

Towards a Coherent Theory of Physics and Mathematics

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As an approach to a Theory of Everything a framework for developing a coherent theory of mathematics and physics together is described. The main characteristic of such a theory is discussed: the theory must be valid and sufficiently strong, and it must maximally describe its own validity and sufficient strength. The mathematical logical definition of validity is used, and sufficient strength is seen to be a necessary and useful concept. The requirement of maximal description of its own validity and sufficient strength may be useful to reject candidate coherent theories for which the description is less than maximal. Other aspects of a coherent theory discussed include universal applicability, the relation to the anthropic principle, and possible uniqueness. It is suggested that the basic properties of the physical and mathematical universes are entwined with and emerge with a coherent theory. Support for this includes the indirect reality status of properties of very small or very large far away systems compared to moderate sized nearby systems. Discussion of the necessary physical nature of language includes physical models of language and a proof that the meaning content of expressions of any axiomatizable theory seems to be independent of the algorithmic complexity of the theory. Gödel maps seem to be less useful for a coherent theory than for purely mathematical theories because all symbols and words of any language must have representations as states of physical systems already in the domain of a coherent theory.

1 Introduction

The goal of a final theory or a Theory of Everything (TOE) is a much sought after dream that has occupied the attention of many physicists and philosophers. The allure of such a goal is perhaps most cogently shown by the title of a recent book "Dreams of a Final Theory"⁽¹⁾. The large effort and development of superstring theory in physics to unify quantum mechanics and general relativity⁽²⁾ also attests to the desire for such a theory.

The existence of so much effort shows that it is not known if it is even possible to construct a TOE. However the conclusion of one study⁽³⁾ that there do not seem to be any fundamental limits on scientific knowledge at least lends

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support to searching for a TOE, even if it turns out that there are limits to scientific knowledge.

Here the emphasis is on approaching a TOE from a direction that emphasizes the close connection between mathematics and physics. The idea is to work towards developing a coherent theory of mathematics and physics by integrating mathematical logical concepts with physical concepts. Whether such a coherent theory can be constructed, and, if so, whether it is or is not a suitable theory of everything, is a question for the future.

The approach taken here is perhaps closest to that of Tegmark⁽⁴⁾ in that he also emphasizes general mathematical and physical aspects of a TOE along with mathematical logical properties. However this work differs in not using the ensemble aspect. Also the equivalence between mathematical and physical existence of systems is not used here.

The plan of the paper is to first provide a background discussion on the relation between physics and mathematics at a foundational level. Some the problems are outlined along with a summary of the main foundational approaches to mathematics. There is also a very brief description of other work on the foundational aspects of the relation between physics and mathematics.

This is followed in Section 3 by a description of a general framework for a coherent theory of mathematics and physics. Much of the discussion is centered on the main and possibly defining requirement for a coherent theory: *The theory must be valid and sufficiently strong and it must maximally describe its own validity and sufficient strength.*

This requirement is discussed in the context of the assumption that a coherent theory can be, in principle at least, axiomatized using the first order predicate calculus. In this way the tools that have been extensively developed in mathematical logic for treating first order theories, such as Gödel's completeness theorem and his two incompleteness theorems^(5,6,7) and other results, can be used and brought into close contact with physics and help to support a coherent theory.

The restriction to first order theories would seem to be a minimal restriction as it includes all mathematics used by physics so far. For instance, Zermelo Frankel set theory, which includes all mathematics used to date by physics,^(8,5,9) is a first order theory. If it turns out to be necessary to consider second order theories⁽¹⁰⁾, then it is hoped that the ideas developed here could be extended to these theories.

Validity is given the usual mathematical logical definition, that all expressions of a coherent theory that are theorems are true. The important role of physical implementability for the definition of validity is noted. It is also seen that even though sufficient strength cannot be defined, it is both a necessary and a useful concept, especially for incomplete theories such as arithmetic and set theory. It is expected that any coherent theory will also be incomplete, but it should be sufficiently strong to be recognized as a coherent theory.

The potential role of the requirement that a coherent theory maximally describe its own validity and sufficient strength in restricting or limiting candidate coherent theories is noted. In a more speculative vein a possible use of

the requirement is suggested in which it restricts paths of theories of increasing strength generated by iterated extensions of a theory.

Other aspects of a coherent theory that are discussed include universal applicability, the strong anthropic principle, and the possible uniqueness of a coherent theory. Problems in how to exactly define universal applicability are considered. It is noted that a coherent theory should include all mathematical systems used by physics, physical systems as complex as intelligent systems, and it should exclude the possibility of pointing to a specific physical system that is not included.

Emergence of the physical and mathematical universes and a coherent theory of mathematics and physics is discussed in Section 4. The position taken is that the basic properties of the physical and mathematical universes and a coherent theory are emergent together and mutually determined and entwined. They should not be regarded as having an a priori independent existence. To support this point the reality status of very small or very large far away systems is compared with that of moderate sized systems. The indirectness of properties of the former, in their dependence on many layers of supporting theory and experiment, is contrasted with directly experienced properties of moderate sized and nearby systems.

The importance of the fact that language is physical for a coherent theory is discussed in Section 5 and the Appendix. The fact that all languages, formal or informal, necessarily have physical representations as states of physical systems is emphasized by describing quantum mechanical models of language. Included is a discussion of the importance of efficient implementability for operations such as generating text. A specific model of a multistate system moving along a lattice of quantum systems is used to illustrate this requirement.

Other aspects of the physical nature of language discussed in Section 5 include a proof that the relation between the meaning of language expression states and their algorithmic information content is, at best, complex and is probably nonexistent. The section closes with a discussion of Gödel maps and the observation that they play a more limited role in coherent theories than in any purely mathematical theory. The reason is that physical representations of language, which must exist, are already in the domain of a coherent theory.

2 Background

Physics and mathematics have a somewhat contradictory relationship in that they are both closely related and are also disconnected. The close relation can be seen by noting the mathematical nature of theoretical physics and the use of theory to generate predictions as the outcomes of mathematical calculations that can be affirmed or refuted by experiment. The validity of a physical theory is based on many such comparisons between theory and experiment. Agreement constitutes support for the theory. Disagreement between theoretical predictions and experiment erodes support for the theory and, for crucial experiments, may

result in the theory being abandoned.

The disconnect between physics and mathematics can be seen by noting that, from a foundational point of view, physics takes mathematics for granted. In many ways theoretical physics treats mathematics much like a warehouse of different consistent axiom systems each with their set of theorems. If a system needed by physics has been studied, it is taken from the warehouse, existing theorems and results are used, and, if needed, new theorems are proved. If theoretical physics needs a system which has not been invented, it is created as a new system. Then the needed theorems are proved based on the axioms of the new system.

The problem here is that physics and mathematics are considered as separate disciplines. The possibility that they might be part of a larger coherent theory of mathematics and physics together is not much discussed. For example basic aspects such as truth, validity, consistency, and provability are described in detail in mathematical logic which is the study of axiom systems and their models⁽⁵⁾. The possibility that how these concepts are described or defined may affect their use in physics, and may also even influence what is true in physics at a very basic level has not been considered. (A very preliminary attempt to see how these concepts might be used in quantum mechanics is made in⁽¹¹⁾.)

The situation in mathematics is different. Here the problem is that most work in mathematics and mathematical logic is purely abstract with little attention paid to foundational aspects of physics. The facts that mathematical reasoning is carried out by physical systems subject to physical laws, and symbols, words, and formulas in any language, formal or not, are physical systems in different states, is, for the most part, ignored. In some ways the various constructivist interpretations of mathematics, ranging from extreme intuitionism⁽¹²⁾ to more moderate views^(13,14) (see also⁽⁸⁾) do acknowledge this problem.

However most mathematicians and physicists ignore any limitations imposed by constructivist viewpoints. Their activities appear to be based implicitly on the ideal or Platonic viewpoint of mathematical existence, i.e. that mathematical entities and statements have an ideal existence and truth status independent of any physical limitations⁽¹⁵⁾ or an observers knowledge of them^(16,17). The "luscious jungle flora"⁽⁸⁾ aspect of this view of mathematical existence compared to the more ascetic landscape⁽⁸⁾ of more constructivist views is hard to resist.

However, this viewpoint has the problem that one must face the existence of two types of objects. There are the ideal mathematical objects that exist outside space-time and the physical objects that exist inside of and influence the properties of space-time. The existence of two types of objects that appear to be unrelated yet are also closely related is quite unsatisfactory.

Another approach to mathematics is that of the formalist school^(16,17), Here mathematics is considered to be in essence like a game in which symbol strings (statements or formulas) are manipulated according to well defined rules. The goal is the rigorous proof of theorems. Mathematical entities have no independent reality status or meaning.

In one sense the formalist school is related to physics in that provability and

computability are closely related. In work on computability and computational complexity⁽¹⁹⁾ it is clearly realized that computability is related closely to what can be carried out (in an ideal sense) on a physical computer. Yet, as has been noted⁽²⁰⁾, the exact nature of the relationship between computations carried out by real physical computers and abstract ideal computers, such as Turing machines, is not clear.

The influence of physics on mathematics is perhaps most apparent in recent work on quantum information theory and quantum computing. Here it has been shown^(21,22) that there exist problems that can in principle be solved more efficiently on a quantum computer than by any known classical computational algorithm. Also the increased efficiency of simulation of physical quantum systems on quantum computers^(23,24,25,26,27) compared to simulation on classical systems is relevant to these considerations.

The problems on the relationship between physics and mathematics have been considered by others. In his insistence that "Information is Physical" Landauer⁽²⁸⁾ also recognizes the importance of this relationship. His reference to the fact that, according to Bridgman, mathematics should be confined to what are in essence programmable sequences of operations, or that mathematics is empirical⁽²⁹⁾, supports this viewpoint. Similar views on the need for an operational characterization of physical and set theoretic entities has been expressed⁽³⁰⁾.

Other attempts to show the importance of physics on the foundations of mathematics include work on randomness⁽⁹⁾ and on quantum set theory^(31,32) (see also⁽³³⁾). Recent work on the relationship between the Riemann hypothesis and aspects of quantum mechanics^(34,35) and relativity⁽³⁶⁾, and efforts to connect quantum mechanics and quantum computing with logic, languages, and different aspects of physics should be noted^(37,38,39,40) along with efforts to connect mathematical logic with physics^(4,41,42).

