	I'm trying to how many ways can you make 18	I need to each train never has more than 18 cars

Show how you solved the problem using pictures, numbers and words.

I learned that you can have numbers lik 3+2+5 (3+2+5=10) and arange them and they will still = the same thing.

FIGURE 1.1



What do you know for sure?	What are you trying to find out?	Are there any special conditions? (Special rules? Tricks to watch out for? Things to remember?)
I know that 18 cars. 10 cars per kind. livestalk. vegtables. lumber	I'm trying to Find out how many ways I can make 18	I need to only 18 cars. don't do the same thing twice. don't mix colors. use all three

Show how you solved the problem using pictures, numbers and words.

I learned that you can make a lot of thing out of eighteen. One of the ways is 6+6+6. I could go on and on and on. But I dont have time and room on this paper.

FIGURE 1.2

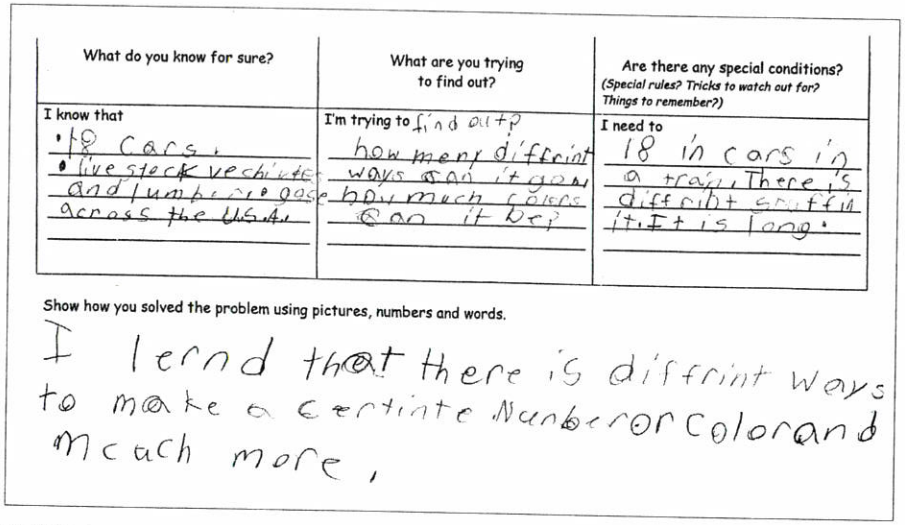


FIGURE 1.3

in it. When they had done that, they were to cut out each separate train. They now had a color-picture record of their work.

I asked for all the kids who had made a train where there was the same number of each kind of cars to hold up that train. It was a 6 + 6 + 6 = 18 train. I took one of these from a student near me and taped it on to a sheet of newsprint. It had 6 green cars on the left, 6 brown in the middle, and 6 red on the right. I asked if any one else had exactly this train. Several said yes. I asked them hold the train up and asked if these were the same as the one we had just put on the chart. Several children immediately said no and their classmates were puzzled.

I asked the students holding up 6-6-6 trains to compare and to contrast their train with the one on the chart, "Tell us what is the same and what is different." Several kids gave partial answers; this was tricky for them. They did explain eventually that the numbers were all the same but the colors from left to right were different. Betty told them that we could say that the "order" of the colors was different. I asked them to bring all the 6-6-6s forward and we taped them up. I asked them to figure out how many different trains we had on the chart. They checked carefully and found only six: (B, R, G); (B, G, R); (R, G, B); (R, B, G); (G, B, R); (G, R B). I then suggested that if the order of the colors (kinds of cars) does not matter, we could say these are all from the same "family"—the 6-6-6 family.

Then I asked if anyone had found a 5-6-7 train and we went through much the same "debriefing" as with the 6-6-6s, comparing and contrast-

