An Introduction to Intraclass Correlation that Resolves Some Common Confusions

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Abstract

This article resolves common confusions about intraclass correlations in a way that starts from Weldon's (2000) introduction of the usual linear correlation and regression in terms of standardized variables and minimum squared distances, but also makes use of principal component analysis and refers to analysis of variance. The intraclass correlation for a set of classes in which the order of the values is arbitrary (and independent from one class to the next) is a non-negative quantity. Negative estimates are possible and can be interpreted as indicating that the true intraclass correlation is low, that is, two members chosen randomly from any class vary almost as much as any two randomly chosen members of the whole population.

1. Introduction

Intraclass correlation (hereon: ICC) refers to a number of quantities, but the simplest form is the usual linear (Pearson product-moment) correlation among a set of pairs of values when the order in each pair is arbitrary. That would be the case if we wanted to know the correlation of heights in same sex couples or the agreement of two independent ratings of an exam done by a set of students (where the two raters differed from one student to the next). This form of ICC—which is the subject of this article—can be calculated after a "double-entry" of the set of pairs, where each is included twice in the calculation of the linear correlation: once in one order; once in the other. The double-entry method can be generalized to classes of size k, where each of the k(k-1)/2 possible pairs for each class is entered in both orders. Indeed, this was the original method used to calculate the ICC, but Harris (1913; as described in Haggard 1958) showed that it is equivalent to finding the ratio of the variance of the means of the classes to the variance of the entire set of values, which is easier to calculate.

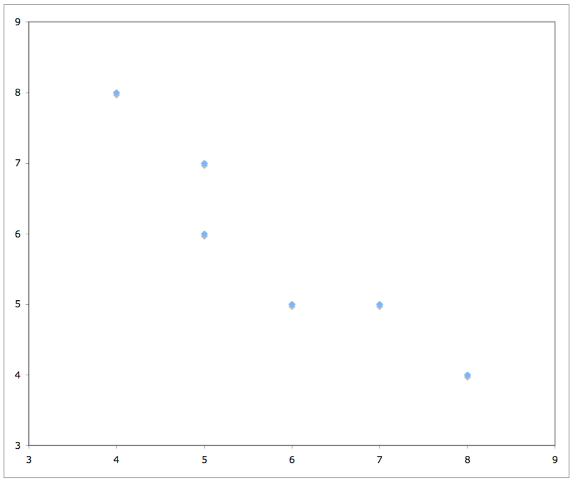
It is easy to be confused by published discussions of ICC. As a ratio of variances, the ICC should be in the interval [0,1]. Yet, without comment, negative estimates are included in Field's (2005) encyclopedia entry. Of course, estimates of variances can sometimes be negative. (If that were not the case and the true variance were 0, estimates could not average out to 0, i.e., they would be biased.) Is this fact sufficient to account for the ICC value of -.94 derived from the double-entered data from Kenny (2008) that is presented in Table 1 and Figure 1? Or do we find negative values because, "[l]ike the interclass [Pearson] correlation, the intraclass correlation for paired data will be confined to the interval [-1, +1]" (Wikipedia n.d.)? In their recent text on analysis of paired (dyadic) data, Kenny et al. (2006, 33) take this second position and go on to explain how to intepret the negative values:

> Intraclass correlations for dyads are interpreted in the same fashion as Pearson correlations. Thus, if a dyad member has a high score on a measure and the intraclass correlation is positive, then the other dyad member also has a relatively high score; if the intraclass correlation is negative, then the other dyad member also has a relatively low score.

For classes of size greater than 2, it is also said that the most negative ICC is -1/(k-1) (Wikipedia n.d.). Yet, if the ICC is to be interpreted in the same fashion as the usual linear correlation, what restricts the negative ICC values to a smaller range when the class sizes are bigger than 2?

Table 1 and Figure 1. Double-entered data that have a negative ICC

Original unordered pairs of values	5	7
	8	4
	5	6
Double entries of pairs in the reverse order	7	5
	4	8
	6	5



This article resolves the confusions in a way that starts from Weldon's (2000) introduction of the usual linear correlation and regression in terms of standardized variables and minimum squared distances.