The existence of axiomatizations of physical theories in the literature also suggests the significance of foundational aspects in the relation between physics and mathematics. In particular axiomatizations of quantum mechanics and quantum field theory have been much studied. These include algebraic approaches^(43,44), quantum logic approaches^(45,46) and others⁽⁴⁷⁾. These axiomatizations are often quite mathematical and rigorous, but they refer to specific physical theories and not to a general theory of physics and mathematics.

In spite of this progress both the lack of and a need for a coherent theory of mathematics and physics together remain. One view of this is expressed by the title of a paper by Wigner⁽⁴⁸⁾ "On the unreasonable effectiveness of mathematics in the natural sciences". One does not know why mathematics is so effective and an explanation is needed. Even though the paper was published in 1960, it is still relevant today. A related question, "Why is the physical world so comprehensible?"⁽⁴⁹⁾ also needs to be answered.

These questions are still relevant today. It is hoped that the following material will help to answer these questions by providing part of a framework that can be used to construct a coherent theory of mathematics and physics together.

3 A Coherent Theory of Physics and Mathematics

The basic idea is that a coherent theory of mathematics and physics must be a description of both the mathematical and physical components together of the universe. They should not be treated as separate universes as has been done so far. Whether such a theory would include all or just some mathematics is not known, and one suspects that only some mathematics would be included. However it is reasonable to require that mathematical systems that are used or are potentially usable by physics should be included in the domain of a coherent theory.

At present it is not known how to construct a coherent theory of mathematics and physics. One purpose of this paper is to suggest that a way to accomplish this is to work towards a combination of mathematical logical aspects with physics at a foundational level. This approach, which is implied by other work⁽⁴⁾, seems worthwhile since mathematical logic is the study of basic aspects of mathematical systems in general. This includes a study of properties such as consistency, completeness, truth, and validity. By combining mathematical logic with physics one may hope to make these properties an integral part of physics at a basic level. Furthermore such a combination may even influence what is true in physics at a basic level.

A basic and possibly defining requirement of a coherent theory is that it must be valid and sufficiently strong, and it must be able to maximally describe both its own validity and sufficient strength. Much of this paper is devoted to discussing this requirement and showing why it may be quite important.

3.1 Validity and Sufficient Strength

It is first necessary to discuss properties that any satisfactory first order axiomatizable theory should have^(5,6). Some of this subsection is a review of well known material. A good theory should be such that all properties predicted by the theory should be true in its domains of applicability. Also the theory should be sufficiently strong in terms of its predictive power to be regarded as a satisfactory or useful theory.

Since theories differ by having different sets of axioms, the interest is in the properties of a sets of axioms for theories, including a coherent theory of physics and mathematics. One property that a theory must have is that it is valid. A theory is defined to be *valid* for a structure⁽⁵⁾ if all theorems of the theory are true in the structure. For a coherent theory the structure is that part of the physical and mathematical universe to which the theory applies (this will be discussed more later on) and truth in the structure has the usual intuitive informal meaning. Following the usual practice in mathematical logic, no definition of truth is provided as it is an intuitive concept¹ Even so various

¹As is well known, one can define the truth of formulas in a language of a theory by induction on the logical depth of the formulas. However the definition depends on the intuitive

properties of the set of true formulas of theories such as arithmetic can be derived⁽⁶⁾.

The requirement of validity is essential as it connects the formal notion of proof and theoremhood of a theory with the notion of truth in a structure. But it is not sufficient. To see this note that the definition of validity says that for all formulas F in the language of the theory, if F is a theorem, then it is true. The problem is that this definition admits, as valid, theories that are too weak. Consider for example an empty theory with no (nonlogical or logical) axioms. Since it has no theorems it is valid. This follows from the truth table of if-then statements in that such statements are true if the "if" part is false. Another example of a theory that is too weak is first order predicate calculus as a theory with just logical axioms and no nonlogical axioms. This theory is valid as all the theorems are logical consequences of tautologies⁽⁵⁾ and are true in any universe. The logical axioms would be present in any axiomatization of a coherent theory.

As is well known in mathematical logic, weak theories can be strengthened by adding more nonlogical axioms. Extension of a theory by adding more axioms gives a theory that is stronger in that it has more theorems and predictive power than the unextended theory. However, if the added axioms are just axioms that define new symbols for existing terms in the language of the old theory, then the new theory is not stronger than the old.

These considerations show that one must require that a coherent theory is sufficiently strong. This means that the set of axioms should be sufficiently large and appropriate so that the resulting theory will be recognized as a suitable coherent theory of mathematics and physics.

It must be emphasized that this is not a definition of sufficient strength. It is not known at present how to give a definition. However it can be shown that the idea has merit and is in fact already in use for incomplete theories such as arithmetic and set theory.

The first question to consider is whether a coherent theory will be complete or incomplete. A theory is complete if for each closed formula (a formula with no free variables) in the language of the theory, either the formula or its negation (but not both if the theory is consistent) is a theorem of the theory⁽⁵⁾. A complete theory is of maximum strength. There is no way to increase its strength by adding more axioms. Examples of complete theories include those for the real numbers as a real closed field and atomless Boolean algebras^(5,50).

As was first shown by Gödel^(7,6) there are also incomplete theories. Good examples are arithmetic and any other theory, such as Zermelo Frankel Set theory, sufficiently strong to include arithmetic. Gödel first showed incompleteness by exhibiting a formula that asserted its own unprovability. If arithmetic is valid, then neither this formula nor its negation can be a theorem of arithmetic.

This argument was later extended by Chaitin⁽⁵¹⁾ to show that arithmetic, and any axiom system that includes the natural numbers, is quite incomplete. In particular it follows from his work that almost all formulas that express the randomness or algorithmic complexity of numbers are not theorems of arith-

notion that is used to define the truth of the atomic formulas at the base of the induction.

metic. This follows from the observation that almost all length n bit strings \underline{g} are random in that the algorithmic complexity of \underline{g} is equal to $n + c$ which is the length m in bits of the shortest program that generates \underline{g} as output. Here c is a constant. Also the number of random \underline{g} increases exponentially with n . In addition Chaitin's incompleteness theorem⁽⁵¹⁾ states that no formal axiom system whose axioms have algorithmic complexity p can be used to prove formulas stating that a specific \underline{g} has complexity m with $m \geq p$. Proofs of complexity or randomness are limited to statements about complexity values $< p$.

Since arithmetic is so incomplete it follows that there is much room for extending the strength of arithmetic in many ways. One such path is the non-terminating extension provided by Gödel's second incompleteness theorem^(7,6). This theorem says that in any theory T strong enough to express by a formula in the language of T , the consistency of T , then that formula is not a theorem of T . Also if one strengthens T by adding new axioms so that the consistency of T can be proved in the stronger theory, then the same incompleteness result hold for the stronger theory.

Besides this example there are many other ways or paths to follow in strengthening arithmetic by adding new axioms. In view of this it is surprising that the relatively simple axiomatization of arithmetic, that is in use, is sufficiently strong for most uses made of the theory. The reason, which is in many ways remarkable, is that arithmetic, and mathematics in general and physics is really concerned with small numbers with complexities of at most a few hundred bits and with arithmetic operations on these numbers. Really large numbers, such as those of the order of 2^n where $n = 10^{20}$ with correspondingly large complexities, seem to play no role in arithmetic or physics.

Zermelo Frankel set theory with the axiom of choice (ZFC) is also incomplete. It is also powerful enough to include all the mathematics used so far by physics. However, unlike the case with arithmetic, changes in the axioms of the theory have been considered that depend on how the theory is to be used. One example is the extension of the theory to include proper classes⁽⁸⁾ to include objects that are not sets, such as the class of all sets that are not members of themselves and the class of all ordinals.

Other changes are based on the existence of several easily formulated mathematical statements whose truth value is unknown. The main example is the continuum hypothesis (CH) which has been shown to be independent of the other axioms of ZFC⁽⁵²⁾. Thus one can study the properties of ZFC plus CH and of ZFC plus the negation of CH. Other ways of extending or changing ZFC include the addition of large cardinal axioms or replacing the axiom of choice by the axiom of projective determinacy⁽⁵³⁾. A recent review of some of these aspects is provided in⁽⁵⁴⁾.

This and other work on arithmetic and set theory shows that these incomplete theories, axiomatized as Peano arithmetic and as ZFC set theory, are sufficiently strong for almost all uses in mathematics and physics even though they are quite incomplete. These results should also apply to a coherent theory of physics and mathematics. Any axiomatization of a coherent theory is expected to be incomplete as arithmetic is included as part of the mathematical

component. A coherent theory should also be valid and be sufficiently strong for almost all uses.

These arguments all assume that it is possible to axiomatize a coherent theory of physics and mathematics. Whether this is possible or not, or if there are many different coherent theories each with their own axiom sets, will have to await further work.

Other aspects of the requirement of validity and sufficient strength for a coherent theory need discussion. It was noted that validity means that all formulas in the language of a coherent theory that are theorems must be true in the domain of applicability of the theory, which includes physical and mathematical systems. Applied to physical systems this means that all properties of physical systems that are predicted by theorems of the theory and are capable of experimental verification or refutation, are verifiable by experiment. Sufficient strength means that there must be sufficiently many properties of physical systems that are predicted by theorems of the theory and can be experimentally tested.

One should emphasize the role that physical procedures play in this requirement. The requirement of validity means that all theoretically predictable properties of systems that are experimentally testable, are true. Experimental testability of any prediction requires the existence of physical procedures for preparing a system in a specified state and for carrying out the required measurement.

This requirement is problematic for quantum systems in that there is no way so far to define physical implementability for preparation of quantum systems in various states and for measuring observables, represented by self adjoint operators in an algebra of operators. It may well be that there are many properties of systems that are predicted to be true for physical systems, but there do not exist any corresponding experimental physical procedures for actually testing the prediction that are implementable in an efficient manner.