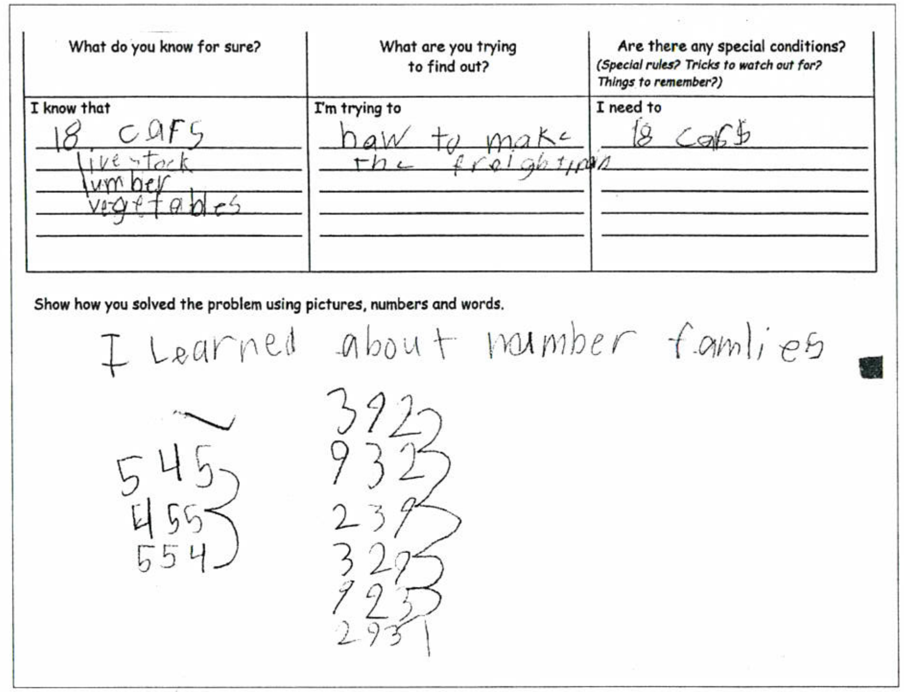


FIGURE 1.4

ing trains. This time I suggested that the order of the colors and the order of the numbers really does not matter; we can say these are all from the same family. We talked about it in terms of the number sentences. Betty reminded them that they could add together any of the addends and then add the third one and always get the same sum. I asked, "If we ignore colors and just look at the number sentences, how many members are in this family?" They found six: (5, 6, 7); (5, 7, 6); (6, 5, 7); (6, 7, 5); (7, 5, 6); (7, 6, 5). We decided that they were all in the same family. I told them that mathematicians name this family the 5-6-7 family, listing the numbers in order.

I then told them that we were going to try to determine what the different families were. How many were there? And what were they? At this point a girl who was sitting near me came over to me and very quietly asked, "Do you mean we are supposed to find all the combinations?" I whispered yes to her.

We asked them to save their paper trains and the next day Betty continued to work on finding all the families (combinations) with them.

DEBRIEFING THE ACTIVITY

What follows are my comments on why Betty and I did what we did. She put them into small groups according to her perception about who would work well together on this task. I have done this activity with second graders before and was able to describe the typical behavior and what I hoped to see. The small groups, in this case pairs, would give the students maximum opportunities to talk about their conceptions of the problem. The materials and their use in creating representations (i.e., the Unifix cubes, the paper strips, and the crayons, markers, colored pencils) could be shared more easily with two rather than three students.

We told them that we would read a story aloud and they could read along with us on the newsprint. Then they'd have to answer some questions about it, math questions. They were fine with that. Since math educators call them "story" problems, I give my problems a title. Therefore, the students' first encounter with the problem had the title of "The Freight Trains." By asking them to think about the meaning of the title, we were treating this experience like one of their reading activities, which, if positive, would carry over into the math activity. They had lots to say and many experiences to relate to this reading. They were motivated; they had bought in.

When we introduce the KWC to kids, it is not just to stimulate engagement (although that is valuable). They need to activate the schemata they have that will help them with the problem. For instance, we wanted them to imagine trains with an engine car pulling a great many freight cars. As we slid the paper down to reveal each sentence separately, we posed questions and also asked the kids to explain their responses. When teachers introduce the KWC they often make an overhead transparency of the problem, separating each sentence on different lines. Some kids get overwhelmed by the text of an entire problem. The overhead transparency allows the kids to focus on just one sentence at a time as the teacher covers what comes next.

After we had read and talked through the entire problem, they began to write information on the graphic organizer. Betty had added the second row of "prompts" to the KWC questions: "I know that . . . ," "I'm trying to . . . ," and "I need to . . ." She also added the italicized prompts in the first row under C (*Special rules? Tricks to watch out for? Things to remember?*). I have encouraged teachers to work with their classes to modify wording of the KWC questions in ways that will help their kids.

Writing things down under K is not a problem for kids, except when there is an abundance of information and they have not yet worked with the strategy for determining the most important information. Unlike

at the very end. Students know this text structure, and they are on the lookout for a question or a question mark. They have no trouble discerning the W and what to enter onto their sheet. The C is always tricky and this case was no exception.