2. From linear correlation to double-entry intraclass correlation

Following Weldon (2000), consider pairs of values of two standardized variables, x and y, each with mean of 0 and variance of 1. The linear correlation between x and y is the same as their covariance; if we call this ρ, the variance/covariance matrix is

$$\begin{vmatrix} 1 & \rho \\ \rho & 1 \end{vmatrix}$$

The eigenvalues of this matrix are $1 + \rho$ and $1 - \rho$, with corresponding eigenvectors or "principal components" $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ and $\begin{vmatrix} 1 \\ -1 \end{vmatrix}$

If the (x, y) pairs are viewed as points in two dimensions, the eigenvalue/vector result is equivalent to saying that the variance of the projections of the points onto the line $y = x$ is $1 + \rho$ and the variance of the projections onto the line $y = -x$ is $1 – \rho$. Because these lines are orthogonal, the result is also equivalent to saying that the expected squared perpendicular distance from the line $y = x$ is $1 – \rho$ and from the line $y = -x$ is $1 + \rho$. The larger the positive correlation, the tighter the packing of the points to the line $y = x$, or the larger the negative correlation, the tighter the packing of the points to the line $y = -x$ (Figure 2).

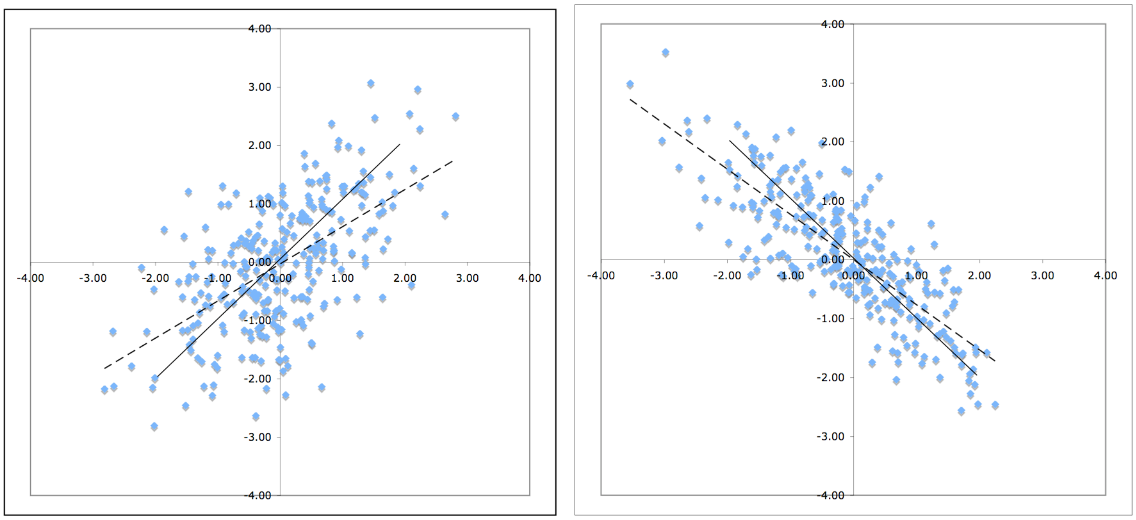


Figure 2. Scatter plots of sets of two variables with correlation 0.5 and -0.8. The solid lines are y = x and y =-x, respectively; the dashed lines are y = .5x and -.8x, that is, the linear regression lines.

Linear correlation for the standardized variables is directly interchangeable with linear regression—the line of slope ρ is the regression line for y on x (Weldon 2000) and the line of slope 1/ ρ is the regression line for x on y. (This can be shown by showing that the expected vertical distance from those lines is the minimum. For the regression of y on x the expected vertical distance to a line with slope t, is, where E denotes expected value:

$$E\,[(y - tx)^2] = E\,[y^2] - 2t\,E\,[xy] + t^2\,E\,[x^2] = \mathrm{Var}\,(y) - 2t\,\rho + t^2\,\mathrm{Var}\,(x) = 1 - 2t\,\rho + t^2$$

This last expression has a minimum value when t = ρ.) The interchangeability of linear correlation and regression means that, even though regression analysis is conventionally conceived as derivation of the best predictor of one variable by the other, such analysis can generate no more insight about the data than is contained in the tightness of packing of the points.

Now consider the ICC and start with the simplest case of classes of two, that is, a set of unordered pairs of values. If the full set of values is standardized so that it has a mean of 0 and variance of 1, and the data are double-entered so that for every pair (c, d) there is also a pair (d, c), then the linear correlation of the double-entered set—call this ρ_I—has the properties described above. The more tightly packed the double-entered points are around the line y = x (in this case symmetrically packed), the higher the ICC. (Also, the better the regression lines $y = \rho_I x$ and $y = x/\rho_I$ predict the second value from the first and vice versa.) One difference from the usual linear correlation, however, is that the points are never more tightly packed around the line $y = -x$, than around $y = x$, that is, the ICC values is a non-negative quantity.