For example, it is clear that for many states of complex quantum systems and for many observables it is very unlikely that there exist efficiently implementable physical procedures for preparing the states and measuring the observables. Examples of these states include complex entangled states of multicomponent systems of the type studied by Bennett et al⁽⁵⁵⁾. Examples of observables include projection operators on these entangled states. A related example is based on the observation that efficient physical implementability is not preserved under arbitrary unitary transformations of self adjoint operators⁽⁵⁶⁾. Thus if the observable \hat{O} is efficiently implementable it does not follow that $U\hat{O}U^\dagger$ is physically implementable for arbitrary unitary U . It has also been noted that there is a problem in determining exactly which logical procedures or algorithms are physically implementable⁽²⁰⁾. This problem is especially relevant for quantum computer algorithms as it is not at all clear which are efficiently physically implementable and which are not.

The same requirement of physical implementability applies to computers as physical realizations of any one of the equivalent representations of abstract computability. A much studied and quite useful abstract concept is that of

Turing machines⁽⁵⁷⁾. These machines, described by a multistate head that moves along a tape of cells and interacts locally with the cells, are both quite simple and very powerful. In particular, by the Church Turing thesis, any computable function is computable by a Turing machine. So there are Turing machine equivalents for any existing computer. Also Turing machines provide a convenient venue for proving the unsolvability of the halting problems and the existence of universal computers, both quantum and macroscopic^(57,58,59).

As is well known, limitations of the computation process make many calculations of properties of systems predicted by theory very hard or impossible. In this case various model assumptions and simplifications are used to make the calculations more tractable. The use of quantum computers, if and when they are developed, to make some of these calculations may help in that some problems^(21,22) are much more tractable on quantum computers than on classical computers. The observation that quantum computers are much more efficient at simulating the behavior of quantum systems than are classical computers^(26,27) may also help.

3.2 Maximal Description of Validity and Sufficient Strength

It was noted earlier that it is not known if it is possible to axiomatize a coherent theory or if there are many axiomatizable coherent theories. The other part of the basic requirement of a coherent theory, that it describe to the maximum extent possible its own validity and sufficient strength, may be relevant to these considerations.

The meaning of this requirement is that there are one or more formulas in the language of a coherent theory that can be interpreted to express to some extent that the theory is itself valid and sufficiently strong. In addition these formulas must be true. This condition is expressed by the requirement that a coherent theory is valid and sufficiently strong. It is also desirable, but unlikely, that these formulas be theorems of the coherent theory they are describing; one way of approaching this aspect by theory extension will be discussed soon.

Now it may be the case that there are many candidate coherent theories, each described by a different set of nonlogical axioms, that differ in their ability to express their own validity and sufficient strength. In some the true formulas express very little or none of the validity and strength of the theory. In other coherent theories there may be true formulas that express more of the validity and strength of the theories.

The condition of description of their own validity and strength *to the maximum extent possible* serves an essential role in that it limits or restricts the choice of candidate theories. Only those candidate theories that maximally describe their own validity and strength are acceptable as coherent theories. However, use of this maximality condition to limit or restrict the candidate theories, requires that one have a measure of the extent of a theory's description of its own validity and sufficient strength. Whether such a measure exists and, if so whether it has a maximum or not, are questions for the future.

It was noted that formulas of a coherent theory that describe to some extent

the validity and sufficient strength of the theory must be true. In addition one would like them to be theorems of the theory. In view of Gödel's second incompleteness theorem this is unlikely. However the theorem suggests a possible way out that throws new light on the condition that a theory *maximally* describe its own validity and sufficient strength.

To begin it is expected that in any coherent theory in which it is possible to express the maximal validity and sufficient strength of the theory, then the formula or formulas expressing it are not theorems of the theory. The reason is that expression of validity of a theory is similar to the expression of consistency when Gödel's completeness theorem is taken into account. (A theory is consistent if and only if it has a model⁽⁵⁾.) If such a coherent theory T_0 exists then it may be possible to extend and strengthen the theory by adding axioms so that the formula expressing the validity and sufficient strength of T_0 is a theorem of the extended theory T_1 . But then the formula or formulas expressing the validity and sufficient strength of T_1 are not theorems of T_1 .

This suggests the consideration of an iterated extension of theories where the formula or formulas $\{F_n\}$ expressing the validity of T_n are theorems in T_{n+1} but the formulas $\{F_{n+1}\}$ are not theorems of T_{n+1} . Since there is no upper bound to n , one is faced with a nonterminating extension of theories. It may also be the case that there are many possible iterated extensions, with the theorems of T_{n+1} describing to some degree the validity and strength of T_n . In this case one limits the paths or extensions by requiring that at each stage in the extension T_{n+1} must be such that the formulas that *maximally* describe the validity of T_n are theorems of T_{n+1} . In this way the condition of *maximal* description limits or restricts the extension paths.

In an even more speculative vein, it may even be possible to at least partly axiomatize the extension process itself. In this way one might hope to obtain one or a few coherent theories that describe to the maximum extent possible the process of extension of parts of themselves. Each of these "limit coherent theories" would describe a nonterminating extension of parts of themselves. It might be that the qualification "to the maximum extent possible" is very useful in restricting an iterated selection process that leads to or selects a "limit coherent theory". For example it may be that the theory at each stage of the iterated extension is algorithmically more complex than the theories at earlier stages. The theory at each stage may also maximally describe its own validity and sufficient strength by maximally describing and proving the corresponding formulas for the theories in all preceding stages².

It is easy to dismiss all of the above as idle speculation with no basis in fact. This may indeed be the case. However it is worth noting that a coherent theory of physics and mathematics, by virtue of including both physical systems in space-time and mathematical systems in its domain, is quite different and may be potentially more powerful than is a purely mathematical theory whose domain is limited to mathematical systems. It is hoped that the reader will ap-

²One wonders if the maximal axiomatization of this extension process, as a restriction on the possible paths of extension, has any overlap with the least action principle used in physics to restrict the paths taken by systems as described by Feynman path integrals.

preciate this possibility after reading the following sections, especially Sections 4 and 5.

3.3 Universal Applicability

Another distinguishing property of a coherent theory is that the theory should be universally applicable. This means that the domain of applicability should include a "sufficiently large" component of the physical and mathematical universes. These two requirements, inclusion of both physical and mathematical systems and universal applicability, might be expected to yield some new results not obtainable from purely mathematical theories or purely physical theories that ignore foundational aspects of mathematics. It will be seen later that this may be the case.

At this point the precise definition of "sufficiently large" must be left to future work. However, it is worth repeating that, as is well known, there is no single theory that is universally applicable to all of mathematics. One such attempt, Zermelo Frankel set theory⁽⁸⁾, founded on the Russell antimony such as the "set of all sets that do not contain themselves", and other similar "sets". Extensions of set theory to include proper classes⁽⁸⁾ may solve these problems but they leave untouched other aspects.

It is also not clear at this point how to exactly define universal applicability for the physical component of the theory, or whether one should even separate the theory into mathematical and physical components. In the case of quantum mechanics, one view⁽⁶⁰⁾ is that defining universal applicability to include all physical systems means that one must accept the Everett interpretation^(61,62) of quantum mechanics. This interpretation assumes that the whole universe is described as a closed system by a quantum state evolving according to quantum dynamical laws.

One way to avoid this may be to assume that universal applicability means that the theory is applicable only to systems that are subsystems or part of other larger systems that are in turn subsystems of other still larger systems.³ This includes both open and closed subsystems including those that may be isolated for a period of time.

At present it is not clear if an exact definition is needed. However a definition should be such to include subsystems described by a finite number of degrees of freedom and many systems, such as quantum fields, that are described by an infinite number of degrees of freedom. One should not be able to point to or describe a physical system occupying a finite region of space and time that is not included in the theory.

An essential aspect of this is that intelligent systems should be included in the domain of a coherent theory. If quantum mechanics is universally applicable

³A more precise statement of this might be: (1) If the theory is applicable to subsystem A , then there exist many subsystems B that contain A and to which the theory is applicable. (2) There are many subsystems A to which the theory is applicable. Furthermore the definition of "many subsystems" must be sufficiently broad to include all subsystems accessible to state preparation and experiment.

it then follows that intelligent systems are quantum mechanical systems. The observation that the only known examples (including the readers of this paper) of intelligent systems are macroscopic, with about 10^{25} degrees of freedom, does not contradict the quantum mechanical nature of these systems. This may well be a reflection of the possibility that a necessary requirement for a quantum system to be intelligent is that it is macroscopic. However, whether this is or is not the case, is not known at present.

That intelligent observers are both conscious self aware systems and quantum systems has been the basis for much discussion on consciousness in quantum mechanics^(15,63,64,65). Included are discussions on interactions between two quantum observers^(66,67). These avenues will not be pursued here as they do not seem to be the best way to progress towards developing a coherent theory of mathematics and physics.

It follows that the dynamics of the quantum systems carrying out the validation of quantum mechanics must be described by quantum dynamical laws. Thus quantum mechanics must be able to describe the dynamics of its own validation process. However validation of a theory involves more than just describing the dynamics of the systems carrying out the validation. Validation includes the association of meaning to the results of theoretical derivations and computations carried out by quantum systems (as computers). Meaning must also be associated to the results of carrying out experiments by quantum systems (as robots or intelligent systems).

This association of meaning to the results of quantum processes is essential. It is basic to determining which processes are either computational or experimental procedures. These processes are a very small fraction of the totality of all processes that can be carried out, most of which have no meaning at all. They are neither computations or experiments.

This association of meaning includes such essentials as the (nontrivial) assignment of numbers to the results of both computational and experimental process. A computation process or an experiment that halts produces a complex physical system in a particular physical state. What numbers, if any that are associated to the states depend on the meaning or interpretation of the process^(68,69). That is, the process must be a computational or experimental procedure. If it is, then one must know the property to which it refers. This is needed to know the association between theoretical computations and experimental procedures.

For example in quantum mechanics for some observable \check{O} and state Ψ , one must be able to determine which procedure is a computation of the expectation value $\langle \Psi | \check{O} | \Psi \rangle$. One must also know which experimental procedure corresponds to a measurement of this expectation value. This assumes that requisite experimental and theoretical calculation procedures exist for \check{O} and Ψ . As is well known the experiment must in general be repeated many times to generate the expectation value as a limit as $n \rightarrow \infty$ of the average of the first n repetitions of the experiment. Association of meaning to these procedures also includes all the components involved in determining that appropriate limits exist for both the computation procedures and experimental procedures.