In mathematical problem solving we have long had "givens, the goal, and the constraints." I will discuss in Chapter 5 how some believe that math story problems are a genre and we will examine alternatives to this genre. "Givens" have been established by the author of the story problem. In technical terms, one is in a "given state" and the problem solver is to figure out/find out how to get to the "goal state." The KWC asks the question, What do I Know for sure? so that students will think about the problem and generate the givens. Then the KWC asks, what do I Want to find out or figure out?

Constraints are special conditions, often limitations on what the problem solver/mathematician can do, or what possible values are allowed. These constraints are often the most difficult of the three things to discern. Sometimes they involve drawing an inference or making an assumption. Sometimes they are only obvious to an expert in the context of the problem and hard for a novice to see because they are only implied. On the other hand, sometimes students will generate all the necessary information in the problem, including the constraints, while looking for the K. Students and teachers usually need some examples of what we mean by C.

In the freight train problem the students had the opportunity of oral discussion as a whole group with the teacher modeling the thinking that goes on when we try to answer the three questions: What do you know for sure? What are you trying to find out? Are there any special conditions? They read through each of the sentences of the story one at a time as a class and then they wrote down their answers on their graphic organizer.

However, five of the twenty kids did not mention that each train had three different kinds of cars. Sometimes, kids will pick up a piece of information like that under C rather than K, but in this case they just did not mention it anywhere on their papers. However, all five students did *make* trains that had three different kinds and/or three different colors. They "knew" to do so. These second graders were learning how to write down information. The physical representation of the trains made it easy to realize the need for three colors.

In this problem we scaffolded the multiple representations and helped the kids go back and forth between them. The term *scaffolding* has several meanings. For some it means making sure that the task is challenging, but within their capabilities; I'd say, pitched just right to make them have to stretch to do it. Scaffolding can also mean that the teacher provides some process, structure, or device (tangible materials or a good question to prompt thinking) that enables the students to actually do the task. Scaffolding does not necessarily make the problem easier, and the teacher does not do the work for students or show them how to do it.

safely work on the outside wall, the scaffolding does not do the work. It enables the person to do it.

In the freight train problem we provided some scaffolding by requiring that: (1) they use oral and written language to describe the situation; (2) they physically make each train, then place it on the paper to check that they had 18; and (3) they had to make a color-picture record of each train. Finally, they counted the squares and wrote number sentences/equations with three addends that symbolically represented what they had done.

This progression of representations was moving from concrete to abstract: language and object (most concrete) to picture to symbols (most abstract). In the final stage, which began on day one and continued into day two, the students found different number families and made a table of them (used an abstract representation to look for patterns). The girl who recognized that we were looking for combinations might have been able to handle a more abstract version of the problem from the beginning. "What are all the combinations of three different addends that sum to 18 with the constraint that none can be greater than 10?"

The girl's insightful question brings up a related issue. There are general questions that kids can ask themselves that would fit with virtually any story problem (e.g., the three KWC questions). There are also questions that are very context dependent. For example, in the freight train problem kids often ask, "Can we alternate the colors like stripes?" Although it does not literally say "alternating colors is prohibited," the prohibition is implied and kids should realize that this is an inference they should have drawn from the statement, "Freight cars that are the same kind are always connected together." We will deal with inferences in a subsequent chapter. Here simply note that to a child who asked about alternating colors, we might ask things like, "What does the problem actually say?" "What do you think that means?" "Are you sure?" "Are you making an inference?" Asking questions keeps their thinking going. When we answer definitively, we give them permission to stop thinking.

Another kind of question is more *content* or *concept* dependent. The girl asked about "combinations," a very slippery concept when approached in the abstract. When kids study the concept of combinations (versus permutations), they use the KWC but we also help them learn to ask themselves several critical questions that are strictly related to the concept.

> *How many different combinations did you find?*
> *Did you check for duplicates/repeats? How?*
> *Did you find all the combinations?*

And the really important question is:

> ***How do you know when you've found them all?***

There are many different directions that we could have gone with Betty's second graders once they had generated a good number of trains with number sentences. For instance, this problem can be a very nice basis for the associative and commutative properties of addition. It can help students see the difference between combinations and

permutations (where order does matter). A longer discussion of what makes two things "different" might have been profitable. We might have used the paper trains to show the patterns as the numbers of each color were systematically changed in order, which can be a precursor to an organized list—keep one color the same number (start with the most it can be) and change the other two (e.g., 10, 7, 1; 10, 6, 2; and so on). When you place the paper trains side by side in that order the visual impact is considerable.