To appreciate the non-negativity of the ICC, consider a variable generated by model 1:

$z_{ij} = a_i + w_{ij}$,

where a_i, the mean of the ith class, and w_{ij} are distributed with means of 0 and variances of σ_a^2 and σ_w^2 respectively and independently.

Setting σ_w^2 to $1- \sigma_a^2$ (so that the variance of the z_{ij} values is 1), the ICC calculated as the ratio of variance among the means of the classes to the variance of the z_{ij} values is simply σ_a^2, which is non-negative. This result can be seen graphically in terms of the double-entry correlation. Take the simplest case: classes of size 2, with a_i values of $\pm\sigma_a$ and w_{ij} values of $\pm\sigma_w$. There are four possible pairs of values from each class: $(a_i + \sigma_w, a_i + \sigma_w)$, $(a_i + \sigma_w, a_i - \sigma_w)$, $(a_i - \sigma_w, a_i + \sigma_w)$, $(a_i - \sigma_w, a_i - \sigma_w)$, where a_i is either σ_a or $-\sigma_a$. Double-entered, the midpoints of the pairs of points are, respectively, $(a_i + \sigma_w, a_i + \sigma_w)$, (a_i , a_i), (a_i , a_i), $(a_i - \sigma_w, a_i - \sigma_w)$. The variance of all the midpoints along the line $y = x$ is $2\sigma_a^2+ \sigma_w^2$. The variance of the distance of the double-entered points from their midpoints is σ_w^2. No matter how small σ_a^2 is, the variance along the line $y = x$ is greater than the variance perpendicular to it. In other words, $1 + \rho_I$ is greater than $1- \rho_I$, so the double-entered points are never more closely packed around the line $y = -x$ than they are around the line $y = x$, and the correlation of the double-entered points, i.e., the ICC, is non-negative. The preceding features are illustrated in Figure 3 (which consists of two snapshots from a spreadsheet downloadable from http://www.faculty.umb.edu/pjt/ICC.xls).

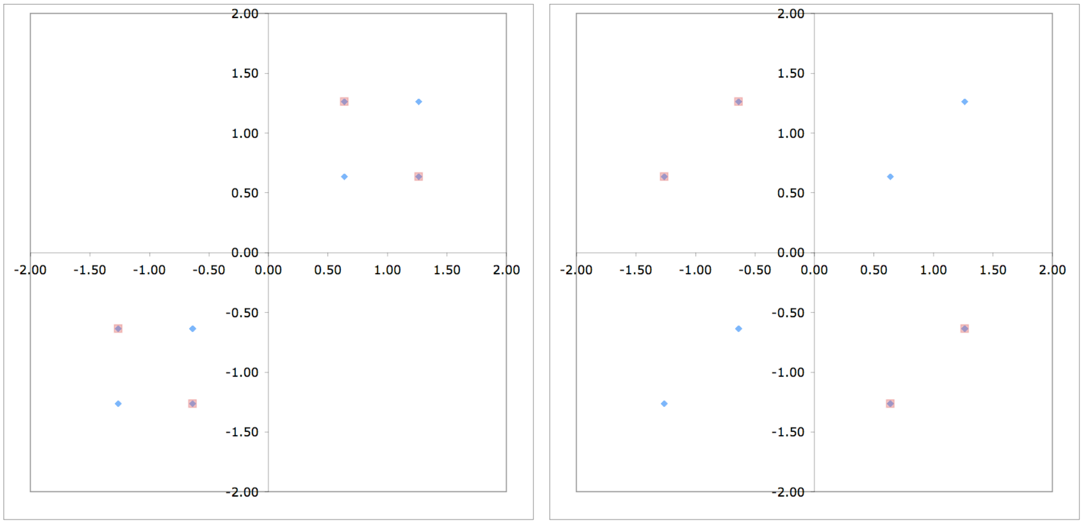


Figure 3. Double-entry pairs (blue dots) compared to projections of the class means onto y = x and the within-class deviations perpendicular to that (red squares). σ_a^2 = .90 and .10

This illustration can be readily generalized to classes with more than two members, to data from more than two classes, to classes with unequal variances, and, with loss of some visual clarity, to distributions other than the binomial having the same mean of 0 and variance, namely, σ_a^2 or σ_w^2.