A coherent theory of mathematics and quantum mechanics must be able to express as much of this meaning as is possible. Not only must it be able to express the dynamics of its own validation, but it must be able to express the meaning associations described above to the maximum extent possible. This is part of being able to maximally describe its own validation.

A similar argument applies to sufficient strength. That is, there must be a sense in which a coherent theory is sufficiently strong to include all properties that are predictable and capable of experimental test. Of course the problem here lies in the exact meaning of "all properties that ...". One may hope and expect that a coherent theory would be able to express to the maximum extent possible the meaning of "all properties that ...". And it should also be able to express the condition that it is also sufficiently strong.

These arguments are part of the basis for the requirement that the coherent theory maximally describe its own validation and sufficient strength. However, being able to generate such a description does not guarantee that the coherent theory *is* valid and sufficiently strong. A theory may be interpreted to express that it has some property, but it does not follow that it actually has that property. This possibility is taken care of by the other component of the basic requirement of a coherent theory, that it is valid and sufficiently strong.

3.4 A Coherent Theory and the Strong Anthropic Principle

The conditions that a coherent theory include both physics and mathematics and that it satisfy the requirement of maximal description of its validity and sufficient strength and be maximally valid and sufficiently strong, suggest that there may be a very close relation between the theory and the basic properties of the physical universe. It may be the case that at a very basic level the basic properties of the physical universe are entwined with and may even be determined by a coherent theory that satisfies the requirements.

Examples of such basic properties that may emerge from or be determined by the coherent theory include such aspects as the reason for three space and one time dimension (See Tegmark⁽⁴⁾ for another viewpoint), the strengths and reason for existence of the four basic forces, why quantum mechanics is the valid physical theory, etc.. Even if few or none of these properties are determined, one may hope that the theory will shed new light on already explained basic properties.

These possibilities suggest that a coherent theory with the requirement is related to the strong anthropic principle^(70,71,72). This principle can be stated in different ways. One statement is that "The basic properties of the universe must be such that [intelligent]life can develop"⁽⁷⁰⁾. Wheeler's interpretation as quoted by Barrow and Tipler⁽⁷⁰⁾ is that "Observers are necessary to bring the universe into being". A stronger statement is the final anthropic principle⁽⁷⁰⁾ "Intelligent information processing must occur and never die out".

The relation between this principle and a coherent theory can be seen by recasting the statement of the maximal validity and sufficient strength require-

ment into an existence statement or condition: *There exists a coherent theory of physics and mathematics that is valid and sufficiently strong and maximally describes its own validity and sufficient strength.* In this case the basic properties of the physical universe emerge from or are a consequence of the existence statement. That is, the basic properties of the physical universe must be such that the existence statement is true.

Another way to state this is that the basic properties of the physical universe must be such that a coherent theory is creatable. Since intelligent beings are necessary to create such a theory, it follows that the basic properties of the physical universe must be such as to make it possible for intelligent beings to exist. Since the intelligent beings, as physical systems, are part of the physical universe, the theory must, in some sense, also refer to its own creatability.

None of this implies that intelligent beings must exist, only that it must be possible for them to exist. Of course existence of intelligent beings is a necessary condition for the actual creation of such a coherent theory.

3.5 The Possible Uniqueness of a Coherent Theory

The requirement that a coherent theory of mathematics and physics is valid and sufficiently strong and maximally describe its own validity and sufficient strength would seem to be quite restrictive. Indeed one may speculate that the condition is so restrictive that there is just one such theory.

One reason this might be the case is that if there were several different coherent theories each satisfying the requirement, then there might be several different physical universes, with the basic physical properties of each universe determined by one of the theories. Yet we are aware of just one physical universe, the one we inhabit, with the basic properties determined by both physical theory and experiment. It follows that if the basic properties of the physical universe are determined by a coherent theory satisfying the requirement, then the existence of just one physical universe implies that there is just one coherent theory satisfying the requirement.

Viewed from this uniqueness perspective, the basic statement that there exists just one coherent theory of physics and mathematics that is valid and sufficiently strong and maximally describes its own validity and sufficient strength becomes a quite powerful restrictive condition. The reason is that it can be used with the arguments given above to obtain the result that there is just one physical universe with basic properties determined by the unique theory. And this should be our universe.

This possibility was discussed earlier as a restriction on the process of iterated extensions of a theory where the theory at each stage of the iteration was required to be such that formulas in the theory maximally described the validity and sufficient strength of the theories in the prior stages. Also the possibility of "limit coherent theories" was considered. Here the possibility that this restriction results in just one coherent theory is being considered.

If this line of reasoning is indeed valid and there exists just one coherent theory satisfying the basic requirement, then it would be very satisfying as it

answers the question, "Why does our physical universe have the properties it does?". Answer: The physical universe could not be otherwise as it is the only one whose properties emerge from or are determined by and determine a coherent theory. No other universe is possible because there is just one coherent theory satisfying the maximality requirement and each such theory is associated with just one physical universe.

At present this argument, although appealing, must be regarded as speculation. Whether it is true or not must await development of a coherent theory of physics and mathematics, if such is even possible.

4 Emergence of the Physical and Mathematical Universes

At present the main approach to physics seems to be that one assumes implicitly a physical universe whose basic properties exist independent of and a priori to a theoretical description, supported by experiment, of the universe. This is implied by reference to experiments as "discovering properties of nature". An a priori, independent existence of the physical universe is also implied in the expression used above "theoretical description, supported by experiment, of the universe".

The approach to mathematics is much more variable as there are many different interpretations of the meaning of existence in mathematics^(16,17,18). However, the Platonic viewpoint that is widely accepted, at least implicitly, is that mathematical objects exist a priori to and independent of a theoretical description of them with their properties to be discovered by mathematical research.

Here the position is taken that one should regard the basic properties of the physical and mathematical universes as very much entwined with a coherent theory of mathematics and physics. Neither the mathematical universe, physical universe, nor the coherent theory should be considered to be a priori and independent of the other two components. The basic properties of all three components should be considered to be emergent together, and mutually determined and entwined.

This means that, for the relation between the physical universe and the coherent theory, the basic physical aspects of the physical universe should be considered to emerge from and be determined by the basic properties of a coherent theory of physics and mathematics. Also the basic properties of a coherent theory should, in turn, emerge from and be determined by the basic properties of the physical universe.

It must be strongly emphasized that the emergence noted above does *not* mean that there is any arbitrariness to the basic physical properties and that an observer can choose them as he pleases. Rather the viewpoint taken here suggests that a coherent theory that is valid and sufficiently strong and maximally

describes its own validity and sufficient strength, is also maximally objective. The reason is that a maximally self referential theory refers to as much of its own consistency, validity, and strength as is possible, and the role of an observer or intelligent being is thereby minimized in determining the basic properties of the theory. In this case the basic properties of the universe as described by a coherent theory must appear to any observer to be objective and real and maximally independent of the existence and activities of an observer. That is what one means by objectivity.

4.1 Dependence of Reality Status on Theory

In one sense the idea of the emergence of basic physical properties of the universe is already in use. This is based on the observation that the more fundamental properties of the physical universe require many layers of theory supported by experiment to give them meaning. Their reality status is more indirect as it depends on many layers of theory supported by experiment.

For example the existence and properties of atoms is indirect in that it is based on all the experimental support for the many theoretical predictions based on the assumed existence and properties of atoms. One does not directly observe individual atoms. Pictures of individual atoms taken with an electron microscope depend on many layers of theory and experiment to determine that a complex physical system is an electron microscope and that the output patterns of light and dark shown on film or a screen are not meaningless but have meaning as pictures of individual atoms.

The physical reality and properties of more fundamental systems, such as quarks and gluons, are even more indirect than for atoms and depend on more intervening layers of theory and experiment. The same holds for neutrinos as fundamental systems whose reality status and properties are quite indirect. Experimental support for the existence of these particles depends on the layers of theory, which may include quantum electrodynamics, and all the supporting experiments needed to describe the proper functioning of large particle detectors and assigning meaning to the output of the detectors.

A similar situation exists for large, far away objects such as quasars. The reality status and physical properties of these systems are based on the theories of relativity and interactions of electromagnetic fields, etc.. These are needed to interpret the observations made using telescopes and to describe the proper functioning of telescopes and other equipment used.

On the other hand the reality status and some properties of other physical systems require little or no theoretical or experimental support. For example, the existence and hardness of rocks or the existence of the sun and the facts that it is hot, bright and round, are directly observed properties. Little theory with supporting experiment is needed to make these observations. Other properties of these objects are more indirect. An example is the description of the sun as a gravitating body generating energy by thermonuclear fusion of hydrogen.

It should be emphasized that none of the above implies that systems such as quarks, atoms, and quasars are any less real and objective than are rocks

and the sun. Rather the point is that the reality of their existence and their properties are more indirect in that they depend on more intervening layers of theory and experiment than is the case for rocks and the sun. Also the reality of all the properties of quarks, atoms and quasars, is indirect in its dependence on layers of theory and supporting experiment. For nearby moderate sized objects some of the properties are quite direct and some are more indirect. For example, as noted above, direct properties of the sun are that it is hot bright and round. Indirect properties include the source of its energy.

It is necessary to emphasize the importance of the intervening layers of theory and experiment needed to support the proper functioning and interpretation of complex equipment. Since most equipment involves the electromagnetic interactions between systems or between fields and systems, the theory of these interactions must be well understood to ensure that a given physical system is a properly functioning piece of equipment. This is needed to ensure that certain properties of the system represent output and that the output has meaning.

5 Language is Physical

Additional support for the close relation between a coherent theory and physics and mathematics is based on the essential nature of language. This applies to formal languages such as those studied in mathematical logic and informal languages such as English that are used for communication or transmission of information among intelligent beings or for thinking.

The essential point to make here is that language is physical. All language expressions must be represented by physical systems in some states. This applies irrespective of whether the language is written or spoken and of how the basic units of the language are organized. Depending on the physical representation the basic units can be a set of symbols or characters organized into strings as in written language, or a set of syllables appropriate for a spoken language.