We chose to focus on making sure that the children were not overwhelmed by the sheer number of trains they had made. We wanted them to have a way of grouping them to be a more manageable number. This also served to lay a foundation for other ideas Betty would address later, such as associative and commutative properties. Just as important, they were developing their cognitive and metacognitive processes. They were learning how to attack problems by breaking them down analytically, by translating between representations they had created, and by talking through with teammates what was going on.

In the KWC, the teacher models for the whole class the process of asking herself the three critical questions. Then the teacher leads the whole class in asking the questions. Then the students work in small groups and ask each other those questions. Finally, each individual student should internalize these questions and use them when solving problems alone (on a test, at home, by oneself).

FRONTLOADING TO UNDERSTAND THE PROBLEM

In the introduction I mentioned George Polya, the grandfather of problem solving. He saw problem solving in four phases:

1. understanding the problem
2. developing a plan/considering various problem-solving strategies
3. implementing a strategy or plan
4. looking back to see if your answer makes sense

For Polya and many others who write on problem solving, the first phase is critically important and deserves to have significant time devoted to it—perhaps up to 75 percent of the time available. Does this sound excessive? Have you ever had a project given to you that was ill-defined or that made you initially uncertain how to do it? That would be a problem. Or have you ever worked on a group project that was not structured, where the participants had to make a lot of decisions about what to do and how to do it? In both these cases, spending initial time on *task definition*—understanding exactly what you have to do—is the smart way to proceed. If one "frontloads" the discussion to gain clarity on the task, it is time well spent because there is always danger in *assuming* one knows what is the problem and charging ahead to solve

The KWC is designed to encourage students to consciously attend to understanding the problem. It is the best device we have found to help with the understanding phase. It requires kids to stop and think, to get their minds around the problem. It is quite versatile and can be used with many different kinds of problems. The reading folks use the expression "activating relevant schemata," in which a reader brings to conscious awareness the prior knowledge that is related to the text at hand. If a student, even a very capable math student, goes charging ahead without really thinking about all elements of the story problem, there is a high probability of misreading or misunderstanding and operating in error. Each of the questions in the KWC asks the students to focus attention on a somewhat different but related part of the problem.

The students use the KWC to sort out basic factual information. In so doing, they must read the problem and think about the information. The teacher may ask them to jot down some notes, or to do it orally for the whole class, and she records the facts on the chalkboard. Although this recording may sound tedious, it establishes that the teacher believes it is important for the students to actively think about what they know to be true. They get the message that this is an important step. Notice that they cannot tell what facts are relevant or important until they have read or seen the question. I'll have more to say about this in Chapter 5. The key idea here is that they read carefully, attending to any information. As we saw in the freight train problem, while they are developing their ability in problem solving, the teacher may require them to consider only one sentence at a time by having the problem on an overhead transparency and by displaying a single sentence to be understood.

After students have read the question, they do the W— What are we trying to find out or prove? They should restate the problem in their own words. Next they must consider the special conditions. For some problems these are very real constraints on the possible values. In other cases, the information is so evident that students glean all they could possibly get from simply doing the K phase. Nevertheless, thinking about possible C statements keeps them thinking. It slows down the impulsive students who like to charge on ahead, willy-nilly, and gets them to be more careful and thoughtful. For those who need more guidance in problem solving, it provides a structure.

HOW THE K AND THE C WORK TOGETHER

The interplay of the K and the C can be seen in a different kind of problem done in the third-grade classroom of Beverly Kiss in Deerfield, Illinois, a number of years ago. We were working on fractions and gave the students what we called the Canteen Problem. As usual we had them ask the KWC questions. The problem was:

> A small plane carrying three people makes a forced landing in the desert. The people decide to split up and go in three different directions in search of an oasis. They agree to divide equally the food and water they have, which includes 15 identical canteens, 5 full of water, 5 half-full of water, and 5 empty. They will want to take the empty canteens with them in case they find an oasis. How can they equally divide the water and the canteens among themselves?