3. Interpretation and Estimation

The preceding theory seems clear. The ICC as the linear correlation of the double-entered points or as the variance of the class means, σ_a^2, divided by the total variance (in this case 1) has values that lie in the range [0, 1]. If there is full agreement in every class, then $\sigma_w^2 = 0$ and the ICC =1. If there is no agreement, that is, if two members of any class vary as much as any two randomly chosen members of the whole population, then $\sigma_w^2 = 1$ and the ICC =0. (Indeed, this parallels the usual linear correlation. Full agreement between x and y in a linear correlation occurs either when y = x or when y = -x, i.e., when $\rho = 1$ or -1. Agreement declines as the correlation moves towards 0 from either the positive or the negative direction.)

The negative ICC value for the data in Table 1 still needs to be explained. Has the theory overlooked a category for ICC of worse-than-random agreement? As the earlier quote from

Kenny et al. (2006) suggests, negative values would mean that the process, whatever it is, that brings pairs together makes them less similar than any two randomly chosen members of the whole population. For height of same sex couples we could imagine a world in which this were the case, but how would less-similar-than-random happen for independent ratings of exams? Moreover, how would a process of unlikes coupling play out for classes of three or more? After all, if A seeks to be distant from both B and C, then there must be limits to how distant B can be from C. Before looking for a hole in the theory, then, let us consider the possibility that the ICC of -.94 is simply an extreme case of a negative estimate of a non-negative quantity.

Suppose that the data in Table 1 were generated by a variant of model 1 in which the variance of w_{ij} can vary from one class to the next and the variable, z_{ij}, is not standardized. If we assume that the w_{ij} values are binomial, i.e., $\pm\sigma_{wi}$, the actual a_i and σ_{wi} values can be calculated (Table 2). Using these values in the variant of model 1, we could generate sets of three pairs of values that include pairs such as (5, 5) and (7, 7) for the first class, (8, 8) and (4, 4) for the second class, and so on. The set of pairs (5, 5), (8, 8), (6, 6) is no less likely to be observed than the data in Table 1, and for this set the intraclass correlation is +1. Estimated ICC values of -.94 and +1 are both possible for the same underlying model. Indeed, for class sizes of 2 and a low ICC—in this case .03 is the true value—widely disparate estimates are possible. The negative value does not call for a special category and explanation.

Table 2. Data from Table 1 analyzed using the variant of Model 1 described in the text

Class	Pairs of values, z_{ij}		a_i	σ_{wi}
1	5	7	6	1
2	8	4	6	2
3	5	6	5.5	.5

σ_z^2 = 65/36; σ_a^2 = 2/36; average σ_{wi}^2 = 63/36; ICC = 2/65 = .03

If we do not assume that the w_{ij} values are binomial, the true values cannot be determined directly from the data. In that case, a one-way Analysis of Variance (ANOVA) can be used to generate estimates of the variances σ_a^2 and σ_w^2 (where σ_w^2 is assumed in the ANOVA to be equal from one class to the next). From these estimates the ICC can be calculated. Because estimation using the ANOVA is introduced well elsewhere (e.g., Field 2005, Howell 2002), it will not be discussed here, except to note three points:

1. The double-entry correlation and ANOVA produce the same estimate; if one if negative, so is the other.
2. The calculation of ICC as a ratio of two variances is biased even if the estimation of the variances is unbiased. The bias, which is negative, can be corrected (Donoghue and Collins 1990).
3. The expected mean squares in the ANOVA table have a close relationship with the variance of the double-entered points projected, after subtracting the sample mean, onto the lines y = x and y = -x, which are also the principal components of the unstandardized variance/covariance matrix (Table 3). (The factor 1/2 enters because every unit step along the lines y = x and y = -x corresponds to an increment of 1/√2 along the axes.)

Table 3. Correspondences between one-way ANOVA and Principal Components of Double-entered Variance/Covariance Matrix

Source	**Expected Mean Square in ANOVA**	**Variance**	**Principal Components (PC)**
Among Class Means	$k\sigma_a^2 + \sigma_w^2$	$\sigma_a^2 + \sigma_w^2/k$	$= PC_1/2$
Within Class Deviations from the means	σ_w^2	σ_w^2	$= k/(k-1)*PC_2/2$

4. Conclusion

The ICC for a set of classes in which the order of the values is arbitrary (and independent from one class to the next) is a non-negative quantity. Negative ICC estimates are possible and can be interpreted as indicating that the true ICC is low, that is, two members chosen randomly from any class vary almost as much as any two randomly chosen members of the whole population.

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