The importance of this aspect is emphasized by the observation that, if it were not possible to represent language by states of physical systems, it would not be possible to communicate or acquire knowledge, or even think. It is an essential part of the existence of intelligent observers, as language is an essential part of the communication of information.

5.1 Physical Models of Language

There are many possible representations for languages based on symbols, words, and word strings. Examples include printed text, modulated waves moving through some medium as is the case for spoken language, or language transmitted optically by use of photons. As a more specific example, consider the text of this paper. Each letter, word, paragraph, etc. is represented by physical systems in different physical states. This is the case whether the paper appears as printed material on pages of paper or as patterns of light and dark regions on a

computer screen. If the text is read, as in a lecture, the language is represented by time variations in phase and amplitude of sound waves. It also applies if the paper is represented as a large tensor product state of quantum systems where each letter of the language is represented by a state of a component quantum system in the tensor product.

Some details of a representation of language text by arrangements of ink molecules located on a two dimensional lattice of potential wells are described in the Appendix. This representation is just one of many possible. Another one, based on spin projection states of systems, will be outlined below. More generally, one can use any physical observable with a discrete spectrum and eigenstates that can be associated with the language units such as symbols or syllables. Also it must be possible to actually physically prepare systems in these eigenstates and to measure the properties of these systems corresponding to different properties of the language text.

A simple way to construct a quantum mechanical product state of a language is based on the well known use of tensor products of qubit states as a binary representation of numbers. Extension of this model to k -ary representations of numbers is done by considering each component to be a qukit or k dimensional system. The states of each qukit in some chosen basis represent the k digits or numerals of the representation. This can easily be taken over to a language representation by letting the basis states of each qukit correspond to the k symbols in the alphabet of the language. (From now on the basic language units will be referred to as symbols.)

To be specific let the language alphabet consist of the 13 symbols $0, 1, c, v, f, r, =, \vee, \exists, \neg, (,), \#$. 0 and 1 are constants; c, v, f, r are constant, variable, function and relation symbols; $=$ is the equality symbol; \vee, \exists, \neg are logical symbols for conjunction, exists, and not; $(,)$ are left and right parentheses used to distinguish components of words; and $\#$ is a spacer symbol. This language is generic and can be used to construct a language suitable for any theory for which a denumerable number of constants in the language suffices. Details of how to implement this will not be done as it is not relevant here.

Consider a 1 dimensional lattice of points $j = \dots - 1, 0, 1, \dots$ with a spin system located on each point j of the lattice. The spin σ of each system is such that $2\sigma + 1 \geq 13$. If I is a one-one map from the symbol set to the spin projection eigenstates, then spin projection basis states of the form $|m, j\rangle$ where $-\sigma \leq m \leq \sigma$ and m is in the range set of I correspond under I to symbol states at site j . That is, the symbol state $|s, j\rangle$ corresponds to $|I(s), j\rangle$ where s is any of the 13 symbols.

Different types of symbol string states will be referred to in the following. Expression states correspond to multisymbol states of the form $|\underline{s}, [a, b]\rangle = \otimes_{j=a}^b |\underline{s}_j, j\rangle$ Here $[a, b]$ is an interval of points on the lattice and \underline{s} is a function from the set of lattice points to the set of 13 symbols with \underline{s}_j the value of \underline{s} at j . Word states are expression states with no spacers included. They have the form $|w, [a, b]\rangle = |\underline{s}, [a, b]\rangle$ where $\underline{s}_j \neq \#$ for $a \leq j \leq b$. Formula states are word states with the symbols combined according to specific spelling rules. In English these would correspond to the words in a dictionary. In formal languages studied in

mathematical logic they are sometimes referred to as well formed formulas.

Each of these types of expression states, $|\underline{s}, [a, b]\rangle$, corresponds, under I , to a product state of spin projections on the interval $[a, b]$. Since the lattice is infinite the system has infinitely many degrees of freedom and is best treated by quantum field theoretic methods. Here this will be avoided by restricting the set of basis states to have the form $|\underline{s}, [-\infty, \infty]\rangle \equiv |\underline{s}\rangle$ where at most a finite number of values of \underline{s} are different from $\#$. In the spin projection model this corresponds to an arbitrary but finite number of spin systems on the lattice having spin projections different from that corresponding to $I(\#)$. The Hilbert space spanned by this basis is separable as there are denumerably many such basis states.

Because $\#$ denotes a spacer, one is limited to expression states corresponding to an arbitrary but finite number of word states where each word state is of finite length. If $|\underline{\#}, [a, b]\rangle$ denotes a spacer string state in the interval $[a, b]$, then all the basis states can be written as alternating word and spacer string states:

$$|\underline{s}, [a, b]\rangle = |\underline{\#}, [-\infty, a]\rangle \otimes |w_1, [a + 1, \alpha_1]\rangle \otimes |\underline{\#}, [\alpha_1 + 1, \alpha_2]\rangle \otimes |w_2, [\alpha_2 + 1, \alpha_3]\rangle \otimes \cdots \otimes |w_m, [\alpha_{2m} + 1, b]\rangle \otimes |\underline{\#}, [b + 1, +\infty]\rangle. \quad (1)$$

This state consists of m word states separated by spacer string states with the lattice intervals occupied by each word or spacer string state shown by the subscripted as .

Pure states of the lattice systems can be represented in general as linear sums of word string states, Eq. 1, where the sums range over the lengths and number of word and spacer string states in $|\underline{s}, [a, b]\rangle$. Sums over the interval variables a, b are also included.

5.2 Efficient Implementability

A basic requirement on any physical model such as the spin systems or the ink molecule systems to be a representation of a language under some interpretation I is that transformations corresponding to basic syntactic operations must be efficiently implementable. Efficient implementability means that there must exist an actual physical procedure for carrying out each operation and the operation must be efficient. That is, the space-time and thermodynamic resources required to implement the operations on states with n symbols must be polynomial in and not exponential in n .

Syntactic operations are state transformations that create new expression states or change existing ones where the dynamics of the operations depends only on the syntactic properties of the states as expression states of a language and on the physical properties of the systems representing the symbols, words, and word strings. The dynamics is independent of any meaning the physical states might or might not have under some interpretation as text with meaning on a language or as formula states in a (consistent) theory described by axioms.

Examples of syntactic operations include concatenation of two expression states, addition of a negation symbol state to some formula state, generation or

enumeration of formula states or term states according to the rules for generating terms or formulas in a language, enumeration of the theorems of a theory, etc.. These operations depend only on the symbols represented by the model states and how they are combined into expression states and word string states. If the dynamics of the physical system does not result in efficient implementation of these operations, then then it is not an admissible model of the language.

It is useful to discuss efficient implementability in terms of a simple dynamical model. Let $U_M(t) = e^{-iH_M t}$ denote a unitary dynamics of a multistate system M moving along a lattice $[-\infty, \infty]$ and interacting locally with spin systems at each lattice point under the action of a Hamiltonian H_M . If the complete initial system state at time 0, $|\underline{\#}\rangle|i, 0\rangle$, describes the lattice system with the spin projections at all sites in the state corresponding to the spacer symbol, and M in internal state $|i\rangle$ and at lattice site 0, then $\psi(t) = U_M(t)|\underline{\#}\rangle|i, 0\rangle$ is the state at time t . In this case $\psi(t)$ can be written as a linear sum over all internal states and positions of M and over all word string states of the form of Eq. 1^(11,73).

The requirement that H_M is physically implementable means that there must exist actual physical procedures, as part of the experimental setup, that can be carried out to ensure that H_M is the correct Hamiltonian description of the dynamics of M interacting with the lattice spin system. Also it must be possible to physically prepare M and the spin lattice in the initial state corresponding to $|\underline{\#}, i, 0\rangle$.

In general $U_M(t)$ describes the dynamics of M starting at position 0 and generating linear superpositions of word string states as linear superpositions of spin projection string states of increasing length on the lattice. H_M may be such that M can move in either direction. Also M may return at any time to change the states of systems in a lattice interval and might even return periodically to any lattice site to make changes.

The requirement that H_M efficiently implements some operation excludes this periodic return possibility. H_M must be such that the probability of a change in the spin projection state of a system at any lattice site decreases as time increases and that the state of lattice systems in any finite lattice region is asymptotically stable.

The requirement of efficiency has two components. One is asymptotic stability and the other is the rate at which stability is approached. As noted, stability is necessary because random or fluctuating changes in the states of spin systems in any lattice interval mean that the spin projection states of systems in an interval do not correspond to a specific word or formula in a language. In this case all parts of the text could be changed at any time irrespective of when they were generated.

The requirement of efficiency can now be defined in terms of the rate at which the states of systems in any lattice region approach asymptotic stability. The system is asymptotically stable under the dynamics $U_M(t)$ if the limit

$$\langle \underline{\#}, i, 0 | P_{\underline{s}, [a, b]}(\infty) | \underline{\#}, i, 0 \rangle \equiv \lim_{t \rightarrow \infty} \langle \underline{\#}, i, 0 | P_{\underline{s}, [a, b]}(t) | \underline{\#}, i, 0 \rangle \quad (2)$$

exists for each interval $[a, b]$ and each state $|\underline{s}, [a, b]\rangle$. Here the Heisenberg rep-

resentation is used where

$$P_{\underline{s},[a,b]}(t) = U_M^\dagger(t)P_{\underline{s},[a,b]}U_M(t). \quad (3)$$

and $P_{\underline{s},[a,b]}$ is the projection operator for the state $|\underline{s}, [a, b]\rangle$.

The existence of the limit is expressed by the statement that for each m there is a time τ such that for all $t, t' > \tau$

$$|\langle \underline{\#}, i, 0 | P_{\underline{s},[a,b]}(t) | \underline{\#}, i, 0 \rangle - \langle \underline{\#}, i, 0 | P_{\underline{s},[a,b]}(t') | \underline{\#}, i, 0 \rangle| < 2^{-m}.$$

In addition this must hold for all intervals $[a, b]$ and all \underline{s} in $[a, b]$, i.e. for all states $|\underline{s}, [a, b]\rangle = \otimes_{j=a}^b |\underline{s}_j, j\rangle$.