Some kids jumped right in with the KWC. Others seemed perplexed and were hesitant to volunteer K information. They finally asked some questions like, "How much money do they have with them?" "Will the people just follow the highway? Maybe a car will stop for them." Schemata were being activated all right, but not what Bev and I expected. By discussing the problem, their schema for "oasis" surfaced; it was a rest stop on the highway with a Baskin-Robbins and a Wendy's. They lived near an interstate highway with the Deerfield Oasis. The value of activating relevant schemata cannot be overestimated. Often students have in their heads accurate and useful information relevant to a problem, but fail to access it or to connect it to the problem at hand. Good problems, rich in mathematical ideas, often reside in specific contexts or real-life situations. After this incident, we started calling this problem "Meet Me at the Oasis," and before reading the problem, we asked the kids to tell us what they thought the story would be about. The question of what is an oasis always seems to pop up. I have shown a five-minute clip from a video of the movie *Jewel of the Nile* showing actor Danny DeVito cavorting with Sufis at the local oasis. That has started a lively dialogue about oases!

There are always some students who need the C with this problem. The problem includes explicit statements about dividing the canteens, even the empty ones. And it gives an explanation of why. Nevertheless, there are always a few students who do not catch this special condition or constraint. They try to divide only the water. Seriously considering the C question helps most students slow down, go back, read and reread, and catch the other idea. Once the students understand the problem and want to try solving it, we suggest drawing a picture. Some students quickly say, "Just pour half of each full canteen into an empty one. Then you'll have 15 half-full canteens and each person can take 5 of them."

When they tell me this, I respond, "How will you know you have exactly half?" Or "You don't want to pour because you might spill or some water might evaporate. Is there a way to solve this problem without pouring anything?" [A simple solution is for one person to take 2 full, 2 empty, and 1 half full—which is 5 of the 15 canteens, and one-third of the water; a second person does exactly the same thing; that will leave for the third person 5 canteens, each of which has the same total amount of water. They start with FFFFF HHHHH EEEEE (full, half, and empty). The use of manipulatives shows this relationship readily. {F F H E E } {F F H E E} or {F H H H E}.

SURFACING PRIOR KNOWLEDGE

A different type of problem illustrates a variation in using the KWC. In the Eight Shapes Problem, pairs of students (fifth grade and up) are given oaktag cutout versions of the shapes in Figure 1.5. The KWC begins with the W: "What do you know for sure about these shapes?" The question is often followed with, "What are the 'attributes' (or features, characteristics, properties) of these eight shapes?" I tell the kids, "I just gave you some manipulatives. Take a close look at them." Either I write down the information they share on the chalkboard or they write it on their graphic organizers. I will sometimes ask, "How many different shapes do you have?" There are only four different shapes, one pair of each. The key concept is congruence. The kids can discern congruent shapes by laying one on top of another.

Depending on grade level and experience, the students may fill the chalkboard with their prior knowledge. Thus, this little problem may be quite a valuable diagnostic activity. If they do not mention a key attribute that I want them to surface for this activity, I might ask, "What can you tell me about the corners (angles, vertices) of these shapes?" Usually most will realize that all the shapes have one square corner (some say right angle,

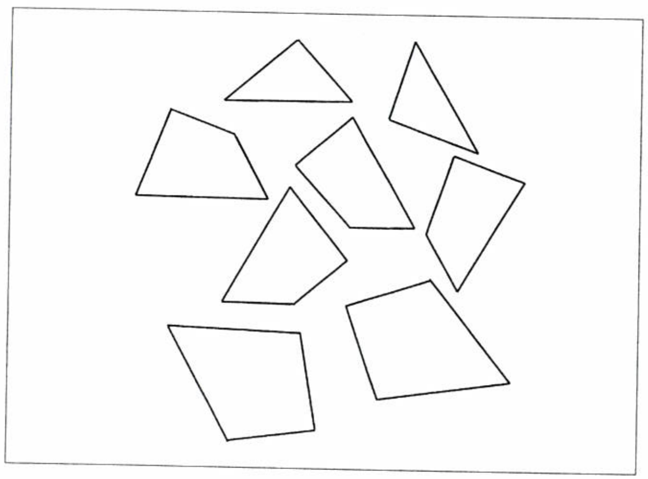


FIGURE 1.5

some say 90-degree angle). With continued questioning the teacher can get a feel for what the students understand about angles and degrees.

There are four different shapes; three are quadrilaterals, of no special name. The fourth shape is a triangle that has one right angle (and is called a right triangle). I ask them, "How would we prove that each shape has a right angle?" They often lay the shapes on top of each other. To this I respond, "That shows me that the angles are congruent, they are the same. But are they right angles?"