Let $n = b - a + 1$ be the length of \underline{s} . It follows from the existence of the limit that for each n, \underline{s}, m there exists a smallest $\tau \equiv \tau(n, \underline{s}, m)$ that satisfies the above expression. The definitions and properties of the dynamical system give the result that

$$\begin{aligned} \tau(n, \underline{s}, m) &\leq \tau(n, \underline{s}, m') && \text{if } m < m' \\ \tau(n, \underline{s}, m) &\leq \tau(n', \underline{s}', m) && \text{if } \begin{cases} \underline{s} \text{ is an initial part of } \underline{s}' \\ \text{and } n' \text{ is the length of } \underline{s}'. \end{cases} \end{aligned}$$

The dependence on \underline{s} can be removed by defining $\tau(n, m) = \max_{\underline{s}: L(\underline{s})=n} \tau(n, \underline{s}, m)$ where the maximum is over all \underline{s} whose length $L(\underline{s}) = n$. From the above one has $\tau(n, m) \leq \tau(n', m')$ if $n < n'$ or $m < m'$. Also the definition of the limit holds if $\tau(n, m)$ replaces τ .

The requirement of efficiency can be expressed in terms of the n dependence of $\tau(n, m)$ for each m . The dynamics $U_M(t)$ is efficient if for each m , the dependence of $\tau(n, m)$ on n is polynomial in n , or $\tau(n, m) = K_m n^{\ell_m}$. The dynamics is inefficient if the dependence is exponential in n , or $\tau(n, m) = C_m 2^{n^{\mu_m}}$. Here K_m, C_m, ℓ_m, μ_m are m dependent positive constants with μ_m usually ~ 1 . (The possible presence of $\log n$ factors is ignored here.) In most applications the requirement of efficiency also means that ℓ_m is not too large, and is of order unity.

The above discussion has ignored the dependence of τ on a , i.e. $\tau = \tau(n, a, \underline{s}, m)$. Such a dependence exists because the values of τ depend on where \underline{s} is located in the lattice. However this dependence, which can be expressed by a location dependence of the constants $K_m = K_{m,a}$ and $C_m = C_{m,a}$ does not affect the rate of convergence.

So far H_M has been required to be such that it is physically implementable and that $\langle \underline{\#}, i, 0 | P_{\underline{s},[a,b]}(t) | \underline{\#}, i, 0 \rangle$ is asymptotically stable for any state $|\underline{s}, [a, b]\rangle$ and that stability be approached efficiently. However no specific operation has been mentioned.

If H_M is to efficiently generate a specific word string state, $|\underline{X}\rangle$ where $\underline{X}(j) = \#$ for all j outside an interval $[a, b]$, then one requires that the dispersion of the limit probability $\langle \underline{\#}, i, 0 | P_{\underline{X}',[a,b]}(\infty) | \underline{\#}, i, 0 \rangle$ about $\langle \underline{\#}, i, 0 | P_{\underline{X},[a,b]}(\infty) | \underline{\#}, i, 0 \rangle$ is small for $|\underline{X}', [a, b]\rangle \neq |\underline{X}, [a, b]\rangle$ and that $\langle \underline{\#}, i, 0 | P_{\underline{X},[a,b]}(\infty) | \underline{\#}, i, 0 \rangle = 1 - \epsilon$ with ϵ small.

In this type of model, H_M is required to be such that the probability distribution is concentrated around a particular word string. Whether H_M has this property or not depends on the physical model and Hamiltonians that are efficiently implementable for the model. The ink molecules on a lattice (Appendix) are likely to be a model of this type. This is indicated by the wide use of printing and typing. Implementation of such a model by a dynamics that is unitary, if such is possible, requires the presence of many auxiliary systems, including a supply of ink molecules.

For microscopic physical systems, such as the spin projection systems, it is less clear if this type of model giving a small dispersion around just one state is efficiently implementable. In this case it may be appropriate to consider models closer in spirit to those considered in quantum computation. Then one requires that H_M is such that a linear superposition of word string states is generated where the probability is high that each component in the superposition has a specific property.

One example would be a theorem enumeration process for an axiomatizable theory where each word string state in the superposition is a string of formula states that corresponds to a proof in the language of the theory. Each formula is either an axiom (logical or nonlogical) of the theory or a formula derived from an already generated formula in the string by application of one of the logical rules of deduction. Thus the state generated by $U_M(t)$ becomes a linear superposition of proof states in which each formula state in each string is a theorem of the theory. Since the enumeration process does not halt, the length of each formula string state in the superposition increases with increasing t . Of course the existence of such a model depends on the existence of a Hamiltonian such that the dynamics with the required properties is efficiently implementable.

The usefulness of the simple model of a system M moving along a lattice of systems derives from the observation that there are several interesting examples of dynamical systems generating output that may represent strings of words in some language. Any quantum system generating text, such as the writing of a research paper, is an example. Another would be a complex quantum system, such as a quantum robot⁽⁷⁴⁾ or intelligent system, moving about in a complex environment of quantum systems and generating output in the form described above. Relevant questions for this example include "Does the output have meaning to us as external observers?; If so, what is the meaning?". "Does the output also have meaning to the system M generating the output? If so, do the two meaning interpretations coincide?"

A simple example of such a system M moving along a one dimensional lattice initially in the constant spacer state $|\# \rangle$ and generating states that are linear superpositions of word string states in a very simple language has been described^(11,73). Here some word states were taken to have meaning in that they denoted the appearance or nonappearance of other specified word states in the word string states.

It was seen that the requirements that the dynamics of M be valid (any word that has meaning must be true) and complete (each word with meaning must appear in some word string state at some time) imposed restrictions on $U_M(t)$.

Here the requirement of sufficient strength becomes that of maximal strength which is the same as completeness for this example. This quantum mechanical model differs from a classical model in that the meaning domain of the word states with meaning was limited to those strings containing the word state. No word state $|\underline{W}\rangle$ was interpreted as implying any property of word string states not containing $|\underline{W}\rangle$. It was also possible for both a word state with meaning and its negation to appear and not violate consistency. In this case with multiple branching paths of word string states, consistency merely required that no word state with meaning and its negation state appear on the same path (or word string state). It did not prevent them from appearing on different paths. This option is not available in the classical case with only one path present.

5.3 Meaning and Information

The emphasis here on the physical nature of language is to be contrasted with Landauer's point that information is physical⁽²⁸⁾. Landauer's point stresses the physical nature of the information content of the states representing expressions in a language. Here the emphasis is directly on the physical nature of language independent of the information content of any word string state or formula string state.

The independence of the physical nature and information content of language is emphasized by the observation that the relationship between the meaning, if any, of language expression states and their algorithmic information content, is, at best, complex and may be nonexistent. The proof, which follows that given elsewhere⁽⁷³⁾, consists of showing the unitary dynamics for simulation of theorem enumerations for two theories with about the same algorithmic complexity. But the theorems of one of the theories have meaning and those of the other do not.

Let U_1 and U_2 represent the unitary dynamics for a single time step (for simplicity a discrete time step model is used here) of simulations of theorem enumeration machines for two different theories, T_1 and T_2 . Such machines can be modelled as described above by a complex head M moving on a one dimensional lattice of quantum systems, i.e. as a quantum Turing machine.

There are different ways to describe what is meant by simulation^(59,75). However here it is sufficient to require that for each $k = 1, 2$ the dispersion of the probability $|\langle \underline{s}(n) | (U_k)^n | \# , i, 0 \rangle|^2$ around the specific expression state $|\underline{s}_k(n)\rangle$ is required to be less than ϵ for all $n < N \equiv N(\epsilon)$. The state $|\underline{s}_k(n)\rangle$ is a word string state, Eq. 1, in which each word state is a theorem state of T_k .

The n dependence follows from the fact that theorem enumeration systems described by the U_k do not halt and, as M moves along the lattice, the number of theorem states in $|\underline{s}_k(n)\rangle$ increases with increasing n . The n dependence in the states $|\underline{s}(n)\rangle$ shows that as n increases the length of the word string states which are close to $|\underline{s}_k(n)\rangle$ also increases with n . The dependence of n on ϵ shows that the dispersion is expected to increase with increasing n . This increase is expressed by the condition that $N(\epsilon)$ gets smaller as ϵ decreases.

Based on work in the literature^(75,76,77), the quantum algorithmic complexity of the U_k is described as the length of the shortest qubit string states as input to a universal quantum Turing machine U that give a simulation of the U_k . If $|q_k\rangle$ are product qubit states such that $U^n|q_k\rangle$, acting on $|\#\#, i, 0\rangle$, simulates $(U_k)^n$ acting on $|\#\#, i, 0\rangle$ for $k = 1, 2$, then the algorithmic complexity of U_k is the length of the shortest state $|q_k\rangle$ that gives the simulation. Again by simulation it is sufficient to require that the dispersion of the probabilities $|\langle s(n) | (U^n|q_k\rangle|\#\#, i, 0)\rangle|^2$ around expression states $|s_k(n)\rangle$ for $k = 1, 2$ is less than ϵ for all $n < N \equiv N(\epsilon)$ where the $|s_k(n)\rangle$ is a theorem string state for T_k .

This definition extends to quantum Turing machines the definition based on classical machines⁽⁵¹⁾ that defines the algorithmic complexity of a theory as the length of the shortest program as input to a universal machine that enumerates the theorems of the theory. Since it is decidable whether a formula is or is not an axiom, the states $|q_k\rangle$ have finite lengths which are proportional to the algorithmic complexities of the axiom sets Ax_1 and Ax_2 for the two theories⁽⁷⁵⁾.

Let Ax_1 and Ax_2 be two sets of axioms that have about the same algorithmic complexities and are such that the theory T_1 is consistent and T_2 is not consistent. It follows from Gödel's completeness theorem^(7,5) that, since T_1 is consistent, it has a model in which the theorem states have meaning. Since T_2 is inconsistent, it has no model so the theorem states enumerated by U_2 have no meaning. However the algorithmic complexities of U_1 and U_2 are about the same since Ax_1 and Ax_2 have about the same complexities.

The above proof depends on the existence of a theory T_2 that is inconsistent and has about the same algorithmic complexity as T_1 . To support this note that since the number of sets of (nonlogical) axioms increases exponentially with the algorithmic complexity of the set in terms of the number of bits of the shortest program needed to list the axioms⁽⁵¹⁾, the number of theories increases exponentially with the algorithmic complexity of the axiom sets. Also there are at least as many inconsistent axioms sets as consistent ones. To see this let Ax'_1 be obtained from Ax_1 by adding the negation of a formula in Ax_1 to the axioms in Ax_1 .

T'_1 is not a good candidate for T_2 because algorithmic complexity of the proof of inconsistency for T'_1 is small. In particular the algorithmic complexity of T'_1 is bounded above by the length of the shortest proof⁴ of inconsistency of T'_1 . An inconsistency proof is a proof that terminates with a formula that is the negation of a formula appearing earlier in the string. It is clear from this that the length of the shortest proof of inconsistency for T'_1 is quite small, and any formula enumeration procedure is a theorem enumeration procedure.

T_2 must be a theory whose algorithmic complexity, in terms of a decision procedure for the set Ax_2 of axioms, is about the same as that of T_1 and is such that the length of the shortest proof of inconsistency of T_2 is bounded below by the algorithmic complexity of its own set of axioms. This guarantees

⁴A proof is a string of formulas such that each formula in the string is either an axiom or is derived from other formulas in the string by use of the logical rules of deduction.

that any theorem enumeration procedure for T_2 has an algorithmic complexity similar to that of T_1 . This follows from the observation that a suitable theorem enumeration procedure that checks for consistency is such that it checks each terminal formula in a proof to see if it is the negation of a formula appearing earlier in the string. If not, the procedure continues. If it is a negation, the procedure changes to a simple enumeration procedure for all formulas.

additional justification for the existence of a theory T_2 with the requisite properties is based on the observation that the number of random sequences increases exponentially with their length⁽⁵¹⁾. It follows from this that the number of proofs, even with the consistency check inserted as described above, that are random should also increase exponentially with their length. From this one concludes that for large n there should be many theories T_1 and T_2 with the requisite properties.

5.4 Gödel Maps

Gödel maps have an interesting feature for a coherent theory, or for any theory which is universally applicable to physical systems. To see this it is worth a brief review of what a Gödel map is and what it does.

These maps are an important component of the proof of the Gödel incompleteness theorems for arithmetic^(7,6). In this case a Gödel map is a map G from the expressions in the language of arithmetic to the natural numbers. The map extends the domain of arithmetic in the sense that syntactic properties of expressions in the language, which are not numbers, become properties of numbers. In this way metamathematical properties of the theory become mathematical properties of numbers which are in the domain of the theory. Then formulas of arithmetic can be interpreted by anyone who knows G as statements about properties of expressions in the language of arithmetic.

Many different Gödel maps are possible. An especially transparent one was first described by Quine⁽⁷⁸⁾ (See also⁽⁶⁾). The map includes a one-one map $d : Sy \rightarrow 0, \dots, k - 1$ of the k language symbols in Sy onto the k digits of a $k - ary$ number representation. The map also requires an association of lattice sites with powers of k . Typically one considers expression states starting at site 0. Then the site j symbol state $|s, j\rangle$ corresponds to the number state $|d(s), j\rangle$ which corresponds to the number $d(s) \times k^j$. If values of $j < 0$ are allowed then the numbers correspond to nonnegative $k - ary$ rational numbers with the " $k - al$ " point between $j = 0$ and $j = -1$.

Such a map can be used to map expression states as tensor product states of the 13 symbol states of the language described earlier into tensor product states of qukits that correspond to a $k - ary$ representation of numbers. Physically each symbol state or qukit state often corresponds to states of single systems as in the spin projection model described earlier. However models can be considered in which each symbol state in an expression product state correspond to an entangled state of several quantum systems.⁵ Examples of this are shown by

⁵One can also construct representations of numbers in which there is no correspondence

various quantum error correcting codes⁽⁷⁹⁾.

It is worth summarizing some of the main properties of Gödel maps G in a more general context. For any mathematical theory strong enough to include arithmetic, such as Zermelo Frankel set theory, the maps G extend the domain of the theory in the sense that language expressions are mapped into the theory domain. This means that syntactic properties of expressions can be defined through G , by properties of elements in the theory domain.

Gödel maps can also be used in physical theories. However, for these theories, they have some different properties. For a coherent theory of mathematics and physics, or for any physical theory that is universally applicable, a Gödel map does not extend the domain of applicability of the theory. The reason is that, since language is physical, all expressions of any language are already in the theory domain as states of physical systems. For these theories a Gödel map has a more limited function in that it determines which of the many possible physical representations of a language, if any, should be used to interpret physical system states generated by some dynamical process as expressions in some language.

Examples of these dynamical processes include creating written text in a language, speaking, or, for microscopic systems, generating quantum states by the action of some Hamiltonian that can be interpreted as expressions in some language. However other interpretations are possible and are often used. For example the states could be interpreted as k -ary representations of numbers or be given some other interpretation. Which interpretation is appropriate depends on the specific dynamics being considered.

Macroscopic computers, which are in such wide use, are a good illustration of these points. In these machines with a binary code, strings of 0s and 1s can be stored and manipulated as states of small magnetic regions located on some substrate. It makes no difference for the operation of the machine whether these strings are considered as binary numbers or as expressions in a language with a two letter alphabet. This argument extends immediately to k -ary representations in that it makes no difference as far as the dynamics of a particular machine is concerned whether the small magnetic regions on the substrate represent numbers or expressions in a language with a k symbol alphabet.

However, the usefulness of a machine as a computer has everything to do with the interpretation or meaning assigned to the input and output states and with the details of the dynamical evolution of the machine. Dynamical processes or machines that have useful meaning for the number representations need not have a useful meaning for the language expression representation. The converse also holds in that dynamical processes or machines that have meaning for language expressions have no useful meaning for k -ary representations of numbers. An example of the former is a machine that multiplies two numbers

between the tensor product representation of expression states or qubit states and the states representing numbers. An example of this using complex entangled states for $k = 2$ was shown in⁽⁵⁶⁾ for numbers $< 2^n$ with n arbitrary. A physical representation of numbers with entangled state structure representing that in the example is very unlikely to exist. The reason is that a necessary condition that states of quantum systems represent numbers is that the basic arithmetic operations be efficiently physically implementable^(68,69,56).

together to obtain the product, all in a k -ary representation. "Multiplying" two language expressions together to obtain another is not a useful or meaningful concept as far as syntactic properties of language expressions are concerned. Conversely a dynamical process that enumerates the proofs of theorems of a particular theory has no useful meaning in number theory as an enumeration of a string of extremely large numbers.

Interpretative maps play a similar role for microscopic quantum mechanical machines or dynamical processes. However, there is an additional problem in that one must know what basis to use for the interpretation. For example the spin projection model considered earlier requires knowing the direction of the axis of quantization in order to assign meaning to input and output states. Similarly, to interpret states of quantum computers as numbers, one must know the basis to use for the particular physical quantum system under consideration.

This problem is especially important in the case of a complex quantum system such as a quantum robot M moving about in a complex environment and interacting with various systems. To answer the question of whether M is generating physical systems in states with meaning and if so what the meaning is, requires not only knowing which physical systems to examine but what basis state representation to use. If one measures the states of physical systems in the wrong basis then there is a good probability that a misleading result will be obtained. Also, as is well known, the state of the measured system is changed so that it cannot be reread to determine the original state.

As a specific example suppose M is creating a product spin projection state that, as a specific string of spin projections along the z axis, has meaning to him. If an external observer measures the state as a string of spin projections along the x axis, then there is no way the observer can determine from his results if the states generated by M have meaning and, if so, what the meaning is. The same holds for the results obtained by M for any reading of his own output after the measurements by the external observer.

6 Discussion

At this point it is not known how to construct a coherent theory of mathematics and physics. However the material presented here may help in that it should be regarded as a general framework for constructing such a theory. The details and many aspects of a coherent theory remain to be worked out. Also some or many of the points and aspects described may need modification. However some of the points are expected to remain.

First and foremost among the remaining points is the requirement that the coherent theory maximally describe its own validity and sufficient strength and that it be valid and sufficiently strong. This requirement is expected to greatly restrict the range of allowed theories. It may even be so restrictive that just one theory satisfies it.

The requirement also has the advantage that it automatically ensures that

any theory satisfying it agrees with experiment. This follows from the definition of validity, that any physical property that is predictable by the theory and is testable by experiment, is true. Sufficient strength ensures that the theory makes sufficiently many and powerful predictions to be recognized as a coherent theory of physics and mathematics together.

This raises the problem that if the requirement that the theory agree with experiment is built into the structure of the theory itself, then one might think that the theory is not falsifiable or even testable. This is not the case. Even if the maximal validity and sufficient strength requirement is built into the theory it still must be tested. In particular the theory may be interpreted to state, by complicated expressions, that it satisfies the requirement. This would include a statement of maximal agreement with experiment. But is this in fact the case? Is the theory statement of this true or false? One still has to carry out experiments to find out.

One should also keep separate the requirement that the theory maximally agree with experiment from what the actual results are of carrying out the experiments. For instance, incorporation of the validity and sufficient strength requirement into the theory to the maximum extent possible may mean that the theory describes the existence of a map between a set of theoretical predictions and a set of experimental procedures, which are both described by the theory. The theory would also describe general properties of the map that correspond to agreement between theory and experiment.

However existence and general description of such a map in a coherent theory does not mean that a coherent theory is any different than present day physics regarding the need to carry out experiments to test the validity of theoretical predictions and determine detailed properties of the map. A coherent theory that maximally describes its own validity and sufficient strength may deepen the understanding between physics and mathematics, and may even suggest new experiments, but it should not change the status or need to carry out experiments. These are still needed to see if the theory is valid and sufficiently strong.

It also may be the case that the most basic aspects of the physical universe are a direct consequence of the basic requirement that there exist a coherent theory that maximally describes its own validity and completeness and is maximally valid and complete. Included are the reasons why space-time is $3 + 1$ dimensional, why quantum mechanics is the correct physical theory, and predictions of the existence and strengths of the four basic forces. However other aspects of the universe, which are also predictable by the theory, are not in this category and are subject to experimental test. This includes essentially all of the experimental and theoretical work done in physics.