The students use various means to compare the four corners to a standard they "know" is a right angle, such as a piece of paper. Others stand two shapes next to one another on a table top with right angles on the table. Or they place the four quadrilaterals down on the table and move four alleged right angles together at a common point. If the four are equal and completely encircle the point (covering the full 360 degrees), all are 90-degree angles. Now comes the question that makes this a problem.

> Can you use these eight shapes to make a square from some and an equilateral triangle out of the others? And the two shapes must have the same area.

When we ask the kids the W question: What do I Want to do, to figure out, to find out? They readily say, make a square and an equilateral triangle. But there are always some (generally less than a third of the class) who ask, "Can we make a square with all eight pieces?" Then we ask them the C question: "Are there any special conditions here?" The problem is not simply asking for a square and an equilateral triangle. The constraint is that they must be of equal area. How can one guarantee that the two shapes have equal area?

Notice that the critical piece of information that needs to be activated and connected to the problem is all about congruence. If each of these four pieces is congruent to these other four pieces, then any polygon you make with the first four MUST have the same area as any polygon made with the other four shapes. If this critical concept does not surface/emerge when discussing the K, the students are rarely able to solve this problem.

The discussion about the right angles of the four shapes frequently leads students to place the four right angles at the four approximate corners and then interchange them or turn them over until they fit nicely together as a square. See Figure 1.6. However, most kids, in fact most adults, have trouble making the equilateral triangle. While discussing the K, some students activate potentially useful triangle schemata: an equilateral triangle has three 60-degree angles. Then after placing the four right angles in the four corners to help them make the square, they try to find three 60-degree angles, a marvelous use of prior knowledge.

By comparing the angles of shapes (laying them on top of one an-

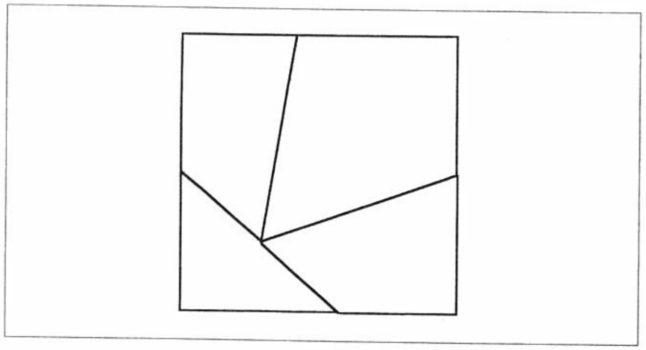


FIGURE 1.6

angles. But are they each 60 degrees? When placed side by side, the three angles will form a straight line against a ruler or piece of paper (three 60s make 180 degrees, a straight line). If some students are having difficulty making the triangle, we provide a little scaffolding in the form of a blackline border of the triangle. See Figure 1.7.

USING REAL-LIFE PROBLEMS: INTERROGATE THE AUTHOR

More and more in recent years, students are asked to work on real problems that are quite different from the often-contrived problems of school math. Some educators applaud this trend and others deride it. In a sense there are two competing philosophies here. One says that children learn mathematics incrementally by working with one piece of the puzzle at a time. And these pieces cannot be as messy as real life. They say that students get too confused by the complexities of real-life situations; there are too many concepts impinging all at once on the children. After they have mastered the skills or understood the concepts from easier, simplified cases, they can transfer their knowledge to other more complex ones and apply their new knowledge to real-life situations. This point of view sounds reasonable, but the transfer of learning from one setting to another is not borne out by research.

The viewpoint that is substantiated by research on transfer is that if students are to use mathematical knowledge in certain situations in their lives, they need to have some experiences that approximate the real setting. If well constructed, those simulations of real situations can not only facilitate subsequent transfers, they can greatly enhance a student's ability to build *initial* understanding. So instead of boiling down the learning of computation to its "naked numbers" that refer to nothing and represent nothing, computation concepts and procedures can be better

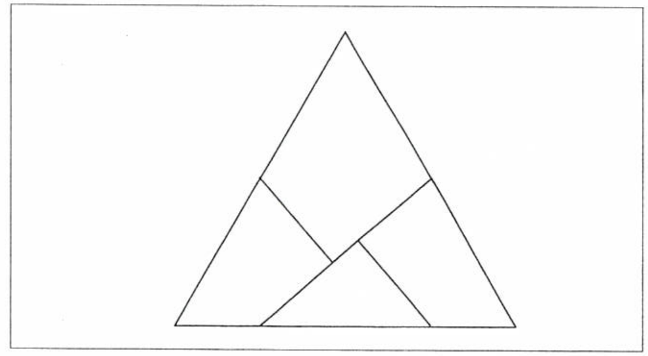


FIGURE 1.7

learned in a context, filled with meaning, where the numbers refer to comprehensible things. Here is an example.