There are also other possibilities to consider. For instance it may be the case that, as discussed in subsection 3.2, there is no single coherent theory. Instead there may be a nonterminating sequence of coherent theories, with each theory more inclusive than those preceding it. If this is the case then the $n + 1st$ theory may include in its domain the requirement that the preceding n theories all maximally agree with experiment. But there may be other theoretical predic-

tions in the $n + 1st$ theory that are not present in the first n theories whose experimental status is outside the domain of the $n + 1st$ theory.

Finally it should be noted that it may be worthwhile to replace validity in the basic requirement with consistency. In this case a coherent theory must maximally describe its own consistency and sufficient strength and it must be consistent and sufficiently strong. The advantage of this change is that, unlike the case for validity, consistency can be defined purely syntactically by reference to proofs only. Also consistency is related to semantic concepts through Gödel's completeness theorem^(7,5): a theory is consistent if and only if it has a model. The usefulness of this alternate approach is a question for the future.

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A Appendix

The physical representation of language considered here is based on the presence or absence of systems in small potential wells located on a two dimensional lattice of points on a solid state matrix. The description is quite simple and will be limited to the representation only. No dynamics corresponding to the generation of word sequences, such as text or those that correspond to proofs in a formal language, will be discussed.

The representation considered here is a model for text on printed pages in that the systems in the potential wells are ink molecules. Each symbol corresponds to a specific pattern of occupied wells surrounded by unoccupied wells. Expressions correspond to paths of symbols on the lattice. A solid state matrix with all potential wells unoccupied corresponds to a blank page. Locations of the wells on the page are given by X, Y coordinates x, y . Multiple pages can be considered by extending the lattice into three dimensions where $X - Y$ planes for different values of Z correspond to different pages.

Each potential well may or may not be occupied by ink molecules. Here an ink molecule is a complex system with many closely spaced internal states of excitation. The molecules are easily excited by absorption of ambient light of all visible wavelengths, and the excited states quickly decay by emitting cascades of infrared photons as heat or by transfer of phonons to the solid state matrix.

The state of an ink molecule in the ground state of the potential well at x, y in thermal equilibrium with an environment at temperature T is given by

$$\rho_{x,y} = \sum_E \frac{e^{-E/kT}}{Z} |E, 0\rangle_{x,y} \langle 0, E|. \quad (4)$$

Here $|E, 0\rangle_{x,y}$ denotes the ink molecule in a state with excitation energy E and in the ground state of the well located at lattice site x, y . Z is the partition function that normalizes the state. It is also assumed that the combination of the shape and height of each potential well and separation of the lattice points are such that the states $|E, 0\rangle_{x,y}$ and $|E, 0\rangle_{x',y'}$ are essentially orthogonal whenever $x \neq x'$ or $y \neq y'$.

To keep things simple the assumption is made that the energy spacing of the potential well states is large compared to kT where k is Boltzmann's constant. Based on this Eq. 4 is a good approximation to the state of the ink molecule in a well at x, y as the probability of being in a state above the ground state of the well is very small. It is also assumed that the internal excitation state of an ink molecule is essentially independent of whether the environment is visibly dark or well illuminated with visible light, provided only that both environments are at the same temperature.

The environmental bath also plays an important role in stabilizing the position states of the individual ink molecules to eigenstates of the individual potential wells. For example ink molecule states of the form

$$\sum_E \frac{e^{-E/kT}}{Z} \sum_{x,y,x',y'} c_{x,y} c_{x',y'}^* |E, 0\rangle_{x,y} \langle 0, E|_{x',y'}$$

would immediately decohere and stabilize^(80,81) to the diagonal form $\sum_{x,y} |c_{x,y}|^2 \rho_{x,y}$ with $\rho_{x,y}$ given by Eq. 4.

Let α be an arbitrary finite set of points on a lattice. The quantum state corresponding to one ink molecule in each well at all locations in α and all other wells unoccupied is given by

$$\rho_\alpha = \bigotimes_{x,y \in \alpha} \rho_{x,y} = \bigotimes_{x,y \in \alpha} \sum_{E_{x,y}} \frac{e^{-E_{x,y}/kT}}{Z} |E_{x,y}, 0\rangle_{x,y} \langle 0, E_{x,y}|. \quad (5)$$

Symbols of a language correspond to sets of different patterns of closely spaced occupied wells. To this end let α_S be the set of occupied locations corresponding to the symbol S . A potentially useful characterization of the set α_S is in terms of a location x, y that serves as a standard fiducial mark or location parameter for the symbol, and a set b of scaling and other parameters needed to uniquely characterize the symbol S . Using this notation, which replaces α_S by $S_{x,y,b}$, Eq. 5 becomes

$$\rho_{S_{x,y,b}} = \bigotimes_{x,y \in S_{x,y,b}} \rho_{x,y}. \quad (6)$$

Some examples will serve to clarify this. The straight vertical line extending for n lattice sites in the Y direction from x, y to $x, y + n - 1$ corresponds to the symbol "|" located at x, y . The point x, y locating one end of the symbol serves as a fiducial location convention for this symbol. For each x, y the physical state of "|" is given by $\rho_{|_{x,y,n}} = \bigotimes_{x'=x}^{x'+n-1} \rho_{x',y}$. Other examples are the symbol " / ", a

diagonal line of length n whose state is $\rho_{/_{x,y,n}} = \rho_\alpha$ with $\alpha = \{x, y; x+1, y+1; \dots; x+n-1, y+n-1\}$, and the "⊥" symbol with horizontal arm of length $2m+1$ and state description $\rho_{\perp_{x,y,n,m}} = \rho_\alpha$ where $\alpha = \{x, y; \dots; x, y+n-1; x-m, y+n-1; \dots; x+m, y+n-1\}$. The values of n, m serve as scale factors for the symbols. For example if "⊥" is described by n, m , then "⊥", which is the same symbol but is twice as large, would be described by $2n, 2m$.

These examples and Eq. 6 show the physical state representation for any printed symbol. This can be extended to give the physical states of words which are strings of symbols. To this end let $W(i)$ denote the i th symbol of the word W containing N symbols. Let the function ℓ be a function from the integers to locations on the lattice such that $\ell(i)$ is the location of $W(i)$. Also $\alpha_{W(i), \ell(i)}$ denotes the set of locations occupied by ink molecules for the symbol $W(i)$ at location $\ell(i)$. The sets $\alpha_{W(i), \ell(i)}$ are disconnected sets where the (unoccupied) spacing between the sets should be larger than the spacing, if any between the individual ink molecule locations within each $\alpha_{W(i), \ell(i)}$.

The state ρ_W for the word W is given by

$$\rho_W = \bigotimes_{i=1}^N \rho_{\alpha_{W(i), \ell(i)}} \quad (7)$$

where $\rho_{\alpha_{W(i), \ell(i)}}$ is given by Eq. 5. In terms of fiducial marks and scale factors

$$\rho_{W, \underline{x}, \underline{y}, \underline{b}} = \bigotimes_{i=1}^N \rho_{W(i), \underline{x}(i), \underline{y}(i), \underline{b}(i)} \quad (8)$$

with $\rho_{W(i), \underline{x}(i), \underline{y}(i), \underline{b}(i)}$ given by Eq. 6. Here $\underline{x}, \underline{y}, \underline{b}$ denote functions from the symbols in W to x and y lattice positions, with $\ell(i) = \underline{x}(i), \underline{y}(i)$, and to a set of scale factors for the symbols.

It is clear that this representation can be extended to strings of words and to text in general. In this case if \underline{W} denotes text or a string of M words and $\underline{\ell}$ denotes a lattice location function for the M words in \underline{W} then

$$\rho_{\underline{W}, \underline{\ell}} = \bigotimes_{j=1}^M \rho_{W(j), \underline{\ell}(j)}. \quad (9)$$

The location function $\underline{\ell}$ determines how the words are organized into text in the same way that the functions $\ell(j)$ for each j determine how the symbols in the j th word are organized into a word. For many languages, texts are often organized into lines of symbols in one space direction with successive lines ordered in an orthogonal direction. Successive pages are then ordered in the third space direction. Here spatial distances between symbols, lines, and pages are used for the ordering. This organization is reflected in the functions $\underline{\ell}$ and $\ell(j)$.

The point of this is to emphasize that, in this model, the functions $\underline{\ell}$ and $\ell(j)$ are given by the rules used to read the words and text. These rules are given by the dynamics of the reading process. The dynamics of this process are not completely arbitrary. They are subject to the requirement of efficient physical

implementation. This requirement, which was discussed elsewhere in the context of representing numbers by states in quantum mechanics^(68,69,56), means that there must be a physical process which can read the text and that the space-time and thermodynamic resources expended to implement the reading must be minimized. In particular the resources expended must not be exponential in the number of symbols read.

It is worth discussing this in a bit more detail. Consider for example symbols scattered about on an infinite $X - Y$ lattice. Any reading rule in which the determination of the location for reading the $n + 1st$ symbol is based on what the first n symbols were requires an exponential amount of resources. This is based on the observation that if there are m symbols in the language, then for each n the rule must distinguish among m^n alternatives to make the determination.

Reading rules in use do not have this property in that determination of the location of the $n + 1st$ symbol from the value of n does not depend on the state of the first n symbols. As such the rules are efficient in that the resources expended are polynomial in the number of symbols read. Since the requirement of polynomial efficiency is quite weak, there are many rules that satisfy this condition, so one would want to pick rules that more or less minimize the free energy resources expended. These are the ones used in practice and include those used to read text on a page.

The model described is clearly robust in the sense that each reading of the text has a very small probability to change the individual symbol states or move them about on the lattice. Thus it can be read many times with the cumulative probability for changing the text remaining small. Physically this is a consequence of the fact that photons in the visible light range excite the ink molecules to internal excited states. The potential wells and interactions with the component atoms in the molecules are such that the amplitudes for exciting an ink molecule to an excited well state, or to move it from one lattice location to another, are very small. However, sufficiently many repeated readings can move ink molecules around and make significant changes in the quantum state of the text.

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