A number of years ago I spotted a data table that I really found fascinating. Ever since, I do a yearly update on the figures to keep it fresh for students. The data table may be seen in Figure 1.8. What do you see?

I let them suggest things. I don't write anything down on the board. I just listen. If no one asks the key questions that I have written on page 36, I will ask the students to consider and respond themselves.

GALLONS OF SOFT DRINKS SOLD IN THE UNITED STATES IN ONE YEAR

	gallons per person		gallons per person		gallons per person		gallons per person
Alabama	59.62	Indiana	46.66	Nebraska	53.30	S. Carolina	63.34
Alaska	47.79	Iowa	46.98	Nevada	55.89	South Dakota	41.31
Arizona	47.14	Kansas	58.16	New Hamp.	46.01	Tennessee	58.97
Arkansas	53.95	Kentucky	57.19	New Jersey	46.49	Texas	58.16
California	52.16	Louisiana	59.45	New Mexico	46.49	Utah	45.36
Colorado	48.60	Maine	47.30	New York	51.35	Vermont	43.09
Connecticut	50.71	Maryland	56.54	N. Carolina	64.64	Virginia	62.05
Delaware	52.65	Mass.	51.19	North Dakota	37.58	Wash. D.C.	58.32
Florida	64.31	Michigan	54.11	Ohio	55.24	Washington	40.66
Georgia	63.83	Minnesota	53.46	Oklahoma	50.22	West Virginia	55.40
Hawaii	50.71	Mississippi	61.88	Oregon	38.56	Wisconsin	46.66
Idaho	33.53	Missouri	58.97	Pennsylvania	42.93	Wyoming	33.37
Illinois	53.78	Montana	37.75	Rhode Island	46.17		

FIGURE 1.8

on a task. However, efforts to break down awareness and monitoring into subcomponents of metacognition in reading comprehension have not obtained consistent results. Meanwhile, many math folks have trouble separating metacognition from cognition. They see cognition including self-regulating awareness and "executive control." Most agree that better problem solving comes when students ask themselves questions about

- the conditions, limitations, and constraints
- if there is sufficient information to get an answer
- if there is one answer, more than one, or no answer
- different ways to represent a problem
- if what you are doing makes sense
- what you have done or where you have been already ("Am I making progress?")
- if answers are reasonable

Using the KWC is an excellent systematic way to accomplish this needed process.

As you plan for your kids to do problem solving, there are several critically important things for you to consider. In the next section you will see some considerations related to the material in this chapter. By the term *consider* (or *consideration*) I mean things that you may need to attend to or deal with. They are not prescriptions for how to do things. They are more like, "I need to check to see if I need to address this issue." There are many different ways to address these considerations, and I have given you some suggestions on how I address them. However, you always will modify and adapt anyone else's ideas to fit your own personality, your teaching style, your school circumstances, and the particular students you have.

In this chapter I have begun to show the *Braid Model of Problem Solving* by describing the KWC. Subsequent chapters will add more features to the model. At the ends of Chapters 2 through 4, I will provide a cumulative picture of the model as it becomes more elaborate and also will offer additional considerations in planning. Therefore, you will have a full model and complete set of considerations by the end of Chapter 4 that you can use in thinking about the problems presented in Chapters 5 and 6.

CONSIDERATIONS IN PLANNING FOR PROBLEM SOLVING

Situation

Big Ideas, Enduring Understandings, and Essential Concepts

What is the concept that I want the students to understand?
To what prior knowledge should we try to connect?
Are there different models of the concept?
Should I break down the concept into its underlying ideas?
Is there a sequence of understandings that the students need to have?
What other mathematical concepts are related?

Authentic Experiences

What are the different real-life situations or contexts in which students would encounter the concept?
Will they see it in science or social studies?
How can I vary the contexts to build up a more generalized understanding?
What version of this situation can I present to start them thinking about the concept?
What questions can I ask to intrigue them and initiate problem solving?